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NOTE ON SYMMETRIC HILBERT SERIES

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Introduction

Let $A = \bigoplus_{n\geq 0} A_n$ be a d-dimensional Noetherian graded ring over an Artinian local ring A_0. In this paper, we study a relationship between a symmetric Hilbert series $P(A, t)$ and its Hilbert coefficients $e_i(A)$. A symmetric Hilbert series is closely related to a Gorenstein property of A. Stanley[9] proves that if A is a domain, then a symmetricity of Hilbert series is equivalent to a Gorenstein property. Also, due to Ooishi[7], if A is a graded ring associate to a maximal primary ideal of some Cohen-Macaulay local ring, then a similar statement holds for A. Moreover, Hyry and Järvilehto[5] gave a characterization of a Gorenstein property of such A in terms of Hilbert coefficients under certain assumptions.

We characterize a symmetric Hilbert series in terms of Hilbert coefficients in (2.1) and determine Hilbert coefficients of Gorenstein graded rings. Applying our result for a graded ring associate to an ideal, we generalize Hyry-Järvilehto[5]'s result in (2.4). Also, we explain conditions of (2.1) by an example arising from well-known formula of Stirling numbers of the second kind. In the last section, we will see that combinatrial informations from Hilbert series are not enough to determine a ring structure of graded rings in general.

1. Preliminary

Let $A = \bigoplus_{n\geq 0} A_n$ be a d-dimensional Noetherian graded ring over an Artinian local ring A_0. For a finitely generated graded A-module M, we denote by $h_M(n) = \ell_{A_0}(M_n)$ $(n \in \mathbb{Z})$ and by $P(M, t) = \sum_{n\in\mathbb{Z}} h_M(n)t^n \in \mathbb{Z}[[t]][t^{-1}]$. h_M (resp. $P(M, t)$) is called the Hilbert function (resp. the Hilbert series) of M. Through out this paper, we always assume that M has the Hilbert series of the form $P(M, t) = \frac{Q_M(t)}{(1-t)^s}$, where $s = \dim(M)$. In this section, we recall some basic properties of Hilbert functions and Hilbert series.

Definition 1.1. For all $i \in \mathbb{Z}$, we define $e_i(M) = \frac{1}{i!}Q_M^{(i)}(1)$ and call it the i-th Hilbert coefficient of M. A Hilbert polynomial $p_M(T)$ of M is defined

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by $p_M(T) = \sum_{i=0}^{s-1}(-1)^{s-1-i}e_{s-1-i}(M)\binom{T+i}{i}$, where $\binom{T+i}{i} = \frac{1}{i!}\prod_{j=1}^{i}(T+j)$. It is well-known that $h_M(n) = p_M(n)$ for all $n >> 0$.

We put $P(M, t^{-1}) = (-1)^s \frac{t^s Q_M(t^{-1})}{(1-t)^s}$. Then we have the following important property, which gives interaction between rational generating functions and graded rings.

Lemma 1.2. *(Serre[8]) The following equalities hold.*

(1) $\sum_{n\in\mathbb{Z}} \{h_M(n) - p_M(n)\} t^n = \sum_{n\in\mathbb{Z}} \left\{\sum_{i=0}^{s}(-1)^i \ell\left(\mathrm{H}^i_{A_+}(M)_n\right)\right\} t^n$,

(2) $P(M, t^{-1}) = \sum_{n\in\mathbb{Z}} \left\{\sum_{i=0}^{s}(-1)^i \ell\left(\mathrm{H}^i_{A_+}(M)_{-n}\right)\right\} t^n$.

(See (2.1) of [7] and (4.4.5) of [2].) □

We denote by $h^*_M(n) = \sum_{i=0}^{s}(-1)^{s-i}\ell(\mathrm{H}^i_{A_+}(M)_{-n})$ for $n \in \mathbb{Z}$ and by

$$p^*_M(T) = \sum_{i=0}^{s-1} e_{s-1-i}(M)\binom{T-1}{i}.$$

Since $\binom{-n+i}{i} = (-1)^i\binom{n-1}{i}$ for all $n > 0$, it follows $p^*_M(T) = (-1)^{s-1}p_M(-T)$. Note that $h^*_M(T) = h_{K_M}(T)$ and $p^*_M(T) = p_{K_M}(T)$, if M is Cohen-Macaulay.

Corollary 1.3. *Put* $\tilde{a}(M) = \max\{n \mid \sum_{i=0}^{s}(-1)^i\ell(\mathrm{H}^i_{A_+}(M)_n) \neq 0\}$ *and* $b(M) = \min\{n \mid M_n \neq 0\}$. *Then we have the following.*

(1) $h_M(n) = p_M(n)$ for all $n > \tilde{a}(M)$.

(2) $h^*_M(n) = p^*_M(n)$ for all $n > -b(M)$.

$\tilde{a}(M)$ plays important role in the theory of Hilbert series as above. Moreover, we also have $\tilde{a}(M) = \deg(Q_M(t)) + s$ (cf. (4.4.1) of [2]).

Definition 1.4. Let $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ be a function on $\mathbb{Z}$. We define functions Δf and ∇f on $\mathbb{Z}$ by $\Delta f(n) = f(n+1) - f(n)$ and $\nabla f(n) = f(n) - f(n-1)$ for $n \in \mathbb{Z}$. It follows that $\Delta^i f(n) = \sum_{j=0}^{i}(-1)^j\binom{i}{j}f(n+i-j)$ and $\nabla^i f(n) = \sum_{j=0}^{i}(-1)^j\binom{i}{j}f(n-j)$ for all i and all $n \in \mathbb{Z}$. This conclude that $\Delta^i f(n) = \nabla^i f(n+i)$ for all i and all $n \in \mathbb{Z}$.

For $n \in \mathbb{Z}$, if $x \in A_1$ is a nonzero divisor of M, then

(*) $$\Delta h_M(n) = h_{M/xM}(n+1) \text{ and } \nabla h_M(n) = h_{M/xM}(n).$$

Taking long exact sequence of local cohomologies with respect to a short exact sequence of the multiplication of x, we have $h^*_M(n+1) = h^*_M(n) + h^*_{M/xM}(n)$ and, thus,

(**) $$\Delta h^*_M(n) = h^*_{M/xM}(n) \text{ and } \nabla h^*_M(n) = h^*_{M/xM}(n-1).$$

Using the identity $\binom{T+i}{i} = \binom{T+i-1}{i} + \binom{T+i-1}{i-1}$, it is easy to compute

$$\nabla^i p_M(T) = \sum_{j=0}^{s-1-i} (-1)^{s-1-i} e_{s-1-i-j}(M) \binom{T+j}{j}$$

and $\Delta^i p^*_M(T) = \sum_{j=0}^{s-1-i} e_{s-1-i-j} \binom{T-1}{j}$. Hence we have

(***) $$\begin{aligned} e_{s-1-i}(M) &= (-1)^{s-1-i} \nabla^i p_M(-1) = \Delta^i p^*_M(1) \\ &= (-1)^{s-1-i} \Delta^i p_M(-i-1) = \nabla^i p^*_M(i+1) \text{ (by (1.4))} \end{aligned}$$

for $0 \le i \le s-1$, since $\binom{T+i}{i}$ have roots $-1, \cdots, -i$.

Example 1.5. For an integer m, a graded A-module $M(m)$ is defined by $M(m)_n = M_{n+m}$ for all $n \in \mathbb{Z}$. Immediately, we have $p_{M(m)}(T) = p_M(T+m)$ and its Hilbert coefficients can be computed as

$$\begin{aligned} e_{s-1-i}(M(m)) &= (-1)^{s-1-i} \nabla^i p_M(m-1) \\ &= \begin{cases} \sum_{j=0}^{s-1-i} (-1)^{s-1-i-j} e_{s-1-i-j}(M) \binom{m-1+j}{j} & (m > 0) \\ \sum_{j=0}^{s-1-i} e_{s-1-i-j}(M) \binom{-m}{j} & (m \le 0) \end{cases} \end{aligned}$$

for $0 \le i \le s-1$. In particular, $e_{s-1-i}(M(-1)) = e_{s-1-i}(M) + e_{s-1-i-1}(M)$.

□

Example 1.6. Let R and S be graded rings such that $\dim(R) = r$ and $\dim(S) = s$. Suppose that $R_0 = S_0$ is a field. Hilbert coefficients of the Segre product $R\#S$ of R and S are described as follows. By (***) and Leibniz's rule of ∇^i,

$$\begin{aligned} e_{r+s-2-i}(R\#S) &= (-1)^{r+s-2-i} \nabla^i (p_R \cdot p_S)(-1) \\ &= \sum_{j=0}^{i} \binom{i}{j} (-1)^{r-1-i+j} \nabla^{i-j} p_R(-1-j)(-1)^{s-1-j} \nabla^j p_S(-1) \\ &= \sum_{j=0}^{i} \binom{i}{j} \Delta^{i-j} p^*_R(j+1) e_{s-1-j}(S) \end{aligned}$$

for $0 \le i \le r+s-2$. Since $\binom{j}{k} = 0$ for $k > j$,

$$\begin{aligned}\Delta^{i-j} p_R^*(j+1) = \sum_{k=0}^{r-1-i+j} e_{r-1-i+j-k}(R)\binom{j}{k} &= \sum_{k=0}^{j} e_{r-1-i+j-k}(R)\binom{j}{k} \\ &= \sum_{k=0}^{j} e_{r-1-i+k}(R)\binom{j}{k}.\end{aligned}$$

Hence we have

$$e_{r+s-2-i}(R\#S) = \sum_{j=0}^{i}\sum_{k=0}^{j}\binom{i}{j}\binom{j}{k} e_{r-1-(i-k)}(R)e_{s-1-j}(S)$$

for $0 \le i \le r+s-2$.

$\square$

In the following, we determine Hilbert coefficients by local cohomologies.

Proposition 1.7. *We put $a = \tilde{a}(M)$ and suppose that $b(M) \ge 0$. Then we have the following;*

$$e_i(M) = \begin{cases} \displaystyle\sum_{j=0}^{s-1-i} (-1)^{s-1-i-j}\binom{d-1-i}{j} h_M^*(j+1) & (0 \le i \le s-1) \\ \displaystyle\sum_{j=0}^{s+a-i} \binom{a-j}{i-s} h_M^*(j-a) & (0 \le a \textit{ and } s \le i \le s+a) \end{cases}$$

for $0 \le i \le s+a$. In particular, $e_s(M) = \sum_{j=0}^{a} h_M^*(j)$(cf. [1],[4]*).*

Proof. Let $A[X_1, \cdots, X_p]$ be a polynomial extension of A. We consider $A[X_1, \cdots, X_p]$ as a graded ring by $\deg(X_i) = 1$ for $1 \le i \le p$. If we put $M[X_1, \cdots, X_p] = M \otimes_A A[X_1, \cdots, X_p]$, then $b(M[X_1 \cdots, X_p]) = b(M)$ and $P(M[X_1, \cdots, X_p], t) = (1-t)^{-p} P(M, t)$. It follows that $Q_{M[X_1,\cdots,X_p]}(t) = Q_M(t)$ and $e_i(M[X_1 \cdots, X_p]) = e_i(M)$ for all $i \ge 0$.

First, we claim that

$$h^*_{M[X_1,\cdots,X_p]}(n) = \sum_{j=0}^{n+a-p} \binom{n+a-j-1}{p-1} h_M^*(j-a)$$

for all n. In fact, it is easy to see that $h^*_{M[X_1\cdots,X_p]}(p-a) = h^*_M(-a)$ and, by induction, we have, for all $n \geq p-a$,

$$
\begin{aligned}
&h^*_{M[X_1\cdots,X_p]}(n) = h^*_{M[X_1\cdots,X_p]}(n-1) + h^*_{M[X_1,\cdots,X_{p-1}]}(n-1)\\
&= \sum_{j=0}^{n-1+a-p} \binom{n+a-j-2}{p-1} h^*_M(j-a)\\
&\quad + \sum_{j=0}^{n-1+a-p+1} \binom{n+a-j-2}{p-2} h^*_M(j-a)\\
&= \sum_{j=0}^{n+a-p} \binom{n+a-j-1}{p-1} h^*_M(j-a).
\end{aligned}
$$

Since $b(M[X_1\cdots,X_p]) \geq 0$, $p^*_{M[X_1\cdots,X_p]}(n) = h^*_{M[X_1\cdots,X_p]}(n)$ for all $n > 0$. Hence, by (***), if $p = \max\{a+1, 0\}$, then

$$
\begin{aligned}
e_i(M) &= \Delta^{s+p-1-i} h^*_{M[X_1,\cdots,\cdots,X_p]}(1)\\
&= \begin{cases} \Delta^{s-1-i} h^*_M(1) & (0 \leq i \leq s-1)\\ h^*_{M[X_1,\cdots,X_{i+1-s}]}(1) & (0 \leq a \text{ and } s \leq i \leq s+a) \end{cases}\\
&= \begin{cases} \displaystyle\sum_{j=0}^{s-1-i} (-1)^{s-1-i-j} \binom{s-1-i}{j} h^*_M(j+1) & (0 \leq i \leq s-1)\\ \displaystyle\sum_{j=0}^{s+a-i} \binom{a-j}{i-s} h^*_M(j-a) & (0 \leq a \text{ and } s \leq i \leq s+a). \end{cases}
\end{aligned}
$$

□

2. A symmetric Hilbert series

In this section, we characterize a symmetric property of $P(A,t)$ in terms of Hilbert coefficients. We call that $P(A,t)$ is symmetric, if $Q_A(t)$ is symmetric (i.e. $Q_A(t) = (-1)^d t^{d+a} Q_A(t^{-1})$). It is easy to see that $P(A,t)$ is symmetric if and only if $P(A,t) = (-1)^d t^a P(A, t^{-1})$. Then we state our result as follows.

Theorem 2.1. *If we put $a = \tilde{a}(A)$, then the following conditions are equivalent.*

(1) *$P(A,t)$ is symmetric.*

(2) *For* $0 \le i \le d+a$,

$$e_i(A) = \begin{cases} \displaystyle\sum_{j=0}^{d-1-i} (-1)^{d-1-i-j} \binom{d-1-i}{j} \ell(A_{j+1+a}) & (0 \le i \le d-1) \\ \displaystyle\sum_{j=0}^{d+a-i} \binom{a-j}{i-d} \ell(A_j) & (0 \le a \text{ and } d \le i \le d+a). \end{cases}$$

(3) *For* $0 \le i \le d+a$,

$$e_i(A) = \sum_{j=0}^{i} (-1)^j e_j(A) \binom{d+a-j}{d+a-i}.$$

In particular, $\{e_i(A) \mid i \text{ is odd}\}$ *is determined by* $\{e_j(A) \mid j \text{ is even}\}$.

□

The following lemma is essential for the symmetricity of $P(A,t)$(cf. (4.24) of [9]).

Lemma 2.2. *We put* $a = \tilde{a}(A)$ *and assume that* $a < 0$. *Then the following conditions are equivalent.*

(1) $P(A,t)$ *is symmetric.*
(2) $p_A^*(T) = p_A(T+a)$.
(3) $p_A^*(i) = p_A(i+a)$ *for* $1 \le i \le d$.
(4) $\Delta^i p_A^*(1) = \Delta^i p_A(1+a)$ *for* $-a-1 \le i \le d-1$.
(5) $\nabla^i p_A^*(i+1) = \nabla^i p_A(i+1+a)$ *for* $-a-1 \le i \le d-1$.

Proof. A symmetricity of $P(A,t)$ is equivalent to $h_A^*(n) = h_A(n+a)$ for all $n \in \mathbb{Z}$ by (1.2), (2). This condition imples the condition (2), since $p_A(T)$ and $p_A^*(T)$ are polynomials. Moreover, since $a < 0$, we have that $h_A^*(n) = 0 = h_A(n+a)$ for all $n \le 0$. Also we have $h_A(n+a) = p_A(n+a)$ and $h_A^*(n) = p_A^*(n)$ for all $n > 0$ by (1.3). This shows (2) $\Longrightarrow$ (1). Both $p_A^*(X)$ and $p_A(X+a)$ have a degree $d-1$, implications (3) $\Longleftrightarrow$ (2) are trivial.

Let $a_0, \cdots, a_n$ be integers. Clearly, $a_0 = \cdots = a_n = 0$ if and only if $\sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} a_j = 0$ for $0 \le i \le n$, by induction. Thus, the condition (3) is equivalent to the condition that

$$\sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} p_A^*(j+1) = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} p_A(j+1+a)$$

for $0 \leq i \leq d-1$. The left hand side of this equation is coinsides with $\Delta^i p_A^*(1) = \nabla^i p_A^*(i+1) = e_{d-1-i}(A)$ by (***). Note that, by definition, $e_{d-1-i}(A) = 0$ and $p_A(i+1+a) = 0$ for $i < -a-1$. Namely, the above equation is automatically satisfied for $0 \leq i \leq -a-1$. Hence, we have that (3) $\Longleftrightarrow$ (4) and (3) $\Longleftrightarrow$ (5). The proof is completed. $\square$

Proof of (2.1). The proof of (2.1) is similar to the proof of (1.7). Let $s = \max\{a+1, 0\}$ and let $B = A[X_1, \cdots, X_s]$ be a polynomial ring over A. We regard B as a graded ring by $\deg(X_i) = 1$ for $1 \leq i \leq s$. Note that $Q_B(t) = Q_A(t)$ and $a(B) = a - s < 0$. Hence, by (2.2), $P(A,t)$ is symmetric if and only if $\Delta^{d+s-1-i} p_B^*(1) = \Delta^{d+s-1-i} p_B(1+a-s)$ for $0 \leq i \leq d+a$. (n.b. $-a+s-1 \leq d+s-1-i \leq d+s-1 \Longleftrightarrow 0 \leq i \leq d+a$.) By (***), $e_i(A) = e_i(B) = \Delta^{d+s-1-i} p_B^*(1)$. On the other hand, by $a-s<0$ and (1.3), $p_B(n+a-s) = h_B(n+a-s)$ for $n > 0$. This implies that $\Delta^{d+s-1-i} p_B(1+a-s) = \Delta^{d+s-1-i} h_B(1+a-s)$ for $0 \leq i \leq d+a$ and, by (*),

$$
\begin{aligned}
&\Delta^{d+s-1-i} h_B(1+a-s) \\
&= \begin{cases} \Delta^{d-1-i} h_A(1+a) & (0 \leq i \leq d-1) \\ h_{A[X_1, \cdots, X_{s-d-s+1+i}]}(d+a-i) & (0 \leq a \text{ and } d \leq i \leq d+a) \end{cases} \\
&= \begin{cases} \displaystyle\sum_{j=0}^{d-1-i} (-1)^{d-1-i-j} \binom{d-1-i}{j} \ell(A_{j+1+a}) & (0 \leq i \leq d-1) \\ \displaystyle\sum_{j=0}^{d+a-i} \binom{a-j}{i-d} \ell(A_j) & (0 \leq a \text{ and } d \leq i \leq d+a) \end{cases}
\end{aligned}
$$

Hence, we have (1) $\Longleftrightarrow$ (2).

Similarly, a symmetricity of $P(A,t)$ is equivalent to$\nabla^{d+s-1-i} p_B^*(d+s-i) = \nabla^{d+s-1-i} p_B(d+a-i)$ for $0 \leq i \leq d+a$. The left hand side of this equation is $e_i(A) = \nabla^{d+s-1-i} p_B^*(d+s-i)$, by (***), and the right hand side is

$$
\begin{aligned}
\nabla^{d+s-1-i} p_B(d+a-i) &= \sum_{j=0}^{i} (-1)^{i-j} e_{i-j}(A) \binom{d+a-i+j}{j} \\
&= \sum_{j=0}^{i} (-1)^j e_j \binom{d+a-j}{d+a-i}.
\end{aligned}
$$

Hence, we have (1) $\Longleftrightarrow$ (3). $\square$

Corollary 2.3. *If A is a Gorenstein graded ring of $a = a(A)$, then*

$$e_i(A) = \begin{cases} \displaystyle\sum_{j=0}^{d-1-i} (-1)^{d-1-i-j} \binom{d-1-i}{j} \ell(A_{j+1+a}) & (0 \le i \le d-1) \\ \displaystyle\sum_{j=0}^{d+a-i} \binom{a-j}{i-d} \ell(A_j) & (0 \le a \text{ and } d \le i \le d+a) \end{cases}$$

for $0 \le i \le d+a$.

□

A symmetricity of $P(A,t)$ is closely related to the Gorenstein property. In fact, by results of Stanley[9], if A is a Cohen-Macaulay domain, then conditions (2) and (3) of (2.1) are equivalent that A is Gorenstein, respectively. Also, by results of Ooishi[7], if A is a graded ring associate to some ideal, then we have similar statements. If we apply (2.1) to $A[X]$, then (2.1), (1) is also equivalent to

$$e_i(A) = \begin{cases} \sum_{j=0}^{d-i} (-1)^{d-i-j} \binom{d-i}{j} \ell(A_{\le j+a}) & (0 \le i \le d) \\ \sum_{j=0}^{d+a-i} \binom{a-1-j}{i-d-1} \ell(A_{\le j}) & (0 < a \text{ and } d+1 \le i \le d+a) \end{cases}$$

for $0 \le i \le d+a$, since $\ell(A[X]_n) = \ell(A_{\le n}) = \sum_{i=0}^{n} \ell(A_i)$. Using this condition, we generalize the result of Hyry-Järvilehto[5] as follows.

Corollary 2.4. *Let $(R, \mathfrak{n})$ be a Gorenstein local ring of dimension d and $\mathfrak{q} \subset R$ be a $\mathfrak{n}$-primary ideal. Put $a = r(\mathfrak{q}) - d$. If $G_R(\mathfrak{q})$ is Cohen-Macaulay, then the following conditions are equivalent.*

(1) *$G_R(\mathfrak{q})$ is Gorenstein.*

(2) *For $0 \le i \le r(\mathfrak{q})$,*

$$e_i(\mathfrak{q}) = \begin{cases} \displaystyle\sum_{j=0}^{d-i} (-1)^{d-i-j} \binom{d-i}{j} \ell(R/\mathfrak{q}^{j+a+1}) & (0 \le i \le d) \\ \displaystyle\sum_{j=0}^{r(\mathfrak{q})-i} \binom{r(\mathfrak{q})-d-1-j}{i-d-1} \ell(R/\mathfrak{q}^{j+1}) & (0 < a \text{ and } d < i). \end{cases}$$

(3) *For $0 \le i \le r(\mathfrak{q})$,*

$$e_i(\mathfrak{q}) = \sum_{j=0}^{i} (-1)^j e_j(\mathfrak{q}) \binom{r(\mathfrak{q})-j}{r(\mathfrak{q})-i}$$

□

Remark 2.5. Suppose that A is Cohen-Macaulay. The right hand side of (2.1), (2) is interpreted as Hilbert coefficients of K_A, in general. In fact $p_{K_A}(T) = p^*_A(T)$ and, by (***), we have

$$\begin{aligned} e_i(K_A) &= (-1)^i \nabla^{d-1-i} p^*_A(-1) \\ &= \Delta^{d-1-i} p_A(1) = \sum_{j=0}^{d-1-i} (-1)^{d-1-i-j} \binom{d-1-i}{j} p_A(j+1) \end{aligned}$$

for $0 \le i \le d-1$.

3. Stirling numbers

In this section, we give a typical example satisfying conditions of (2.1) such that it shows well-known formula of Stirling numbers of the second kind.

Let $\mathcal{F}_A = \{A_{\ge n}\}_{n\in\mathbb{N}}$ be a filtration of homogeneous ideals and $a = \tilde{a}(A)$. We consider a Rees ring $R = R(\mathcal{F}_A) = \bigoplus_{n\in\mathbb{N}} A_{\ge n} x^n \subset A[x]$. Suppose that $\deg(x) = 0$. Then $R_n = A_n^{\oplus n+1}$ for all $n \in \mathbb{Z}$. Thus

$$\begin{aligned} P(R,t) &= \sum_{n\ge 0} (n+1)\ell(A_n)t^n = (tP(A,t))' = \left(\frac{tQ_A(t)}{(1-t)^d} \right)' \\ &= \frac{(1+(d-1)t)Q_A(t) + t(1-t)Q'_A(t)}{(1-t)^{d+1}}. \end{aligned}$$

Here we denote the formal derivation by $(\ \ )'$. By Leibniz's rule, we have

$$Q_R^{(i)}(t) = (t-t^2)Q_A^{(i+1)}(t) + (i+1+(d-1-2i)t)Q_A^{(i)}(t) + i(d-i)Q_A^{(i-1)}(t).$$

Hence we compute Hilbert coefficients of R as

$$e_i(R) = \frac{d-i}{i!} \left\{ Q_A^{(i)}(1) + iQ_A^{(i-1)}(1) \right\} = (d-i)\left\{ e_i(A) + e_{i-1}(A) \right\}$$

for $0 \le i \le d+a+1$.

Now, we define $A^{(0)} = A$ and $A^{(k)} = R(\mathcal{F}_{A^{(k-1)}})$ for $k > 0$, inductively. It follows that $\dim(A^{(k)}) = d+k$, $a(A^{(k)}) = a(A)$, and $\ell(A_n^{(k)}) = \ell(A_n)(n+1)^k$ for all n. We put $e(k,i) = e_i(A^{(k)})$ for $0 \le k$ and $0 \le i$. As above computations, this sequence $\{e(k,i) \mid k,i \in \mathbb{Z},\ k \ge 0\}$ is determined by the following recurrence formula;

$$e(0,i) = \begin{cases} e_i(A) & (0 \le i \le d+a) \\ 0 & (\text{otherwise}) \end{cases}$$

$$e(k+1,i) = (d+k-i)\left\{ e(k,i) + e(k,i-1) \right\}.$$

In particular, $e(k,i) = 0$ for $d+k+a < i$ or $i < 0$.

(2.1) allows that this sequence has a solution, if $A^{(k)}$ is Gorenstein. By results of Goto-Nishida[3], the Gorenstein property of $A^{(k)}$ is determined by the property of A. Since $G(\mathcal{F}_{A^{(k-1)}}) = \bigoplus_{n\geq 0} A^{(k-1)}_{\geq n}/A^{(k-1)}_{\geq n+1} \cong A^{(k-1)}$, $A^{(k)}$ is Cohen-Macaulay if and only if A is Cohen-Macaulay and $a < 0$. Furthermore, $A^{(k)}$ is Gorenstein if and only if A is Gornstein and $a = -2$.

Henceforth, we suppose that A is Gorenstein and $a = -2$. Then $A^{(k)}$ is a $(d+k)$-dimensional Gorenstein ring of $a(A^{(k)}) = -2$. By (2.2), we have $p^*_{A^{(k)}}(T) = p_{A^{(k)}}(T-2)$ and

$$(\clubsuit) \qquad \ell(A_{n-1})n^k = p^*_{A^{(k)}}(n+1) = \sum_{i=1}^{d+k-1} e(k, d+k-1-i)\binom{n}{i}$$

for all $n > 0$. Similarly, since $A[x]$ is Gorenstein and $a(A^{(k)}[x]) = -3$, we also have $p^*_{A^{(k)}[x]}(T) = p_{A^{(k)}[x]}(T-3)$ and

$$(\diamondsuit) \qquad \sum_{i=1}^{n} \ell(A_{i-1})i^k = p^*_{A^{(k)}[x]}(n+2) = \sum_{i=2}^{d+k} e(k, d+k-i)\binom{n+1}{i}$$

for all $n > 0$. Finally, by (2.1), we have solutions

$$e(k, d+k-i) = \sum_{j=1}^{i-1}(-1)^{i-1-j}\binom{i-1}{j}\ell(A_{j-1})j^k$$

for $2 \leq i \leq d+k$.

Remark 3.1. Similarly, we are able to compute h-vecters. We denote the h-vecter of $A^{(k)}$ by $h(k,i) = \frac{1}{i!}Q_{A^{(k)}}(0)$ for $k \geq 0$ and $i \geq 0$. Then we have $h(k+1,i) = \frac{1}{i!}\left\{(i+1)Q^{(i)}_{A^{(k)}}(0) + i(d-i)Q^{(i-1)}_{A^{(k)}}(0)\right\} = (i+1)h(k,i) + (d+k-i)h(k,i-1)$. Namely, $\{h(k,i) \mid k,i \in \mathbb{Z},\ k \geq 0\}$ can be determined by the recurrence

$$h(0,i) = \begin{cases} h_i & (0 \leq i \leq d+a) \\ 0 & (\text{otherwise}) \end{cases}$$
$$h(k+1,i) = (i+1)h(k,i) + (d+k-i)h(k,i-1),$$

where h_i is a h-vector of A. If A is Cohen-Macaulay, then this sequence has a solution as follows. Let $\underline{x} \subset A^{(k)}_1$ be a liner sop. Then $h(k,i) = \ell\left([A^{(k)}/(\underline{x})]_i\right) = \nabla^{d+k}h_{A^{(k)}}(i) = \sum_{j=1}^{i+1}(-1)^{i-j+1}\binom{d+k}{i-j+1}\ell(A_{j-1})j^k$ for $0 \leq i \leq d+k+a$.

Example 3.2. (Stirling numbers arising from power sum formula)

Let us recall that well-known formula of power sums. For all natural number n, $\sum_{i=1}^{n} i^k$ can be written as a polynomial of n in degree $k+1$. Bernoulli-Seki formula is stated as

$$\sum_{i=1}^{n} i^k = \frac{1}{k+1} \sum_{i=0}^{k} \binom{k+1}{i} B_i (n+1)^{k+1-i}$$

where B_n is the number satisfying conditions $B_0 = 1$ and $\sum_{i=0}^{n} \binom{n+1}{i} B_i = 0$ and call it the n-th Bernoulli number. It seems that a Bernoulli number is not able to describe as a "simple" linear combination of binomial coefficients and it is described as a sum of Stirling numbers of the second kind. Stirling numbers of the second kind are defined by recurrence as $S(0,0) = 1$, $S(0,i) = 0 = S(k,0)$ for $k, i \neq 0$, and $S(k,i) = S(k-1,i-1) + iS(k-1,i)$. Then it is known that

$$B_k = \sum_{i=0}^{k} (-1)^{k-i} \frac{i!S(k,i)}{i+1} = \sum_{i=0}^{k} (-1)^{i} \frac{i!S(k+1,i+1)}{i+1}.$$

If we put $A = k[x_0, x_1]$, then $A^{(k-1)} \cong A^{\# k}$ and it is a Gorenstein ring of $\dim(A^{(k-1)}) = k+1$ and $a(A^{(k-1)}) = -2$. Since $\ell([A^{(k-1)}]_n) = (n+1)^k$, the Hilbert function is given by $n^k = p_{A^{(k-1)}}(n-1) = \sum_{i=0}^{k} (-1)^{k-i} e(k-1, k-i)\binom{n+i-1}{i}$. Then, by $\clubsuit$, we have

$$n^k = \sum_{i=1}^{k} e(k-1, k-i) \binom{n}{i}.$$

This shows that our Hilbert coefficient is essentially same as the Stirling number, namely $e(k-1, k-i) = i!S(k,i)$. Also, by $\diamondsuit$, we describe power sums as

$$\sum_{i=1}^{n} i^k = \sum_{i=2}^{k+1} e(k-1, k+1-i) \binom{n+1}{i} = \sum_{i=1}^{k} e(k-1, k-i) \binom{n+1}{i+1}.$$

Finally, we obtain Hilbert coefficients $e(k,i)$ by $e(0,0) = 1$, $e(0,i) = 0$ $(0 < i)$ and

$$e(k-1, k-i) = \sum_{j=1}^{i} (-1)^{i-j} \binom{i}{j} j^k$$

for $0 < k$ and $1 \leq i \leq k$.

$\square$

Remark 3.3. As above example, Stirling numbers give a base change of $\mathbb{Q}[T]$ between $\{T^n\}_{n\geq 0}$ and $\{\binom{T+n}{n}\}_{n\geq 0}$. Namely, for $f(T)=\sum_{k=0}^m a_k T^k = \sum_{k=0}^m b_k \binom{T+k}{k} \in \mathbb{Q}[T]$, we can describe b_k in terms of $\{a_k\}$ and $\{e(k,i)\}$.

Applying (1.5) for $A^{(k-1)}$ and $m=-1$, $e_{k-i}(A^{(k-1)}(-1)) = e(k-1,k-i)+e(k-1,k-i-1) = \frac{e(k,k-i)}{i+1}$ and

$$n^k = p_{A^{(k-1)}(-1)}(n) = \sum_{i=0}^{k} (-1)^{k-i} \frac{e(k,k-i)}{i+1}\binom{n+i}{i}.$$

Replacing T^k by this equation, we have

$$\begin{aligned} f(T) &= \sum_{k=0}^m a_k \sum_{i=0}^k (-1)^{k-i} \frac{e(k,k-i)}{i+1}\binom{T+i}{i} \\ &= \sum_{i=0}^m \frac{1}{i+1} \sum_{k=0}^{m-i} (-1)^k a_k e(k+i,k) \binom{T+i}{i}. \end{aligned}$$

Hence we have

$$b_k = \frac{1}{k+1}\sum_{i=0}^{m-k} (-1)^i a_i e(k+i,i).$$

Using the same shifting trick for $A^{(k-1)}(-1)[X]$, the k-th power sum of 1 to n can be described as a combination of $\binom{n}{1},\cdots,\binom{n}{k+1}$.

Since $p_{A^{(k-1)}(-1)[X]}(n) = \sum_{i=1}^{k+1} (-1)^{k+1-i}\ \frac{e(k,k+1-i)}{i}\binom{n+i}{i}$, we have

$$\begin{aligned} \sum_{i=1}^n i^k &= p_{A^{k-1}[X]}(n-1) = p^*_{A^{k-1}[X]}(n+2) = p^*_{A^{(k-1)}(-1)[X]}(n+1) \\ &= \sum_{i=1}^{k+1} \frac{e(k,k+1-i)}{i}\binom{n}{i} \end{aligned}$$

Remark 3.4. As in (3.3), it is able to express a_k in terms of b_k using Stirling numbers $s(k,i)$ of the first kind. In fact, if we put $d(k,i)=\frac{s(k,i)}{k!}$, then

$$a_k = \sum_{i=k}^m \left\{ \sum_{j=i}^m (-1)^{m-j} b_j d(j,i) \right\} \binom{i}{k}.$$

However, we don't know that it is necessary to take double summations as above.

4. Hilbert series of the polynomial type

In this section, we consider that graded rings possess Hilbert serires of the form $\frac{e_0(A)}{(1-t)^d}$. We call that A has a Hilbert series of the polynomial type, if $P(A,t) = \frac{e_0(A)}{(1-t)^d}$. Clearly, a polynomial ring has a Hilbert series of the polynomial type . However, we will see that a Hilbert series of the polynomial type does not imply a polynomial ring. Namely, it is not enough to determine an algebra structure on A, even if A has such a typical Hilbert series. This is the purpose of this section.

First, we give an easy example. Let $A = k[x_1, \cdots, x_n]/(x_1, \cdots, x_{n-1})^2$. We regard A as a graded ring by $\deg(x_i) = i$ $(i = 1, \cdots, n)$. Then $P(A,t) = \frac{1}{1-t}$ and A does not have a sop in A_1 (or $\sqrt{A_1} \neq \sqrt{A_+}$). This example shows that our problem make sense only on standard graded rings. Henceforth, we assume that $A = A_0[A_1]$. In the following, we characterize polynomial rings by Hilbert series of the polynomial type with extra assumptions.

Theorem 4.1. *Let A be a standard graded ring of* $\dim(A) = d$. *We denote* $P_A(T) = \sum_{i=0}^{d}(-1)^{d-i}e_{d-i}(A)\binom{T+i}{i}$ *and* $a(A) = max\{n \in \mathbb{Z} \mid \mathrm{H}^d_{A_+}(A)_n \neq 0\}$. *Then the following conditions are equivalent.*

(1) $A \cong A_0[X_1, \cdots, X_d]$.

(2) *A is Cohen-Macaulay and* $a(A) = -d$.

(3) $\mathrm{P}(A,t) = \dfrac{e_0(A)}{(1-t)^d}$ *and* $a(A) = -d$.

(4) $P_A(T) = e_0(A)\binom{T+d}{d}$ *and* $a(A) = -d$.

(5) $a(A) = -d$ *and, if* $d > 0$, *then* $\sum_{i=1}^{d-1}(-1)^i\ell(\mathrm{H}^i_{A_+}(A)_{-n}) = 0$ *for* $1 \leq n \leq d-1$ *and* $e_d(A) = 0$.

(6) $a(A) = -d$ *and, if* $d > 0$, *then* $\sum_{i=1}^{d-1}(-1)^i\ell(\mathrm{H}^i_{A_+}(A)_{-n}) = 0$ *for* $1 \leq n \leq d-1$ *and depth*$(A) > 0$.

Proof . To prove our result, we may assume that A_0 has an infinite residue field, without loss of generality. Implications (1) $\Leftrightarrow$ (2), (3) $\Rightarrow$ (4), (2) $\Rightarrow$ (5), (6) are trivial. If A is Cohen-Macaulay, then we have $a(A) = \tilde{a}(A)$ and $\deg Q_A(t) = a(A) + d$. This shows the implication (2) $\Rightarrow$ (3).

(4) $\Longrightarrow$ (5) Since $p_A(T) = \nabla P_A(T) = e_0(A)\binom{T+d-1}{d-1}$, $p^*_A(T) = e_0(A)\binom{T-1}{d-1}$ and it vanishes at $T = 1, \cdots, d-1$. Hence, by (1.3), $\sum_{i=1}^{d-1}(-1)^i\ell(\mathrm{H}^i_{A_+}(A)_{-n})$ $= (-1)^d h^*_A(n) = 0$ for $1 \leq n \leq d-1$.

We prove (5) $\Longrightarrow$ (2) by induction on d. If $d = 0$, then $A = A_0$, since $a(A) = 0$. Suppose that $d > 0$ and the statement is true for $d-1$. Let

$A' = A/\mathrm{H}^0_{A_+}(A)$ and $B = A'/xA'$, where $x \in A_1$ is a nonzero divisor of A'. Note that $\mathrm{H}^i_{A_+}(A') \cong \mathrm{H}^i_{A_+}(A)$ for $1 \leq i \leq d$ and $a(B) = a(A') + 1 = a(A) + 1 = -d + 1$. Then we have

$$\sum_{i=1}^{d-2}(-1)^i \ell(\mathrm{H}^i_{B_+}(B)_{-n}) = (-1)^{d-1} h^*_B(n) = (-1)^{d-1}\Delta h^*_A(n) = 0$$

for $1 \leq n \leq d-2$. On the other hand, $\ell(A'_{\leq n}) = \ell(A_{\leq n}) - \ell(\mathrm{H}^0_{A_+}(A)) = P_A(n) - \ell(\mathrm{H}^0_{A_+}(A))$ for all sufficiently large n and, by the uniqueness of Hilbert polynomial, $P_{A'}(T) = P_A(T) - \ell(\mathrm{H}^0_{A_+}(A))$. Since x is a nonzero divisor of A', we have $P_B(T) = \Delta P_{A'}(T) = \Delta P_A(T)$ and

$$e_{d-1}(B) = e_{d-1}(A) = \sum_{i=1}^{d}(-1)^i \ell(\mathrm{H}^i_{A_+}(A)_{-1}) = 0.$$

Hence, by induction hypothesis, B is a Cohen-Macaulay ring of dimension $d-1$. Since A' is Cohen-Macaulay, $\mathrm{H}^i_{A_+}(A) = 0$ for $1 \leq i \leq d-1$ and $\tilde{a}(A) = a(A) = -d$. This implies that

$$0 = e_d(A) = \sum_{i=0}^{d}(-1)^{d-i}\ell_{A_0}(\mathrm{H}^i_{A_+}(A)_{\geq 0}) = (-1)^d \ell_{A_0}(\mathrm{H}^0_{A_+}(A))$$

and $\mathrm{H}^0_{A_+}(A) = 0$, by (1.7). This conclude that $A = A'$ is Cohen-Macaulay.

(6) $\Longrightarrow$ (2) The assertion is clear for $d \leq 1$. We may assume that $d > 1$. Let $x \in A_1$ be a non zero divisor of A and $B = A/xA$. Similar to the proof of (5) $\Rightarrow$ (2), we have $a(B) = -d+1$, $\sum_{i=1}^{d-2}(-1)^i \ell_{B_0}(\mathrm{H}^i_{B_+}(B)_{-n}) = 0$ $(1 \leq n \leq d-2)$ and $e_{d-1}(B) = e_{d-1}(A) = \sum_{i=1}^{d}(-1)^i \ell_{A_0}(\mathrm{H}^i_{A_+}(A)_{-1}) = 0$. Hence, by (5) $\Longrightarrow$ (2), B is Cohen-Macaulay and so is A. $\square$

Remark 4.2. *Assume that A satisfy one of the following conditions;*

- *A is flat over A_0*
- *A is Cohen-Macaulay*
- *$A \cong G(\mathfrak{p})$ for some parameter ideal $\mathfrak{p}$ of a Noetherian local ring $(R, \mathfrak{n})$*
- *$A \cong G(\mathfrak{q})$ for some $\mathfrak{n}$-primary ideal $\mathfrak{q}$ of a Cohen-Macaulay local ring $(R, \mathfrak{n})$ (cf. [6])*

Then $\frac{e_0(A)}{(1-t)^d}$ implies that A is a polynomial ring without any other condition.

As above Remark, Hilbert series of the polynomial type allow polynomial rings in usual situations. However, in general, the condition on $a(A)$ is necessary. In the following, we give non polynomial rings having Hilbert series of the polynomial type.

Let $B = B_0 \oplus B_1 \oplus \cdots \oplus B_a$ be an Artinian local graded ring such that $B = B_0[B_1]$. We construct a graded ring A such that $A_n = B$ for all $n \geq 0$ and $a(A) = a - 1$. Put $A_0 = B$ and $\mathfrak{a} = B_+ \subset A_0$. Here we regard A_0 as an Artinian local ring without gradings. Then we define a graded ring A' by $A' = R_{A_0}(\mathfrak{a})[Y] = A_0[\mathfrak{a}X, Y] \subset A_0[X, Y]$ and put $A = A'/\mathfrak{a}YA'$. The grading on A' (and A) is given by $\deg(X) = \deg(Y) = 1$. For all $n \geq 0$, we have $A'_n = A_0[\mathfrak{a}X, Y]_n = \sum_{i=0}^{n} B_{\geq i} X^i Y^{n-i}$ and $(\mathfrak{a}YA')_n = B_1 Y \cdot A_0[\mathfrak{a}X, Y]_{n-1} = \sum_{i=0}^{n-1} B_{\geq i+1} X^i Y^{n-i}$. Hence that $A_n = B_0 Y^n \oplus B_1 X Y^{n-1} \oplus B_2 X^2 Y^{n-2} \oplus \cdots \oplus B_{n-1} X^{n-1} Y \oplus B_{\geq n} X^n$. It is clear that $[0 :_A Y]_n = B_{\geq n+1} X^n$ and there is a short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [0 :_A Y] & \longrightarrow & A & \longrightarrow & B[y] & \longrightarrow & 0 \\ & & & & Y & \longmapsto & y & & \end{array}$$

via an isomorphism $B \cong B_0 \oplus B_1 X \oplus \cdots \oplus B_a X^a$. Here $B[y]$ is a graded polynomial ring with $\deg(y) = 1$. This conclude that A is a 1-dimensional graded ring such that $depth(A) = 0$, $e_0(A) = \ell(B)$, $P_A(T) = e_0(A)(T + 1)$, $\mathrm{P}(A, t) = \dfrac{e_0(A)}{1 - t}$, and $a(A) = a - 1$. Through the polynomial extension of A, we have the following.

Proposition 4.3. *Let e, d be positive integers and let a be an integer such that $e - d \geq a > -d$.*

(1) *A non Cohen-Macaulay graded ring A such that $\mathrm{P}(A, t) = \dfrac{e}{(1 - t)^d}$ and $a(A) = a$ exists.*
(2) *A non Cohen-Macaulay graded ring A such that $P_A(T) = e\binom{T+d}{d}$ and $a(A) = a$ exists.*
(3) *If $d \geq 2$, then there exist a non Cohen-Macaulay graded ring A such that $depth(A) > 0$, $a(A) = a$ and $\sum_{i=1}^{d} (-1)^i \ell(\mathrm{H}^i_{A_+}(A)_{-n}) = 0$ for $1 \leq n \leq d - 1$.*

□

Example 4.4. Let k be a field and $k[a, x, y]/((a, x)^2, ay)$ with $\deg a = 0$ and $\deg x = \deg y = 1$. Then $P(A, t) = \frac{2}{1-t}$ and $a(A) = 0$.

□

Example 4.5. Let k be a field and let $B = k[x, y, z, w]/(x, y)(x, y^2, z, w) + (z^3)$, with $\deg x = \deg y = \deg z = \deg w = 1$. It is easy to see that w is a parameter of B and $B/[0:_A w] \cong k[z,w]/(z^3)$. Hence we have

$$\begin{aligned} P(B,t) &= tP(B/[0:w],t) + P(B/(w),t) \\ &= \tfrac{t+t^2+t^3}{1-t} + 1 + 3t + 2t^2 = \tfrac{1+3t-t^3}{1-t} \end{aligned}$$

and $P_B(T) = 3\binom{T+1}{1}$. This shows that (4.1), (4) does not imply (4.1), (3) without $a(A) = -d$.

Now, we define $\mathfrak{a} = (a(a^2,x,y,z,w,v^2,av),(x,y)(x,y^2,z,w),z^3)$ and $A = k[a,x,y,z,w,v]/\mathfrak{a}$, with $\deg a = 0$ and $\deg x = \deg y = \deg z = \deg w = \deg v = 1$. Then $\dim A = 2$, $depth A = 0$ and $\mathrm{H}^0_{A_+}(A) = aA = ka + ka^2 + kv$. Also, we have $A/\mathrm{H}^0_{A_+}(A) \cong B[v]$ and, thus, $\mathrm{P}(A,t) = 2+t+\frac{1+3t-t^3}{(1-t)^2} = \frac{3}{(1-t)^2}$.

$\square$

Remark 4.6. Our example shows that a class of graded rings having Hilbert series of the polynomial type is a relatively large as same as a class of Artinian graded rings. Also, we can find such a graded ring from any 1-dimensional Cohen-Macaulay graded rings. Maybe it frequently occurs in the following sense.

We call that a graded ring A is obtained from a graded ring B by the standard procedure, if there are graded rings $A(0), \cdots, A(n)$ such that $A = A(n)$, $B = A(0)$ and $A(i+1) = A(i)/\mathrm{H}^0_{A(i)_+}(A(i)) + (x_i)$ where $x_i \in A(i)_1$ and x_i is a nonzero divisor of $A(i)/\mathrm{H}^0_{A(i)_+}(A(i))$.

(**Question**) Let A be a graded ring such that $p_A(T) = e\binom{T+d-1}{d-1}$. Does there exist a graded ring B having a Hilbert series of the polynomial type such that A is obtained from B by the standard procedure?

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