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A Note on Qualitative Economics for Univalence of Nonlinear Mappings

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1. Introduction

Qualitative economics suggested in Samuelson (1947) is supposed to be concerned with comparative statics, and as such the study was started by Lancaster (1962) in a systematic way. A survey covering the period until 1970 is provided in Allingham and Morishima (Morishima et al. (1973, pp. 3–69)), and a recent article by Quirk (1997) summarizes the researches with a special emphasis on stability hypothesis based upon Maybee's approach (Maybee (1966)). (Quirk and Saposnik (1968) present introductory materials.)

On the other hand, Bandyopadhyay and Biswas (1994) obtained theorems on univalence assuming some sign patterns of nonlinear functions. (Their first theorem, however, is included in a more general proposition by Moré (1972, Theorem 3.3, p. 365).) Quirk (1997) also pays attention to L-matrices, i.e., matrices whose regularity is determined by the sign pattern alone.

In this note, we take up nonlinear mappings and investigate the

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relationship between the sign-regularity of Jacobian matrices and the univalence of those mappings. The idea is extremely simple, and probably our first result reported here has been stated elsewhere. The method can, however, be applied to much wider class of mappings, e. g., discontinuous mappings on ordered integral domains. In the final section, we will explain our future research projects. In the next section, our framework is given, and in section 3, the main theorem in the differentiable case is proved.

2. Assumptions and Notation

We consider a nonlinear mapping $f \equiv (f_1, f_2, \dots, f_n)'$ from a domain D in the n-dimensional Euclidean space R^n into R^n. The prime means the transposition of a vector. We make the following assumptions :

A 1. The domain D is convex.

A 2. Each f_i is Gateaux-differentiable.

Let us denote by $J(x)$ the Jacobian matrix at $x \in D$, with $J_{ij}(x)$ or J_{ij} being the (i,j)-element of $J(x)$. We classify the sign patterns of J_{ij} on D into four categories.

(1) $J_{ij} = +$: that is, f_i is always increasing with respect to x_j.

(2) $J_{ij} = -$: that is, f_i is always decreasing with respect to x_j.

(3) $J_{ij} = 0$: that is, f_i is not dependent upon x_j.

(4) $J_{ij} = *$: the remaining cases complementary to the above (1) to (3).

Now we give the definition of sign-regular Jacobian matrices.

Definition : A given Jacobian matrix is sigh-regular on the domain if its regularity is judged only by its sign pattern classified in the above four categories +, –, 0, and *.

NB. A sign-regular matrix, defined in a similar manner, is sometimes called an L-matrix or an SNS-matrix (sign-non-singular matrix).

Here are three examples of sign-regular pattern :

$$\begin{pmatrix} + & * \\ 0 & - \end{pmatrix}, \quad \begin{pmatrix} + & + & + \\ - & + & + \\ 0 & + & - \end{pmatrix}, \text{ and } \begin{pmatrix} - & + & - & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{pmatrix}.$$

The last example is from Quirk (1997, p. 133) with the (1, 3)-element changed from 0 to $-$. When verifying the sign-regularity of matrices, simply regard the sign $*$ as taking any value, positive, zero, or negative.

3. Main Theorem

Our main result in this note is :

Theorem 1. If the Jacobian matrix is sign-regular, the mapping is univalent on the domain.

Proof. Suppose there exist two vectors x, y in D such that

$x \neq y$ and $f(x) = f(y)$.

This implies that $f_i(x) = f_i(y)$ for all i. Let $w \equiv y - x \neq 0$. Then for each i, by the mean value theorem (Ortega and Rheinboldt (1970, p. 68)) (or more specifically Rolle's theorem), there is a vector z_i, which is an interior point of the line segment $[x, y]$, such that

$f_i'(z_i) \cdot w = 0,$

where $f_i'(x)$ is the gradient of f_i at x, i.e., $\left(\frac{\partial f_i}{\partial x_i}, \frac{\partial f_i}{\partial x_2}, \ldots, \frac{\partial f_i}{\partial x_n}\right)$, and the symbol $\cdot$ stands for inner product. By arranging the above equations for all i's in a vertical way, we have

$$\begin{pmatrix} f_1'(z_1) \\ f_2'(z_2) \\ \vdots \\ f_n'(z_n) \end{pmatrix} \cdot w = 0.$$

This linear equation contradicts the sign–regularity because $w \neq 0$ and each f_{ij} follows the sign pattern stipulated for the Jacobian matrix. This completes the proof. □

4. Research Topics

Various generalizations are possible.

(1) We can introduce new categories of sign pattern +0, that is, f_i is either increasing or keeps the same value as x_j increases : similarly another new category –0.

(2) More importantly, a given mapping need not be differentiable, nor even continuous. Bandyopadhyay and Biswas (1994) noted in section 1 proved their first theorem on univalence assuming semi–qualitative sign patterns of nonlinear functions without requiring continuity. (Their first theorem, however, is included in a more general proposition by Moré (1972, Theorem 3.3, p. 365).) Note that in the above classification of sign patterns, we do not use the continuity of each component function : we depend merely on whether a function is increasing, decreasing, or independent of x_j. We may prove the injectiveness by mathematical induction, using a method similar to the above differentiable case.

(3) A given space need not be Euclidean. All we need is an ordered vector space which accommodates the construction of matrix as

well as determinant theory where $Mx = 0$ implies the vector $x = 0$ provided the determinant of M is not zero. Since we do not use inverse matrices, the above requirement is satisfied by any integral domains, more precisely ordered integral domains. By this extension we can deal with the univalence problem for models with indivisibility of commodities and/or processes.

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Abstract

This note is aimed at presenting an easy and simple proposition on the univalence of a given nonlinear differentiable mapping whose Jacobian matrix has sign-regularity. First the notion of sign-regularity of Jacobian matrix on a domain is defined. We classify the sign patterns into four categories : plus, minus, zero, and the rest. The plus sign is given to the (i, j) entry of the Jacobian matrix when the i-th component function is always increasing with respect to the j-th coordinate variable, the negative sign when the function is always decreasing, and the sign of zero when the function does not include the j-th coordinate variable. Otherwise, the sign is set as an asterisk *. Our proof is simple and elementary by use of the mean value theorem.

In the final section, we give a list of our future research topics, some of which are under way. Especially a generalization to discontinuous mappings should be interesting.