Math. J. Okayama Univ. **53** (2011), 1–37

ASYMPTOTIC ANALYSIS FOR GREEN FUNCTIONS OF AHARONOV-BOHM HAMILTONIAN WITH APPLICATION TO RESONANCE WIDTHS IN MAGNETIC SCATTERING

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Abstract. The Aharonov–Bohm Hamiltonian is the energy operator which governs quantum particles moving in a solenoidal field in two dimensions. We analyze asymptotic properties of its Green function with spectral parameters in the unphysical sheet. As an application, we discuss the lower bound on resonance widths for scattering by two magnetic fields with compact supports at large separation. The bound is evaluated in terms of backward scattering amplitudes by a single magnetic field. A special emphasis is placed on analyzing how a trajectory oscillating between two magnetic fields gives rise to resonances near the real axis, as the distance between two supports goes to infinity. We also refer to the relation to the semiclassical resonance theory for scattering by two solenoidal fields.

1. Introduction

In the present paper we develop an asymptotic analysis for the Green functions of magnetic Schrödinger operators with spectral parameters in the unphysical sheet (the lower half plane of the complex plane $\boldsymbol{C}$). As an application, we study the scattering by two magnetic fields with compact supports at large separation in two dimensions, and we analyze how a trajectory oscillating between two magnetic fields gives rise to resonances near the real axis when the distance between two supports goes to infinity. We give a sharp lower bound on resonance widths in terms of backward scattering amplitudes by a single magnetic field. We restrict ourselves to the two dimensional case throughout the whole exposition. This is the most interesting case in magnetic scattering. It is due to the topological feature of dimension two that vector potentials associated with fields compactly supported do not necessarily have compact support but they generally fall off slowly at infinity.

We shall formulate the problem more precisely. We write

$$(1.1)\qquad H(A) = (-i\nabla - A)^2 = \sum_{j=1}^{2} (-i\partial_j - a_j)^2, \quad \partial_j = \partial/\partial x_j$$

Mathematics Subject Classification. Primary 35Q40; Secondary 35P25.

Key words and phrases. Aharonov-Bohm Hamiltonian, Green function, magnetic Schrödinger operator, scattering amplitude, resonance width.

for the magnetic Schrödinger operator with vector potential $A = (a_1, a_2) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$. The magnetic field $b : \boldsymbol{R}^2 \to \boldsymbol{R}$ associated with the potential A is defined by

$$b(x) = \nabla \times A(x) = \partial_1 a_2 - \partial_2 a_1$$

and the magnetic flux of b is defined by

$$\alpha = \frac{1}{2\pi} \int b(x)\, dx,$$

where the integration with no domain attached is taken over the whole space. We often use this abbreviation.

The Aharonov–Bohm Hamiltonian in the title is the energy operator which governs quantum particles moving in a solenoidal field (point–like field) in two dimensions. This model was employed by Aharonov and Bohm ([4]) in 1959 in order to convince us theoretically that a magnetic potential itself has a direct significance in quantum mechanics. This phenomenon unexplainable from a classical mechanical point of view is now called the Aharonov–Bohm effect, which is known as one of the most remarkable quantum phenomena. The first half of this work is devoted to studying asymptotic properties of the Green function with spectral parameters in the unphysical sheet for the Aharonov–Bohm Hamiltonian. We now consider the operator

$$K_{AB} = H(\alpha\Phi), \qquad \alpha \in \boldsymbol{R}, \tag{1.2}$$

where $\Phi : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is defined by

$$\Phi(x) = \left(-x_2/|x|^2, x_1/|x|^2\right) = \left(-\partial_2 \log |x|, \partial_1 \log |x|\right). \tag{1.3}$$

The potential above defines the point–like field

$$\nabla \times \Phi = \left(\partial_1\partial_1 \log |x| + \partial_2\partial_2 \log |x|\right) = \Delta \log |x| = 2\pi\delta(x)$$

with center at the origin, and the real parameter α denotes the magnetic flux of potential $\alpha\Phi$. The symmetric operator K_{AB} defined over $C_0^\infty\left(\boldsymbol{R}^2 \setminus \{0\}\right)$ is not necessarily essentially self–adjoint in the space $L^2 = L^2(\boldsymbol{R}^2)$ because of the strong singularity at the origin of Φ. We know ([1, 8]) that it is a symmetric operator with type $(2, 2)$ of deficiency indices. The self–adjoint extension is realized by imposing a boundary condition at the origin. Its Friedrichs extension has the domain

$$\mathcal{D} = \left\{ u \in L^2 : (-i\nabla - \alpha\Phi)^2\, u \in L^2,\ \lim_{|x|\to 0} |u(x)| < \infty \right\}, \tag{1.4}$$

where $(-i\nabla - \alpha\Phi)^2\, u$ is understood in $\mathcal{D}'\left(\boldsymbol{R}^2 \setminus \{0\}\right)$ (in the sense of distribution). We denote by the same notation K_{AB} this self–adjoint realization. The operator K_{AB} is often called the Aharonov–Bohm Hamiltonian in physics literatures, and the scattering by a solenoidal field is known as one of the exactly solvable quantum systems. We make a quick review on it in section 2. In particular, the scattering amplitude $f_{\mathrm{AB}}(\omega \to \theta; E)$ for the scattering from the initial direction $\omega \in S^1$ to the final one θ at energy $E > 0$ is explicitly calculated as

$$f_{\mathrm{AB}}(\omega \to \theta; E) = \left(\frac{2}{\pi}\right)^{1/2} e^{i\pi/4} E^{-1/4} \sin(\alpha\pi) e^{i[\alpha](\theta-\omega)} \frac{e^{i(\theta-\omega)}}{1 - e^{i(\theta-\omega)}}, \tag{1.5}$$

where the Gauss notation $[\alpha]$ denotes the greatest integer not exceeding α and the coordinates over the unit circle S^1 are identified with the azimuth angles from the positive x_1 axis. We denote by

$$G_{\mathrm{AB}}(\zeta) = (K_{\mathrm{AB}} - \zeta)^{-1} : L^2 = L^2(\boldsymbol{R}^2) \to L^2, \quad \operatorname{Im} \zeta > 0,$$

the resolvent of K_{AB} and by $G_{\mathrm{AB}}(x, y; \zeta)$ the Green function (integral kernel) of $G_{\mathrm{AB}}(\zeta)$. The resolvent $G_{\mathrm{AB}}(\zeta)$ can be seen to be analytically continued over the lower half of the complex plane, and $G_{\mathrm{AB}}(\zeta)$ with $\operatorname{Im} \zeta \le 0$ is well defined as an operator from L^2_{com} to L^2_{loc}, where L^2_{com} and L^2_{loc} denote the spaces of square integrable functions with compact support and of locally square integrable functions over $\boldsymbol{R}^2$, respectively. We write $\gamma(x; \omega) = \gamma(\hat{x}; \omega)$ for the azimuth angle from $\omega \in S^1$ to $\hat{x} = x/|x|$ and use the notation $\cdot$ to denote the scalar product in $\boldsymbol{R}^2$. Then the first main theorem is formulated as follows.

Theorem 1.1. Let $E > 0$ and $\eta_0 > 0$ be fixed and let $\lambda \gg 1$ be large enough. Assume that $\zeta = E - i\eta$ satisfies $0 \le \eta \le \eta_0 (\log \lambda)/\lambda$. Set $k = \zeta^{1/2}$ with $\operatorname{Im} k \le 0$ and

$$c(E) = (8\pi)^{-1/2} e^{i\pi/4} E^{-1/4}. \tag{1.6}$$

If x and y fulfill

$$\lambda/c \le |x| \le c\lambda, \quad \lambda/c \le |y| \le c\lambda, \quad \lambda/c \le |x - y| \le c\lambda$$

for some $c > 1$, and if $\hat{x}$ and $\hat{y}$ satisfies $\hat{x} \cdot \hat{y} \ge 0$, then $G_{\mathrm{AB}}(x, y; \zeta)$ takes the following asymptotic form

$$\begin{aligned} G_{\mathrm{AB}}(x, y; \zeta) &= c(E)|x - y|^{-1/2} e^{ik|x-y|} \left(e^{i\alpha(\gamma(\hat{x}; -\hat{y}) - \pi)} + e_{1N}(x, y; \zeta, \lambda)\right) \\ &+ \ c(E) \left(|x||y|\right)^{-1/2} e^{ik(|x|+|y|)} \left(f_{\mathrm{AB}}(-\hat{y} \to \hat{x}; E) + e_{2N}(x, y; \zeta, \lambda)\right) \\ &+ \ O(\lambda^{-N}), \qquad \lambda \to \infty, \end{aligned}$$

for any $N \gg 1$, where the remainder terms e_{1N} and e_{2N} obey

$$\left|\partial_x^n \partial_y^m e_{1N}\right| + \left|\partial_x^n \partial_y^m e_{2N}\right| \le C_{nmN} (\log \lambda)^2 \lambda^{-1-|n|-|m|} \tag{1.7}$$

uniformly in x, y and ζ.

We prove the theorem in section 3 and derive a similar asymptotic formula for magnetic Schrödinger operators with smooth fields of compact support in section 4. We add a brief comment on the asymptotic formula in the above theorem. The first term on the right side describes the free trajectory which goes from y to x directly without being scattered at the origin, while the second term comes from the scattering trajectory which starts from y and arrives at x after scattered at the origin.

The second half of the work is devoted to studying the resonance problem in magnetic scattering as an application of Theorem 1.1. The resonance is usually defined as a pole of resolvent when it is analytically continued from the upper half of the complex plane over the lower–half plane. We can obtain a sharp bound on resonance widths (imaginary parts of poles) in scattering by two magnetic fields with compact supports at large separation. Let

$$b_j \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}), \qquad \alpha_j = (2\pi)^{-1} \int b_j(x)\, dx, \quad j = 1, 2,$$

be given two magnetic fields with fluxes α_1 and α_2, and let

$$A_j(x) \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}^2), \qquad \nabla \times A_j = b_j, \quad j = 1, 2,$$

be the vector potentials associated with b_j. Then we consider the operator

$$H_d = H(A_d) = H(A_{1d} + A_{2d}), \quad A_{jd}(x) = A_j(x - d_j), \quad d_j \in \boldsymbol{R}^2. \tag{1.8}$$

The second aim is to evaluate the resonance width asymptotically when the distance $|d| = |d_2 - d_1|$ between the two centers d_1 and d_2 goes to infinity. The obtained result is formulated in terms of backward scattering amplitude $f_j(\omega \to -\omega; E)$ for the Schrödinger operator $H_j = H(A_j)$ with potential A_j. We should note that vector potentials are not uniquely determined for given magnetic fields, but it is easily seen that Schrödinger operators with the same magnetic field are unitarily equivalent to one another. We are now in a position to state the second main theorem.

Theorem 1.2. Let the notation be as above and let $E > 0$ be fixed. Set $\hat{d} = d/|d|$ for $d = d_2 - d_1$. Assume that at least one of two fluxes α_1 and α_2 is an integer. Then, for any $\varepsilon > 0$ small enough, there exists $d_\varepsilon(E) \gg 1$ large enough such that $\zeta = E - i\eta$ with

$$0 < \eta < \frac{E^{1/2}}{|d|} \left\{ \log|d| - \log\left| f_1(-\hat{d} \to \hat{d}; E) f_2(\hat{d} \to -\hat{d}; E) \right| - \varepsilon \right\}$$

is not a resonance of H_d for $|d| > d_\varepsilon(E)$.

We prove the theorem in sections 5 and 6. We can obtain the precise information on the distribution near the real axis of resonances in the course of the proof. In fact, we can establish

$$\frac{e^{2ik|d|}}{|d|} f_1(-\hat{d} \to \hat{d}; E) f_2(\hat{d} \to -\hat{d}; E) = 1, \quad k = \zeta^{1/2}, \quad \operatorname{Im} k < 0, \tag{1.9}$$

as the relation which determines the location of resonance approximately. We have to say a word on the assumption that at least one of two fluxes is an integer. Unfortunately, the proof makes essential use of this technical assumption. However, the argument used for proving the theorem is helpful as the first step toward the general case without such a restrictive assumption, and the idea seems to have a room to be improved for this purpose. We make a further comment on this matter in the last section (section 7). Our aim is to study in an elementary way as far as possible how resonances near the real axis are generated by classical trajectories oscillating between two scatters at large separation.

We also explain the motivation of this work in section 7. The original motivation lies in the semiclassical theory for quantum resonances in magnetic scattering by two solenoidal fields. We consider

$$H_h = (-ih\nabla - \Psi)^2, \quad 0 < h \ll 1, \tag{1.10}$$

where $\Psi = \alpha_1 \Phi(x - p_1) + \alpha_2 \Phi(x - p_2)$. This Hamiltonian governs the movement of particles in two solenoidal fields $2\pi\alpha_1\delta(x - p_1)$ and $2\pi\alpha_2\delta(x - p_2)$, and it becomes self–adjoint under the boundary conditions $\lim_{|x-p_j|\to 0} |u(x)| < \infty$ at centers p_1 and p_2. By a simple gauge transformation, the operator turns out to be unitarily equivalent to

$$\tilde{H}_d = H(\Psi_d), \qquad \Psi_d = \beta_1 \Phi(x - d_1) + \beta_2 \Phi(x - d_2), \tag{1.11}$$

where $d_j = p_j/h$ and $\beta_j = \alpha_j/h - [\alpha_j/h]$. Thus the problem is reduced to the case where the distance

$$|d| = |d_2 - d_1| = |p_2 - p_1|/h \gg 1$$

between two centers d_1 and d_2 is sufficiently large. In the special case that at least one of α_1/h and α_2/h is an integer, $\tilde{H}_d$ further becomes unitarily equivalent to a Hamiltonian with a single solenoidal field, and hence $\tilde{H}_d$ has no resonances (see section 2). The classical system of particles moving in two solenoidal fields has closed trajectories oscillating between centers of fields. For this reason, we are interested in the case without the technical assumption to see how such trapped trajectories give rise to resonances near the real axis. In the previous works ([16, 25]), we have developed the semiclassical asymptotic analysis for scattering quantities such as amplitudes and spectral shift functions for the operator H_h above to study the relation between the Aharonov–Bohm effect in quantum mechanics and the trapping phenomenon in classical mechanics. The resonance problem remains open.

The resonance problem is one of the most active fields in scattering theory at present. There are a large number of works devoted to the semiclassical theory on resonances near the real axis generated by closed classical trajectories. An extensive list of references can be found in the book [12], and the paper by [22] is an excellent exposition on this subject. In particular, the semiclassical problem of shape resonances has been studied in detail, and upper or lower bounds on the resonance width and its asymptotic expansion in h have been obtained by many authors [6, 7, 9, 10, 11, 18] under various assumptions. Theorem 1.2 above gives a new type of lower bound in which backward scattering amplitudes are involved.

2. Scattering theory for Aharonov–Bohm Hamiltonian

We here make a quick review on the scattering by a solenoidal field. As stated in section 1, the scattering by such a field is known as one of the exactly solvable models in quantum mechanics. We refer to [1, 2, 4, 8, 15, 21] for more detailed expositions.

We begin by calculating the eigenfunction to the problem

$$K_{\mathrm{AB}}\varphi = E\,\varphi, \qquad \lim_{|x|\to 0} |\varphi(x)| < \infty,$$

with energy $E > 0$ as an eigenvalue for the self–adjoint operator K_{AB} defined by (1.2) with domain (1.4). The operator is rotationally invariant. To do this, we work in the polar coordinate system (r, θ). Let U be the unitary mapping defined by

$$(Uu)(r,\theta) = r^{1/2}u(r\theta) : L^2 \to L^2((0,\infty); dr) \otimes L^2(S^1).$$

If we write $\sum_l$ for the summation ranging over all integers l, then U enables us to decompose K_{AB} into the partial wave expansion

$$K_{AB} \simeq UK_{AB}U^* = \sum_l \oplus\ (k_{l,AB} \otimes Id)\,, \tag{2.1}$$

where Id is the identity operator and

$$k_{l,\mathrm{AB}} = -\partial_r^2 + (\nu^2 - 1/4)r^{-2}, \quad \nu = |l - \alpha|,$$

is self–adjoint in the space $L^2((0,\infty); dr)$ under the boundary condition $\lim_{r\to 0} r^{-1/2}|u(r)| < \infty$ at $r = 0$. We again write $\gamma(x;\omega)$ for the azimuth angle from $\omega \in S^1$ to $\hat{x} = x/|x|$ and use the notation $\cdot$ to denote the scalar product in $\boldsymbol{R}^2$. Then the outgoing eigenfunction $\varphi_{+\mathrm{AB}}(x;\omega,E)$ with ω as an incident direction at energy $E > 0$ is calculated as

$$\varphi_{+\mathrm{AB}}(x;\omega,E) = \sum_l \exp(-i\nu\pi/2)\exp(il\gamma(x;-\omega))J_\nu(E^{1/2}|x|) \tag{2.2}$$

with $\nu = |l - \alpha|$, where $J_\mu(z)$ denotes the Bessel function of order μ. The eigenfunction $\varphi_{+\mathrm{AB}}$ behaves like

$$\varphi_{+\mathrm{AB}}(x;\omega,E) \sim \varphi_0(x;\omega,E) = \exp\left(iE^{1/2}x\cdot\omega\right)$$

as $|x| \to \infty$ in the direction $-\omega$ ($x = -|x|\omega$), and the difference $\varphi_{+\mathrm{AB}} - \varphi_0$ satisfies the outgoing radiation condition at infinity. On the other hand, the incoming eigenfunction $\varphi_{-\mathrm{AB}}(x;\omega,E)$ is given by

$$\varphi_{-\mathrm{AB}}(x;\omega,E) = \sum_l \exp(i\nu\pi/2)\exp(il\gamma(x;\omega))J_\nu(E^{1/2}|x|), \tag{2.3}$$

which behaves like $\varphi_{-\mathrm{AB}} \sim \varphi_0(x;\omega,E)$ as $|x| \to \infty$ in the direction ω.

We decompose $\varphi_{+\mathrm{AB}}$ into the sum

$$\varphi_{+\mathrm{AB}}(x;\omega,E) = \varphi_{\mathrm{in}}(x;\omega,E) + \varphi_{\mathrm{sc}}(x;\omega,E)$$

of incident and scattering waves to calculate the scattering amplitude through the asymptotic behavior at infinity of scattering wave φ_{sc}. The idea is due to Takabayashi ([19]). We set $\sigma = \sigma(x;\omega) = \gamma(x;\omega) - \pi$. Then we have

$$\varphi_{+\mathrm{AB}} = \sum_l e^{-i\nu\pi/2}e^{il\sigma}J_\nu(E^{1/2}|x|), \quad \nu = |l-\alpha|.$$

If we further make use of the formula $e^{-i\mu\pi/2}J_\mu(iw) = I_\mu(w)$ for the Bessel function

$$I_\mu(w) = \frac{1}{\pi}\left(\int_0^\pi e^{w\cos\xi}\cos(\mu\xi)\,d\xi - \sin(\mu\pi)\int_0^\infty e^{-w\cosh p-\mu p}\,dp\right) \tag{2.4}$$

with $\mathrm{Re}\,w \ge 0$ ([26, p.181]), then $\varphi_{+\mathrm{AB}}$ takes the form

$$\begin{aligned} \varphi_{+\mathrm{AB}} &= \frac{1}{\pi}\sum_l e^{il\sigma}\int_0^\pi e^{-i\sqrt{E}|x|\cos\xi}\cos(\nu\xi)\,d\xi \\ &- \frac{1}{\pi}\sum_l e^{il\sigma}\sin(\nu\pi)\int_0^\infty e^{i\sqrt{E}|x|\cosh p}e^{-\nu p}\,dp. \end{aligned} \tag{2.5}$$

We now take the incident wave φ_{in} as

$$\varphi_{\mathrm{in}}(x;\omega,E) = e^{i\alpha\sigma}\varphi_0(x;\omega,E) = e^{i\alpha\sigma}e^{i\sqrt{E}|x|\cos\gamma(x;\omega)} = e^{i\alpha\sigma}e^{-i\sqrt{E}|x|\cos\sigma},$$

which is different from the usual plane wave $\varphi_0(x;\omega,E)$. The modified factor $e^{i\alpha\sigma}$ appears because of the long–range property of the potential $\Phi(x)$ defined by (1.3). Since the azimuth angle $\gamma(x;\omega)$ fulfills the relation

$$\nabla\gamma(x;\omega) = \left(-x_2/|x|^2, x_1/|x|^2\right) = \Phi(x), \tag{2.6}$$

the factor may be interpreted as the change of phase

$$\int_{l_x} \alpha\Phi(y)\cdot dy = \alpha \int_{-\infty}^{0} (d/ds)\,\gamma(x+s\omega;\omega)\,ds = \alpha\,(\gamma(x;\omega)-\pi) = \alpha\sigma(x;\omega)$$

which the potential $\alpha\Phi$ causes to the particle moving in the direction ω according to the Aharonov–Bohm effect, where $l_x = \{y = x + s\omega\}$. The incident wave admits the Fourier expansion

$$\begin{aligned}\varphi_{\mathrm{in}} &= \frac{1}{2\pi}\sum_l e^{il\sigma}\int_{-\pi}^{\pi} e^{i\alpha\xi - i\sqrt{E}|x|\cos\xi} e^{-il\xi}\,d\xi\\ &= \frac{1}{\pi}\sum_l e^{il\sigma}\int_0^{\pi} e^{-i\sqrt{E}|x|\cos\xi}\cos(\nu\xi)\,d\xi,\end{aligned}$$

and this, together with (2.5), yields

$$\varphi_{\mathrm{sc}}(x;\omega,E) = -\frac{1}{\pi}\sum_l e^{il\sigma}\sin(\nu\pi)\int_0^{\infty} e^{i\sqrt{E}|x|\cosh p} e^{-\nu p}\,dp.$$

We compute the series

$$\begin{aligned}\sum_l e^{il\sigma}e^{-\nu p}\sin(\nu\pi) &= \left\{\sum_{l\le[\alpha]} + \sum_{l\ge[\alpha]+1}\right\} e^{il\sigma}e^{-\nu p}\sin(\nu\pi)\\ &= \sin(\alpha\pi)\,(-1)^{[\alpha]}\left\{\frac{e^{-\alpha p}\left(e^{i\sigma}e^{p}\right)^{[\alpha]}}{1+e^{-i\sigma}e^{-p}} + \frac{e^{\alpha p}\left(e^{i\sigma}e^{-p}\right)^{[\alpha]}}{1+e^{-i\sigma}e^{p}}\right\}\end{aligned}$$

for $|\sigma| < \pi$. Thus we have

$$\varphi_{\mathrm{sc}} = -\frac{\sin(\alpha\pi)}{\pi}\,(-1)^{[\alpha]}\,e^{i[\alpha]\sigma}\int_{-\infty}^{\infty} e^{i\sqrt{E}|x|\cosh p}\frac{e^{-\beta p}}{1+e^{-i\sigma}e^{-p}}\,dp$$

with $\beta = \alpha - [\alpha]$. We apply the stationary phase method to the integral on the right side. Since $e^{i\sigma(x;\omega)} = e^{i(\gamma(x;\omega)-\pi)} = -e^{i(\theta-\omega)}$ by identifying $\theta = x/|x| = \hat{x} \in S^1$ with azimuth angle θ, we see that φ_{sc} obeys

$$\varphi_{\mathrm{sc}}(x;\omega,E) = f_{\mathrm{AB}}(\omega\to\hat{x};E)e^{i\sqrt{E}|x|}|x|^{-1/2} + o(|x|^{-1/2}), \quad |x|\to\infty,$$

where $f_{\mathrm{AB}}(\omega\to\theta;E)$ is defined by (1.5) for $\theta\neq\omega$. The quantity $f_{\mathrm{AB}}(\omega\to\theta;E)$ is called the amplitude for the scattering from the initial direction $\omega\in S^1$ to the final one θ at energy $E>0$. If, in particular, the magnetic flux α is an integer, then the amplitude $f_{\mathrm{AB}}(\omega\to\theta;E)=0$.

We calculate the Green function $G_{\mathrm{AB}}(x,y;\zeta)$ of resolvent $G_{\mathrm{AB}}(\zeta) = (K_{\mathrm{AB}}-\zeta)^{-1}$ with $\mathrm{Im}\,\zeta>0$. Let $k=\zeta^{1/2}$, $\mathrm{Im}\,k>0$, and let $k_{l,\mathrm{AB}}$ be as in (2.1). Then the equation $(k_{l,\mathrm{AB}}-\zeta)\,u = 0$ has $\left(r^{1/2}J_\nu(kr), r^{1/2}H_\nu(kr)\right)$ with Wronskian $2i/\pi$ as a pair of linearly independent solutions, where

$H_\nu(z) = H_\nu^{(1)}(z)$ denotes the Hankel function of the first kind. Thus $(k_{l,\mathrm{AB}} - \zeta)^{-1}$ has the integral kernel

$$g_{l,\mathrm{AB}}(r,\rho;\zeta) = (i\pi/2)\, r^{1/2}\rho^{1/2} J_\nu\left(k(r\wedge\rho)\right) H_\nu\left(k(r\vee\rho)\right), \quad \nu = |l-\alpha|,$$

where $r\wedge\rho = \min(r,\rho)$ and $r\vee\rho = \max(r,\rho)$. Hence the Green function of $G_{\mathrm{AB}}(\zeta)$ in question is given by

$$G_{\mathrm{AB}}(x,y;\zeta) = \frac{i}{4}\sum_l e^{il(\theta-\omega)} J_\nu\left(k(|x|\wedge|y|)\right) H_\nu\left(k(|x|\vee|y|)\right) \tag{2.7}$$

with $x = |x|\theta$ and $y = |y|\omega$. This makes sense even for ζ in the lower half of the complex plane by analytic continuation. Then $G_{\mathrm{AB}}(\zeta)$ with $\mathrm{Im}\,\zeta \le 0$ is well defined as an operator from L^2_{com} to L^2_{loc}. If, in particular, $\zeta = E > 0$ is real, then it is known by the principle of limiting absorption that the boundary value

$$G_{\mathrm{AB}}(E) = G_{\mathrm{AB}}(E+i0) = \lim_{\varepsilon\downarrow 0} G_{\mathrm{AB}}(E+i\varepsilon) : L^2_s \to L^2_{-s}, \quad s > 1/2,$$

to the positive axis becomes a bounded operator form the weighted L^2 space $L^2_s = L^2\left(\boldsymbol{R}^2; \langle x\rangle^{2s}\,dx\right)$ to L^2_{-s}, where $\langle x\rangle = \left(1+|x|^2\right)^{1/2}$. Thus the operator $G_{\mathrm{AB}}(\zeta)$ does not have any poles, and we can say that the Aharonov–Bohm Hamiltonian K_{AB} has no resonances. We do not discuss the resonance at zero energy.

3. Green function of Aharonov–Bohm Hamiltonian

The present section is devoted to proving Theorem 1.1. The proof is done by use of the stationary phase method. However phase functions are not necessarily real valued, so that a little attention should be paid in applying the method to oscillatory integrals. For brevity, we write $G(x,y;\zeta)$ for the Green function $G_{AB}(x,y;\zeta)$ defined by (2.7) with $\zeta = E - i\eta$ throughout the section. Before going into the proof, we derive the basic representation for $G(x,y;\zeta)$. The derivation is based on the following formula

$$H_\mu(Z)J_\mu(z) = \frac{1}{i\pi}\int_0^{\kappa+i\infty} \exp\left\{\frac{t}{2} - \frac{Z^2+z^2}{2t}\right\} I_\mu\left(\frac{Zz}{t}\right)\frac{dt}{t}, \quad |z|\le|Z|,$$

for the product of Bessel functions ([26, p.439]), where the contour is taken to be rectilinear with corner at $\kappa + i0$, $\kappa > 0$ being fixed arbitrarily. We apply this formula with $Z = k\left(|x|\vee|y|\right)$ and $z = k\left(|x|\wedge|y|\right)$ to obtain that

$$G(x,y;\zeta) = \frac{1}{4\pi}\sum_l e^{il\psi}\int_0^{\kappa+i\infty} \exp\left\{\frac{t}{2} - \frac{Z^2+z^2}{2t}\right\} I_\nu\left(\frac{Zz}{t}\right)\frac{dt}{t} \tag{3.1}$$

with $\nu = |l - \alpha|$, where ψ is defined by $\psi = \theta - \omega$ for $x = (|x|\cos\theta, |x|\sin\theta)$ and $y = (|y|\cos\omega, |y|\sin\omega)$ in the polar coordinates. We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We fix $M \gg 1$ large enough and take

$$\kappa = M^2 \log\lambda$$

in the contour of integral (3.1). We further take $0 < \delta,\ \varepsilon \ll 1$ small enough. Then we divide (3.1) into the sum of integrals over the following five intervals by the partition of unity : (0) $0 < t < \kappa$ and

$$(1)\ 0 < s < 2\lambda^{1-\delta}, \quad (2)\ \lambda^{1-\delta} < s < 2\varepsilon\lambda, \quad (3)\ \varepsilon\lambda < s < 2M\lambda, \quad (4)\ M\lambda < s \tag{3.2}$$

for $t = \kappa + is$. We evaluate the integral over each interval. The leading term comes from the integral over interval (3). By assumption, $k^2 = \zeta = E - i\eta$ with $0 \le \eta \le \eta_0 (\log\lambda)/\lambda$. If $t = \kappa + is$ satisfies (1), (2) or (3), then

$$\operatorname{Re}\left(\frac{Zz}{t}\right) = \operatorname{Re}\left(\frac{\zeta}{t}\right)|x||y| = \frac{E\kappa - \eta s}{\kappa^2 + s^2}|x||y| > 0 \tag{3.3}$$

for κ as above.

(*i*) We first evaluate the integral

$$G_0(x, y; \zeta) = \frac{1}{4\pi}\sum_l e^{il\psi}\int_0^\kappa \exp\left\{\frac{t}{2} - \frac{Z^2 + z^2}{2t}\right\} I_\nu\left(\frac{Zz}{t}\right)\frac{dt}{t}$$

over the real interval $(0, \kappa)$ and show that it obeys the bound $O(\lambda^{-N})$ for any $N \gg 1$. We calculate the series $\sum_l e^{il\psi} I_\nu(Zz/t)$ in the integral. To do this, we use the integral representation (2.4) for $I_\mu(w)$ with $\operatorname{Re} w \ge 0$. Since $\operatorname{Re} w = E|x||y|/t > 0$ for $w = Zz/t$, (2.4) enables us to decompose the series into the sum

$$\sum_l e^{il\psi} I_\nu(w) = \sum_l e^{il\psi} I_{\mathrm{fr},\nu}(w) + \sum_l e^{il\psi} I_{\mathrm{sc},\nu}(w), \tag{3.4}$$

where $I_{\mathrm{fr},\nu}(w)$ and $I_{\mathrm{sc},\nu}(w)$ are defined by

$$I_{\mathrm{fr},\nu} = \frac{1}{\pi}\int_0^\pi e^{w\cos\xi}\cos(\nu\xi)\,d\xi, \quad I_{\mathrm{sc},\nu} = -\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-w\cosh p - \nu p}\,dp$$

with $\nu = |l - \alpha|$. A simple calculation yields

$$I_{\mathrm{fr},\nu}(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{w\cos\xi} e^{i\alpha\xi} e^{-il\xi}\,d\xi$$

and hence we have

$$I_{\mathrm{fr}}(w) = \sum_l e^{il\psi} I_{\mathrm{fr},\nu}(w) = e^{w\cos\psi} e^{i\alpha\psi}, \quad |\psi| < \pi, \tag{3.5}$$

by the Fourier expansion, where $\psi = \theta - \omega$ is written as $\psi = \gamma(\hat{x}; -\hat{y}) - \pi$. On the other hand, the second series on the right side is computed as in the same way as in section 2, and we see that the series is convergent to

$$I_{\mathrm{sc}}(w) = -\frac{\sin(\alpha\pi)}{\pi} e^{i[\alpha](\psi+\pi)} \int_{-\infty}^{\infty} e^{-w\cosh p} \frac{e^{-\beta p}}{1 + e^{-i\psi}e^{-p}}\, dp \tag{3.6}$$

with $\beta = \alpha - [\alpha]$, $0 \le \beta < 1$. By assumption, $\hat{x} \cdot \hat{y} \ge 0$. This implies that $-\pi/2 \le \psi = \theta - \omega \le \pi/2$, and hence the denominator $1 + e^{-i\psi}e^{-p}$ never vanishes even for $p = 0$. Thus $G_0(x, y; \zeta)$ is decomposed as

$$G_0(x, y; \zeta) = G_{\mathrm{fr},0}(x, y; \zeta) + G_{\mathrm{sc},0}(x, y; \zeta), \tag{3.7}$$

where $G_{\mathrm{fr},0}$ and $G_{\mathrm{sc},0}$ are defined by

$$G_{\mathrm{fr},0}(x, y; \zeta) = \frac{1}{4\pi}\int_0^\kappa \exp\left\{\frac{t}{2} - \frac{Z^2 + z^2}{2t}\right\} I_{\mathrm{fr}}\left(\frac{Zz}{t}\right)\frac{dt}{t}$$
$$G_{\mathrm{sc},0}(x, y; \zeta) = \frac{1}{4\pi}\int_0^\kappa \exp\left\{\frac{t}{2} - \frac{Z^2 + z^2}{2t}\right\} I_{\mathrm{sc}}\left(\frac{Zz}{t}\right)\frac{dt}{t}.$$

By (3.5), we have

$$G_{\mathrm{fr},0}(x, y; \zeta) = \frac{1}{4\pi} e^{i\alpha\psi} \int_0^\kappa \exp\left\{\frac{t}{2} - \frac{Z^2 - 2Zz\cos\psi + z^2}{2t}\right\}\frac{dt}{t} \tag{3.8}$$

and

$$Z^2 - 2Zz\cos\psi + z^2 = k^2\,|x - y|^2 = \zeta\,|x - y|^2.$$

Thus it follows by assumption that

$$\left|\exp\left\{\frac{t}{2} - \frac{\zeta|x - y|^2}{2t}\right\}\right| = O\left(\lambda^{M^2/2}\right)\exp(-c\lambda^2/t), \quad c > 0,$$

for $0 < t < \kappa$. This shows that $G_{\mathrm{fr},0}(x, y; \zeta)$ obeys $O(\lambda^{-N})$. A similar argument applies to $G_{\mathrm{sc},0}(x, y; \zeta)$. We have

$$\left|\exp\left\{\frac{t}{2} - \frac{Z^2 + z^2}{2t} - \frac{Zz}{t}\cosh p\right\}\right| = O\left(\lambda^{M^2/2}\right)\exp(-c\lambda^2/t), \quad c > 0,$$

uniformly in $p \in \boldsymbol{R}$, so that $G_{\mathrm{sc},0}(x, y; \zeta)$ also obeys $O(\lambda^{-N})$.

The integral over interval (1) in (3.2) is controlled in exactly the same way as above. The decomposition (3.7) holds true with interval $(0, \kappa)$ replaced by $(\kappa + i0, \kappa + i2\lambda^{1-\delta})$, and also we have by (3.3) that

$$\left|\exp\left\{\frac{t}{2} - \frac{\zeta|x - y|^2}{2t}\right\}\right| + \left|\exp\left\{\frac{t}{2} - \frac{Z^2 + z^2}{2t} - \frac{Zz}{t}\cosh p\right\}\right| \le \exp\left(-c\lambda^\delta\right)$$

for $t = \kappa + is$ with $0 < s < 2\lambda^{1-\delta}$. Hence the bound $O(\lambda^{-N})$ is obtained for the integral over interval (1) in (3.2).

(ii) We show that the integral over interval (2) in (3.2) is negligible. The integral admits a decomposition similar to (3.7) with interval $(0,\kappa)$ replaced by $(\kappa + i\lambda^{1-\delta}, \kappa + i2\varepsilon\lambda)$. We may assume that the integrand vanishes in a neighborhood of the end points of interval. We write the integrand of the first term as

$$\exp\left\{\frac{t}{2} - \frac{\zeta|x-y|^2}{2t}\right\} = e^{\kappa/2}e^{(u+iv)/2} = \lambda^{M^2/2}e^{(u+iv)/2},$$

where

$$u(t) = -\mathrm{Re}(\zeta/t)|x-y|^2, \quad v(t) = s - \mathrm{Im}(\zeta/t)|x-y|^2.$$

Since $0 \le \eta \le \eta_0 (\log\lambda)/\lambda$ and $\kappa = M^2 \log\lambda$ with $M \gg 1$, $\mathrm{Re}(\zeta/t)$ and $\mathrm{Im}(\zeta/t)$ behave like

$$\begin{aligned} &\mathrm{Re}(\zeta/t) = (E\kappa - \eta s)/\left(\kappa^2 + s^2\right) \sim E\kappa/s^2 > 0 \\ &\mathrm{Im}(\zeta/t) = -(Es + \eta\kappa)/\left(\kappa^2 + s^2\right) \sim -E/s < 0 \end{aligned} \tag{3.9}$$

for $\lambda^{1-\delta} < s < 2\varepsilon\lambda$. We can take $\varepsilon > 0$ so small that $\partial_s v < -c < 0$. As is easily seen, e^u satisfies $|\partial_s^m e^u| = O(|t|^{-m})$. Thus we make repeated use of partial integration to obtain that the integral of the first term obeys $O(\lambda^{-N})$ for any $N \gg 1$. We look at the second term in the decomposition. The integrand has

$$\exp\left\{\frac{t}{2} - \frac{Z^2+z^2}{2t}\right\}\exp\left\{-\frac{Zz}{t}\cosh p\right\} = \lambda^{M^2/2}e^{(u+iv)/2}$$

as an oscillatory term, where

$$\begin{aligned} u(t) &= -\mathrm{Re}(\zeta/t)\left(|x|^2 + |y|^2\right) - 2\mathrm{Re}(\zeta/t)|x||y|\cosh p \\ v(t) &= s - \mathrm{Im}(\zeta/t)\left(|x|^2 + |y|^2\right) - 2\mathrm{Im}(\zeta/t)|x||y|\cosh p. \end{aligned}$$

If we take account of (3.9), we can obtain the bound $O(\lambda^{-N})$ uniformly in p again by repeated use of partial integration.

(iii) We shall show that the integral over interval (4) in (3.2) is also negligible. The representation (2.4) does not make sense any longer, because $\mathrm{Re}\,(Zz/t)$ is not necessarily nonnegative for $t = \kappa + is$ with $s \ge M\lambda$. To see this, we use the other integral representation

$$\begin{aligned} I_\mu(w) &= \frac{e^{-i\mu\pi/2}}{\pi}\int_0^\pi \cos\left(\mu\xi - iw\sin\xi\right)\,d\xi \\ &- \frac{e^{-i\mu\pi/2}}{\pi}\sin(\mu\pi)\int_0^\infty e^{-iw\sinh p - \mu p}\,dp, \quad \mathrm{Im}\,w \le 0, \end{aligned} \tag{3.10}$$

which is obtained from formula $I_\mu(w) = e^{-i\mu\pi/2}J_\mu(iw)$ ([26, p.176]). We insert $I_\nu(Zz/t)$ into (3.1) and evaluate the integral for each l with $|l| < \lambda^2$.

The resulting integral is represented as a sum of three oscillatory integrals. One of these integrals has the term

$$\exp\left\{\frac{t}{2}-\frac{Z^2+z^2}{2t}\right\}\exp\left(-i\frac{Zz}{t}\sinh p\right), \quad p>0.$$

We have taken $\kappa = M^2\log\lambda$ with $M \gg 1$ and $\zeta = E - i\eta$ with $0 \le \eta \le \eta_0 (\log\lambda)/\lambda$. We write the above term in the form $\lambda^{M^2/2}e^{(u+iv)/2}$, where

$$u(t) = -\mathrm{Re}(\zeta/t)\left(|x|^2+|y|^2\right) + 2\left(\mathrm{Im}(\zeta/t) - i\mathrm{Re}(\zeta/t)\right)|x||y|\sinh p$$

and $v(t) = s - \mathrm{Im}(\zeta/t)\left(|x|^2+|y|^2\right)$ with $t = \kappa + is$. Then $\mathrm{Re}(\zeta/t)$ and $\mathrm{Im}(\zeta/t)$ behave like

$$\mathrm{Re}(\zeta/t) = (E\kappa - \eta s)/\left(\kappa^2+s^2\right) \sim -\eta/s \le 0$$
$$\mathrm{Im}(\zeta/t) = -(Es+\eta\kappa)/\left(\kappa^2+s^2\right) \sim -E/s < 0$$

as $s \to \infty$. Thus we have $\partial_s v > c > 0$ and

$$|\partial_s^m e^u| = O(|t|^{-m})(\log\lambda)^m e^{L\log\lambda}$$

for some $L \ge 1$ uniformly in x, y and p. Hence we can show by repeated use of partial integration that the integral obeys the bound $O(\lambda^{-N})$ uniformly in l with $|l| < \lambda^2$. A similar argument applies to the other two integrals having the terms

$$\exp\left\{\frac{t}{2}-\frac{Z^2+z^2}{2t}\right\}\exp\left(\pm\frac{Zz}{t}\sin\xi\right), \quad 0<\xi<\pi.$$

We consider only the term with $(Zz/t)\sin\xi$ and write it as $\lambda^{M^2/2}e^{(u+iv)/2}$, where

$$u(t) = -\mathrm{Re}(\zeta/t)\left(|x|^2+|y|^2\right) + \mathrm{Re}(2\zeta/t)|x||y|\sin\xi$$
$$v(t) = s - \mathrm{Im}(\zeta/t)\left(|x|^2+|y|^2\right) + \mathrm{Im}(2\zeta/t)|x||y|\sin\xi$$

with $t = \kappa + is$. Thus repeated use of partial integration again yields the bound $O(\lambda^{-N})$. To see that the sum over l with $|l| > \lambda^2$ is negligible, we make use of the representation formula ([26, p.172])

$$I_\mu(w) = \frac{(w/2)^\mu}{\Gamma(\mu+1/2)\Gamma(1/2)}\int_{-1}^{1} e^{-wp}\left(1-p^2\right)^{\mu-1/2}dp \tag{3.11}$$

for $\mu \ge 0$. Since $|w| = |Zz/t| = O(\lambda)$ and

$$\left|e^{-wp}\right| = O\left(e^{|\mathrm{Re}(\zeta/t)||x||y|}\right) = O\left(e^{L\lambda}\right), \quad |p|<1,$$

for some $L > 1$, it follows from the Stirling formula $\Gamma(\mu) \sim (2\pi)^{1/2}e^{-\mu}\mu^{\mu-(1/2)}$ for $\mu \gg 1$ that the sum over l with $|l| > \lambda^2$ makes no contribution. Thus we have shown that the integral over interval (4) in (3.2) is negligible.

(*iv*) The proof is completed in this step. We calculate the leading term which comes from the integral over interval (3) in (3.2). Let $g_0 \in C_0^\infty(0,\infty)$ be a nonnegative smooth cut–off function such that

$$\operatorname{supp} g_0 \subset (\varepsilon, 2M), \qquad g_0 = 1 \quad \text{on } [2\varepsilon, M].$$

We now combine all the results obtained in steps $(i) \sim (iii)$ to see that the Green function $G(x,y;\zeta)$ in question behaves like

$$G(x,y;\zeta) = G_{\mathrm{fr}}(x,y;\zeta) + G_{\mathrm{sc}}(x,y;\zeta) + O(\lambda^{-N}), \quad \lambda \to \infty,$$

where

$$G_{\mathrm{fr}} = \frac{i}{4\pi}\, e^{i\alpha\psi} \int_0^\infty \exp\left\{ \frac{t}{2} - \frac{k^2|x-y|^2}{2t} \right\} \frac{g_0(s/\lambda)}{t}\, ds$$

$$G_{\mathrm{sc}} = \frac{i}{4\pi} \int_0^\infty \exp\left\{ \frac{t}{2} - \frac{k^2\left(|x|^2+|y|^2\right)}{2t} \right\} I_{\mathrm{sc}}\left(\frac{k^2|x||y|}{t} \right) \frac{g_0(s/\lambda)}{t}\, ds$$

with $t = \kappa + is$, and $I_{\mathrm{sc}}(w)$ is defined by (3.6).

We analyze the behavior as $\lambda \to \infty$ of $G_{\mathrm{fr}}(x,y;\zeta)$. If we make change of variable $s = \rho\tau$ with $\rho = E^{1/2}|x-y| \sim \lambda$, then $G_{\mathrm{fr}}(x,y;\zeta)$ takes the form

$$G_{\mathrm{fr}} = \frac{i}{4\pi}\, e^{i\alpha\psi} \int_0^\infty \exp\left(i\rho f(\tau)\right) \exp(a(\tau)) g(\tau)\, d\tau,$$

where $f(\tau) = (\tau + 1/\tau)/2$ and

$$a(\tau) = \frac{i}{2}\left(\frac{k^2|x-y|^2}{\rho\tau - i\kappa} - \frac{\rho}{\tau} - i\kappa \right), \qquad g(\tau) = \frac{g_0(\rho\tau/\lambda)}{i\tau + \kappa/\rho}.$$

We apply the stationary phase method ([13, Theorem 7.7.5]) to the integral above. The stationary point is given by $\tau = 1$, so that

$$e^{i\rho f(1)} = \exp(iE^{1/2}|x-y|), \quad \left(\rho f''(1)/2\pi i\right)^{-1/2} = (2\pi)^{1/2} e^{i\pi/4} E^{-1/4} |x-y|^{-1/2}.$$

We calculate the leading term. The functions $g(\tau)$ and $a(\tau)$ take the following values at the stationary point $\tau = 1$:

$$g(1) = e^{-i\pi/2}\left(1 + O((\log\lambda)/\lambda)\right) = -i + O((\log\lambda)/\lambda)$$

and

$$\begin{aligned} a(1) &= (i/2)\left(k^2|x-y|^2/\left(\rho - i\kappa\right) - \rho - i\kappa\right) \\ &= (i/2)\left\{\rho\left((\zeta/E)/\left(1 - i\kappa/\rho\right) - 1\right) - i\kappa\right\} \\ &= (i/2)\left\{\rho\left((1 - i\eta/E)(1 + i\kappa/\rho + O((\log\lambda)^2/\lambda^2)) - 1\right) - i\kappa\right\} \\ &= (i/2)\left\{\rho(1 - i\eta/E + i\kappa/\rho + O((\log\lambda)^2/\lambda^2)) - \rho - i\kappa\right\} \\ &= \left(\eta/2E^{1/2}\right)|x-y| + O((\log\lambda)^2/\lambda), \end{aligned}$$

because $k^2 = \zeta = E - i\eta$ and $|x-y|^2 = \rho^2/E$. Hence it follows that

$$e^{i\rho f(1)} e^{a(1)} = \exp(ik|x-y|)\left(1 + O((\log\lambda)^2/\lambda)\right).$$

Thus we have

$$G_{\rm fr} = c(E) e^{i\alpha\psi} |x-y|^{-1/2} e^{ik|x-y|}\left(1 + e_{1N}(x,y;\zeta,\lambda)\right) + O(\lambda^{-N}),$$

where $c(E)$ is the constant defined by (1.6), and the remainder term e_{1N} obeys $e_{1N} = O((\log\lambda)^2/\lambda)$. As a function of x and y, $g(1)$ and $a(1)$ satisfy

$$\partial_x^n \partial_y^n \left(g(1) + i\right) = O((\log\lambda)\lambda^{-1-|n|-|m|})$$

and

$$\partial_x^n \partial_y^n \left(a(1) - \left(\eta/2E^{1/2}\right)|x-y|\right) = O((\log\lambda)^2 \lambda^{-1-|n|-|m|}).$$

This implies that $e_{1N}(x,y;\zeta,\lambda)$ fulfills (1.7).

Next we analyze the behavior of $G_{\rm sc}(x,y;\zeta)$. We recall the representation (3.6) for $I_{\rm sc}(k^2|x||y|/t)$ and make change of variable $s = \rho\tau$ with $\rho = E^{1/2}(|x|+|y|)$. Then

$$G_{\rm sc} = \frac{i}{4\pi} C_\alpha \int_0^\infty \int_{-\infty}^\infty \exp\left(i\rho f(\tau,p)\right)\exp(a(\tau,p)) g(\tau,p)\, dp\, d\tau,$$

where $C_\alpha = -\left(\sin(\alpha\pi)/\pi\right) e^{i[\alpha](\psi+\pi)}$ and

$$f = \frac{\tau}{2} + \frac{E(|x|^2+|y|^2)}{\rho^2}\frac{1}{2\tau} + \frac{E|x||y|}{\rho^2}\frac{\cosh p}{\tau}, \quad g = \frac{e^{-\beta p}}{1 + e^{-i\psi}e^{-p}} \frac{g_0(\rho\tau/\lambda)}{i\tau + \kappa/\rho}$$

$$a = \frac{i}{2}\left(\left(\frac{k^2}{\rho\tau - i\kappa} - \frac{E}{\rho\tau}\right)(|x|^2+|y|^2) - i\kappa\right) + i\left(\frac{k^2}{\rho\tau - i\kappa} - \frac{E}{\rho\tau}\right)|x||y|\cosh p$$

with $\beta = \alpha - [\alpha]$. We apply the stationary phase method to this integral in the two variables (τ, p). The stationary point is given by $(\tau,p) = (1,0)$. We have

$$\exp(i\rho f(1,0)) = \exp(i\sqrt{E}(|x|+|y|))$$

and the Hessian is calculated as

$$\left(\det\left[\rho f''(1,0)/2\pi i\right]\right)^{-1/2} = (2\pi i) E^{-1/2} |x|^{-1/2} |y|^{-1/2}$$

according to the notation in [13, Theorem 7.7.5]. The function $g(\tau,p)$ takes the values

$$g(1,0) = \frac{-ie^{i\psi}}{e^{i\psi}+1}\left(1 + O((\log\ \lambda)/\lambda)\right) = \frac{ie^{i(\psi+\pi)}}{1 - e^{i(\psi+\pi)}}\left(1 + O((\log\ \lambda)/\lambda)\right)$$

at $(\tau,p) = (1,0)$ and we repeat the same computation as used for calculating $a(1)$ to obtain that

$$\begin{aligned} a(1,0) &= (i/2)\left\{\left(k^2/\left(\rho - i\kappa\right) - E/\rho\right)\left(|x|+|y|\right)^2 - i\kappa\right\} \\ &= (i/2)\left\{\rho\left((\zeta/E)/\left(1 - i\kappa/\rho\right) - 1\right) - i\kappa\right\} \end{aligned}$$

$$= \left(\eta/2E^{1/2}\right)(|x|+|y|) + O((\log\lambda)^2/\lambda).$$

Thus we have

$$e^{i\rho f(1,0)}e^{a(1,0)} = \exp(ik(|x|+|y|))\left(1+O((\log\lambda)^2/\lambda)\right).$$

We recall the representation (1.5) for the amplitude $f_{\mathrm{AB}}(\omega\to\theta;E)$. Then a direct computation yields

$$\frac{i}{4\pi}C_\alpha(2\pi i)E^{-1/2}\frac{ie^{i(\psi+\pi)}}{1-e^{i(\psi+\pi)}} = c(E)f_{\mathrm{AB}}(-\omega\to\theta;E) = c(E)f_{\mathrm{AB}}(-\hat{y}\to\hat{x};E)$$

and hence it follows that

$$G_{\mathrm{sc}} = c(E)\left(|x||y|\right)^{-1/2}e^{ik(|x|+|y|)}\left(f_{\mathrm{AB}}(-\hat{y}\to\hat{x};E)+e_{2N}\right)+O(\lambda^{-N}),$$

where $e_{2N}(x,y;\zeta,\lambda)$ satisfies (1.7). Thus $G(x,y;\zeta)$ has been shown to take the desired asymptotic form, and the proof is now complete. □

Theorem 3.1. Under the same notation as in Theorem 1.1, one has the following statements :

(*i*) If x and y fulfill $\lambda/c\le|x|\le c\lambda$ and $1/c\le|y|\le c$ for some $c>1$, then

$$G_{\mathrm{AB}}(x,y;\zeta) = c(E)e^{ik|x|}|x|^{-1/2}\left\{\overline{\varphi}_{-\mathrm{AB}}(y;\hat{x},E)+e_{3N}(x,y;\zeta,\lambda)\right\}+O(\lambda^{-N}),$$

where e_{3N} obeys $\left|\partial_x^n\partial_y^m e_{3N}\right|\le C_{nmN}\left(\log\lambda\right)^2\lambda^{-1-|n|}$.

(*ii*) If x and y fulfill $1/c\le|x|\le c$ and $\lambda/c\le|y|\le c\lambda$, then

$$G_{\mathrm{AB}}(x,y;\zeta) = c(E)e^{ik|y|}|y|^{-1/2}\left\{\varphi_{+\mathrm{AB}}(x;-\hat{y},E)+e_{4N}(x,y;\zeta,\lambda)\right\}+O(\lambda^{-N}),$$

where e_{4N} obeys $\left|\partial_x^n\partial_y^m e_{4N}\right|\le C_{nmN}\left(\log\lambda\right)^2\lambda^{-1-|m|}$.

The proof uses the simple lemma below.

Lemma 3.1. Let $w\ge0$. Then we can take $L\gg1$ so large that

$$\exp(-Lw)\left(\sum_l|I_\nu(w)|\right) = O(1), \qquad |w|\to\infty.$$

Proof. We have

$$|I_\nu(w)| = \frac{e^w}{2^\nu\Gamma(\nu+1/2)}O(w^\nu), \quad \nu\gg1,$$

by (3.11), and also it is easy to see that $\exp(-Lw)w^\nu\le e^{-\nu}\left(\nu/L\right)^\nu$ for $\nu\gg1$. Thus the lemma follows from the Stirling formula. □

Proof of Theorem 3.1. The theorem is verified in almost the same way as Theorem 1.1, so we give only a sketch for the proof.

(i) We first consider the case (i) with $|x| \gg |y|$. Then $Z = k|x|$, $z = k|y|$ and $G(x, y; \zeta)$ is represented as

$$G = \frac{1}{4\pi} \sum_l e^{il\psi} \int_0^{\kappa + i\infty} \exp\left\{\frac{t}{2} - \frac{\zeta|x|^2}{2t}\right\} \exp\left\{-\frac{\zeta|y|^2}{2t}\right\} I_\nu\left(\frac{\zeta|x||y|}{t}\right) \frac{dt}{t}$$

with $\kappa = M^2 \log\lambda$, $M \gg 1$. We again divide this integral into the sum of integrals over the five intervals $(0) \sim (4)$ as in the proof of Theorem 1.1. The main contribution again comes from the integral over interval (3).

We first consider the integral over $(0, \kappa)$. If $|x| \gg 1$, then it follows from Lemma 3.1 that

$$\left|\exp(-\zeta|x|^2/4t)\right| \left(\sum_l |I_\nu(\zeta|x||y|/t)|\right) = O(1)$$

is bounded uniformly in t, $0 < t < \kappa$. Hence the integral obeys the bound $O(\lambda^{-N})$. A similar argument applies to the integral over interval (1) in (3.2), and we can show that this also obeys $O(\lambda^{-N})$.

The integrals over intervals (2) and (4) in (3.2) are controlled by repeated use of partial integration. In particular, we can show in exactly the same way as in the proof of Theorem 1.1 that the integral over interval (4) obeys $O(\lambda^{-N})$. We skip the proof for it.

We use the representation (2.4) for $I_\nu(w)$ with $w = Zz/t$ to evaluate the integral over interval (2). We insert $I_\nu(Zz/t)$ into the above representation for $G(x, y; \zeta)$ and evaluate the integral for each l. The resulting integral is represented as a sum of two oscillatory integrals. One of the integrals has the integrand

$$\exp\left\{\frac{t}{2} - \frac{\zeta|x|^2}{2t}\right\} \exp\left(\frac{\zeta|x||y|}{t}\cos\xi\right) = \lambda^{M^2/2} e^{(u+iv)/2}, \quad 0 < \xi < \pi,$$

as an oscillatory term, where

$$u(t) = -\mathrm{Re}(\zeta/t)|x|^2 + 2\mathrm{Re}(\zeta/t)|x||y|\cos\xi$$
$$v(t) = s - \mathrm{Im}(\zeta/t)|x|^2 + 2\mathrm{Im}(\zeta/t)|x||y|\cos\xi$$

with $t = \kappa + is$. By (3.9), $u(t) < 0$ for $M \gg 1$, and $\mathrm{Im}(\zeta/t)$ behaves like $\mathrm{Im}(\zeta/t) \sim -E/s$. Thus we have $\partial_s v < -c < 0$ for $0 < \varepsilon \ll 1$ uniformly in ξ, which shows that the integral obeys $O(\lambda^{-N})$ uniformly in l with $|l| < \lambda^2$.

The integrand of the other oscillatory integral has the term of the form

$$\exp\left\{\frac{t}{2} - \frac{\zeta|x|^2}{2t}\right\} \exp\left(-\frac{\zeta|x||y|}{t}\cosh p\right) = \lambda^{M^2/2} e^{(u+iv)/2},$$

where

$$u(t) = -\mathrm{Re}(\zeta/t)|x|^2 - 2\mathrm{Re}(\zeta/t)|x||y|\cosh p$$

$$v(t) = s - \mathrm{Im}(\zeta/t)|x|^2 - 2\mathrm{Im}(\zeta/t)|x||y|\cosh p.$$

Similarly we have $\partial_s v < -c < 0$ uniformly in p, and integration by parts yields the bound $O(\lambda^{-N})$ uniformly in l as above.

The sum over l with $|l| \geq \lambda^2$ is controlled by the Stirling formula combined with (3.11). Thus we see that the integral over interval (2) in (3.2) is negligible.

We consider the integral over interval (3) in (3.2). Let $g_0 \in C_0^\infty(0,\infty)$ be as in the proof of Theorem 1.1. We combine the results above to obtain that $G(x, y; \zeta)$ behaves like

$$G(x, y; \zeta) = G_-(x, y; \zeta) + O(\lambda^{-N}), \quad \lambda \to \infty,$$

for any $N \gg 1$, where

$$G_- = \frac{i}{4\pi}\int_0^\infty \exp\left\{\frac{t}{2} - \frac{k^2|x|^2}{2t}\right\}\exp\left(-\frac{k^2|y|^2}{2t}\right) I\left(\psi, \frac{k^2|x||y|}{t}\right)\frac{g_0(s/\lambda)}{t}\,ds$$

with $t = \kappa + is$, while $I(\varphi, w)$ is defined by

$$I(\varphi, w) = \sum_l e^{il\varphi} I_\nu(w). \tag{3.12}$$

By (3.11), $I(\varphi, w)$ is a smooth function in φ and w, provided that $1/c < |w| < c$ for some $c > 1$. If we make change of variable $s = \rho\tau$ with $\rho = E^{1/2}|x| \sim \lambda$, then $G_-(x, y; \zeta)$ is represented as

$$G_- = \frac{i}{4\pi}\int_0^\infty \exp\left(i\rho f(\tau)\right)\exp(a(\tau))g(\tau)\,d\tau,$$

where

$$f(\tau) = \left(\tau + 1/\tau\right)/2, \qquad a(\tau) = \frac{i}{2}\left(\frac{k^2|x|^2}{\rho\tau - i\kappa} - \frac{\rho}{\tau} - i\kappa\right)$$
$$g(\tau) = \exp\left(-\frac{k^2|y|^2}{2(\kappa + i\rho\tau)}\right)\frac{g_0(\rho\tau/\lambda)}{i\tau + \kappa/\rho}\, I(\psi, w)$$

with $w = \left(k^2|x||y|\right)/\left(\kappa + i\rho\tau\right)$. We apply the stationary phase method to the integral. We have

$$e^{i\rho f(1)} = \exp(iE^{1/2}|x|), \quad \left(\rho f''(1)/2\pi i\right)^{-1/2} = (2\pi)^{1/2}e^{i\pi/4}E^{-1/4}|x|^{-1/2}$$

and

$$a(1) = \left(\eta/2E^{1/2}\right)|x| + O((\log\lambda)^2/\lambda)$$

at the stationary point $\tau = 1$. Since $I_\nu(z/i) = e^{-i\nu\pi/2}J_\nu(z)$ by formula and since

$$e^{il\psi} = e^{il(\theta-\omega)} = e^{il\gamma(\hat{x};\hat{y})} = e^{-il\gamma(\hat{y};\hat{x})},$$

we have by (2.3) that

$$\sum_l e^{il\psi} I_\nu(z/i) = \sum_l e^{-il\gamma(\hat{y};\hat{x})} e^{-i\nu\pi/2} J_\nu(k|y|) = \overline{\varphi}_{-\mathrm{AB}}(y;\hat{x},E) + O((\log\lambda)/\lambda)$$

for $z = k|y|$. Hence

$$g(1) = -i\,\overline{\varphi}_{-\mathrm{AB}}(y;\hat{x},E) + O((\log\lambda)/\lambda)$$

uniformly in y, $1/c < |y| < c$. This yields the desired asymptotic form for $G(x,y;\zeta)$, when x and y ($|x| \gg |y|$) satisfy the assumptions.

(ii) We move to statement *(ii)*. If $|x| \ll |y|$, then $Z = k|y|$, $z = k|x|$ and we have

$$G(x,y;\zeta) = G_+(x,y;\zeta) + O(\lambda^{-N}),$$

where

$$G_+ = \frac{i}{4\pi}\int_0^\infty \exp\left\{\frac{t}{2} - \frac{k^2|y|^2}{2t}\right\} \exp\left(-\frac{k^2|x|^2}{2t}\right) I\left(\psi, \frac{k^2|x||y|}{t}\right) \frac{g_0(s/\lambda)}{t}\,ds$$

and $I(\varphi,w)$ is defined by (3.12). The function $G_+(x,y;\zeta)$ has the representation

$$G_+ = \frac{i}{4\pi}\int_0^\infty \exp\left(i\rho f(\tau)\right)\exp(a(\tau))g(\tau)\,d\tau$$

after making change of variable $s = \rho\tau$ with $\rho = E^{1/2}|y|$, where

$$f(\tau) = \left(\tau + 1/\tau\right)/2, \qquad a(\tau) = \frac{i}{2}\left(\frac{k^2|y|^2}{\rho\tau - i\kappa} - \frac{\rho}{\tau} - i\kappa\right),$$

$$g(\tau) = \exp\left(-\frac{k^2|x|^2}{2(\kappa + i\rho\tau)}\right)\frac{g_0(\rho\tau/\lambda)}{i\tau + \kappa/\rho}\, I(\psi,w)$$

with $w = \left(k^2|x||y|\right)/\left(\kappa + i\rho\tau\right)$. Since

$$\sum_l e^{il\psi} I_\nu(z/i) = \sum_l e^{il\gamma(\hat{x};\hat{y})} e^{-i\nu\pi/2} J_\nu(k|x|) = \varphi_{+\mathrm{AB}}(x;-\hat{y},E) + O((\log\lambda)/\lambda)$$

for $z = k|x|$, the desired asymptotic form is obtained. This proves *(ii)* and the proof is complete. □

We end the section by introducing a smooth non–negative cut–off function $\chi_0 \in C_0^\infty[0,\infty)$ with the properties

$$0 \le \chi_0 \le 1, \qquad \operatorname{supp}\chi_0 \subset [0,2], \qquad \chi_0 = 1 \text{ on } [0,1]. \tag{3.13}$$

This function is frequently used in the future discussion without further references.

4. Green functions of magnetic Schrödinger operators

The aim of this section is to derive the asymptotic form of the Green function corresponding to Theorems 1.1 and 3.1 for Schrödinger operators with smooth magnetic fields of compact support. The main result here is formulated as Theorem 4.1 below.

According to notation (1.1), we denote by $H(A)$ the Schrödinger operator with vector potential $A = (a_1, a_2) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$. Let

$$b_\alpha \in C_0^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}), \qquad \alpha = (2\pi)^{-1} \int b(x)\, dx, \tag{4.1}$$

be a given magnetic field with flux α. For brevity, we assume that b_α has support in $\{|x| < 1\}$. Then we can construct the potential $A_\alpha \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R}^2)$ associated with b_α in such a way that

$$A_\alpha(x) = \alpha\, \Phi(x), \qquad |x| > 2, \tag{4.2}$$

where $\Phi(x)$ is defined by (1.3). See, for example, [24, section 2], for construction of such a potential. In general, magnetic potentials are not uniquely determined for a given field, but Schrödinger operators with the same magnetic field become unitarily equivalent to one another through gauge transformations. Hence it does not matter to the location of resonances whatever potentials are chosen.

We now define

$$K_\alpha = H(A_\alpha). \tag{4.3}$$

Then K_α is self–adjoint in L^2 with domain $\mathcal{D}(K_\alpha) = H^2(\boldsymbol{R}^2)$, $H^s(\boldsymbol{R}^2)$ being the Sobolev space of order s. We further know ([14, 17, 20]) that K_α has the following spectral properties : (1) K_α has no bound states ; (2) The spectrum of K_α is absolutely continuous and the principle of limiting absorption holds true ; (3) The wave operator exists and is asymptotically complete for the pair (H_0, K_α), $H_0 = -\Delta$ being the free Hamiltonian. We denote by $\varphi_{+\alpha}(x;\omega,E)$ and $\varphi_{-\alpha}(x;\omega,E)$ the outgoing and incoming eigenfunctions of K_α with $\omega \in S^1$ as an incident direction at energy $E > 0$, respectively. The amplitude $f_\alpha(\omega \to \theta; E)$ for the scattering from the initial direction ω to the final one θ at energy E is defined through the asymptotic form

$$\varphi_{+\alpha} = e^{i\alpha(\gamma(x;\omega)-\pi)}\varphi_0(x;\omega,E) + f_\alpha(\omega \to \theta; E)e^{i\sqrt{E}|x|}|x|^{-1/2} + o(|x|^{-1/2})$$

as $|x| \to \infty$ in the direction θ ($x = |x|\theta$), where $\varphi_0(x;\omega,E) = \exp(iE^{1/2}x\cdot\omega)$ and $\gamma(x;\omega)$ again denotes the azimuth angle from ω to $\hat{x} = x/|x|$. We further denote by $G_\alpha(\zeta) = (K_\alpha - \zeta)^{-1}$, $\operatorname{Im}\zeta > 0$, the resolvent of K_α and by $G_\alpha(x, y; \zeta)$ the Green function of $G_\alpha(\zeta)$. We often use the same notation

$G_\alpha(\zeta)$ to denote the resolvent obtained by analytic extension over the lower half plane throughout the discussion in the sequel.

Lemma 4.1. Let $E > 0$ be fixed. Then there exists a neighborhood of E in the complex plane where $G_\alpha(\zeta)$ is analytic as a function with values in operators from $L^2_{\rm com}$ to $L^2_{\rm loc}$ in the sense that $vG_\alpha(\zeta)u : L^2 \to L^2$ is bounded for $u,\ v \in C_0^\infty(\boldsymbol{R}^2)$.

Proof. Let $\chi_0 \in C_0^\infty[0,\infty)$ be as in (3.13). We set $u_1(x) = \chi_0(|x|/4)$ and define $K_{\rm com} = H(A_{\rm com})$ with $A_{\rm com} = u_1 A_\alpha$. Then the coefficients of the differential operator $K_{\rm com} - H_0$ have support in $B = \{|x| < 8\}$. Hence the analytic perturbation theory of Fredholm implies that there exists a neighborhood of E in the complex plane where the resolvent $G_{\rm com}(\zeta) = (K_{\rm com} - \zeta)^{-1}$ is analytic as a function with values in operators from $L^2_{\rm com}$ to $L^2_{\rm loc}$. In fact,

$$(K_{\rm com} - H_0)(H_0 - \zeta)^{-1} : L^2_{\rm com}(B) \to L^2_{\rm com}(B)$$

acts as a compact operator, and $G_{\rm com}(\zeta)$ is represented as

$$\begin{aligned} &G_{\rm com}(\zeta) = (H_0 - \zeta)^{-1} \\ - \ &(H_0 - \zeta)^{-1} \left(Id + (K_{\rm com} - H_0)(H_0 - \zeta)^{-1}\right)^{-1} (K_{\rm com} - H_0)(H_0 - \zeta)^{-1}, \end{aligned}$$

where $L^2_{\rm com}(B)$ denotes the space of L^2 functions with support in B and the topology in $L^2_{\rm com}(B)$ is identified with that in the space $L^2(B)$. If we set $u_0 = \chi_0(|x|/2)$, then $K_\alpha = K_{\rm com}$ over the support of u_0, and also it follows from (4.2) that $K_\alpha = K_{\rm AB}$ over the support of $v_0 = 1 - u_0$. We calculate

$$(K_\alpha - \zeta)\left(u_0 G_{\rm com}(\zeta) + v_0 G_{\rm AB}(\zeta)\right) = Id + \Lambda(\zeta),$$

where

$$\Lambda(\zeta) = [K_\alpha, u_0] G_{\rm com}(\zeta) + [K_{\rm AB}, v_0] G_{\rm AB}(\zeta)$$

and $[X, Y] = XY - YX$ denotes the commutator between two operators X and Y. The operator $\Lambda(\zeta)$ acts as a compact operator on $L^2_{\rm com}(B_0)$ with $B_0 = \{|x| < 4\}$ and is analytic as a function with values in bounded operators acting on $L^2_{\rm com}(B_0)$. Hence there exists an inverse

$$(Id + \Lambda(\zeta))^{-1} : L^2_{\rm com}(B_0) \to L^2_{\rm com}(B_0)$$

for ζ in a neighborhood of E by the analytic perturbation theory. Thus we see that

$$G_\alpha(\zeta) = \left(u_0 G_{\rm com}(\zeta) + v_0 G_{\rm AB}(\zeta)\right)(Id + \Lambda(\zeta))^{-1} : L^2_{\rm com}(B_0) \to L^2_{\rm loc}.$$

As stated above, $K_\alpha = K_{\rm AB}$ on the support of v_0. Hence a simple manipulation yields

$$G_\alpha(\zeta) \quad = \quad G_\alpha(\zeta) u_0 + (G_\alpha(\zeta) v_0 - v_0 G_{\rm AB}(\zeta)) + v_0 G_{\rm AB}(\zeta)$$

$$= \quad G_\alpha(\zeta)u_0 + G_\alpha(\zeta)[v_0, K_\alpha]G_{\mathrm{AB}}(\zeta) + v_0 G_{\mathrm{AB}}(\zeta).$$

Thus it follows that $G_\alpha(\zeta) : L^2_{\mathrm{com}} \to L^2_{\mathrm{loc}}$ is well defined in a neighborhood of E in the complex plane. □

Lemma 4.2. Let $u_0 = \chi_0(|x|/2)$ and $u_1 = \chi_0(|x|/4)$. Denote by $(\ ,\ )$ the L^2 scalar product. Then the amplitude $f_\alpha(\omega \to \theta; E)$ has the representation

$$f_\alpha = f_{\mathrm{AB}} + c(E)\,(G_\alpha(E)[K_{\mathrm{AB}}, u_0]\varphi_{+\mathrm{AB}}(\cdot;\omega,E), [K_{\mathrm{AB}}, u_1]\varphi_{-\mathrm{AB}}(\cdot;\theta,E))$$

with $f_{\mathrm{AB}} = f_{\mathrm{AB}}(\omega \to \theta; E)$, where $c(E)$ is the constant defined by (1.6) and

$$G_\alpha(E) = G_\alpha(E + i0) = \lim_{\varepsilon \downarrow 0} G(E + i\varepsilon) : L^2_s \to L^2_{-s}, \quad s > 1/2.$$

Proof. Note that $K_\alpha = K_{\mathrm{AB}}$ outside the support of u_0. Hence we have

$$\varphi_{+\alpha} = (1 - u_0)\,\varphi_{+\mathrm{AB}} + G_\alpha(E)[K_{\mathrm{AB}}, u_0]\varphi_{+\mathrm{AB}}. \tag{4.4}$$

Similarly

$$\varphi_{+\mathrm{AB}} = (1 - u_1)\,\varphi_{+\alpha} + G_{\mathrm{AB}}(E)[K_{\mathrm{AB}}, u_1]\varphi_{+\alpha}.$$

and hence

$$\varphi_{+\alpha} = \varphi_{+\mathrm{AB}} + u_1\varphi_{+\alpha} - G_{\mathrm{AB}}(E)[K_{\mathrm{AB}}, u_1]\varphi_{+\alpha}. \tag{4.5}$$

It follows from Theorem 3.1 (i) with $\lambda = r = |x|$ that the last term on the right side of (4.5) behaves like

$$c(E)\,(\varphi_{+\alpha}(\cdot;\omega,E), [K_{\mathrm{AB}}, u_1]\varphi_{-\mathrm{AB}}(\cdot;\theta,E))\,|x|^{-1/2}e^{iE|x|} + o(|x|^{-1/2})$$

as $|x| \to \infty$ in the direction θ. We insert (4.4) into $\varphi_{+\alpha}$ on the right side. Since

$$((1 - u_0)\varphi_{+\mathrm{AB}}, [u_1, K_{\mathrm{AB}}]\varphi_{-\mathrm{AB}}) = (\varphi_{+\mathrm{AB}}, [u_1, K_{\mathrm{AB}}]\varphi_{-\mathrm{AB}}) = 0,$$

we obtain the desired relation. □

Theorem 4.1. Keep the same notation as in Theorems 1.1 and 3.1. Assume $\zeta = E - i\eta$ with $0 \le \eta \le \eta_0\,(\log\lambda)\,/\lambda$ for $\lambda \gg 1$. Let $k = \zeta^{1/2}$ with $\mathrm{Im}\,k \le 0$. Then we have the following statements :

(i) If x and y fulfill

$$\lambda/c \le |x| \le c\lambda, \quad \lambda/c \le |y| \le c\lambda, \quad \lambda/c \le |x - y| \le c\lambda, \quad \hat{x}\cdot\hat{y} \ge 0$$

for some $c > 1$, then $G_\alpha(x, y; \zeta)$ takes the asymptotic form

$$\begin{aligned} G_\alpha(x, y; \zeta) &= c(E)|x - y|^{-1/2}e^{ik|x-y|}\left\{e^{i\alpha(\gamma(\hat{x};-\hat{y})-\pi)} + \rho_{1N}(x, y; \zeta, \lambda)\right\} \\ &+\ c(E)\,(|x||y|)^{-1/2}\,e^{ik(|x|+|y|)}\,\{f_\alpha(-\hat{y} \to \hat{x}; E) + \rho_{2N}(x, y; \zeta, \lambda)\} \\ &+\ O(\lambda^{-N}), \qquad \lambda \to \infty, \end{aligned}$$

for any $N \gg 1$, where ρ_{1N} and ρ_{2N} obey

$$\left|\partial_x^n \partial_y^m \rho_{1N}\right| + \left|\partial_x^n \partial_y^m \rho_{2N}\right| \le C_{nmN} (\log \lambda)^2 \lambda^{-1-|n|-|m|}.$$

(ii) If x and y fulfill $\lambda/c \le |x| \le c\lambda$ and $|y| \le c$ for some $c > 1$, then

$$G_\alpha(x, y; \zeta) = c(E) e^{ik|x|} |x|^{-1/2} \left\{ \overline{\varphi}_{-\alpha}(y; \hat{x}, E) + \rho_{3N}(x, y; \zeta, \lambda) \right\} + O(\lambda^{-N}),$$

where ρ_{3N} obeys $\left|\partial_x^n \partial_y^m \rho_{3N}\right| \le C_{nmN} (\log \lambda)^2 \lambda^{-1-|n|}$.

Proof. We sometimes write $G_{\rm AB}$ and G_α for $G_{\rm AB}(\zeta)$ and $G_\alpha(\zeta)$ throughout the discussion in the proof.

(i) We set

$$u_0(x) = \chi_0(|x|/2), \quad u_1(x) = \chi_0(|x|/4), \quad v_0 = 1 - u_0, \quad v_1 = 1 - u_1.$$

Let $p,\ q \in \boldsymbol{R}^2$ $(|p|,\ |q| \gg 1)$ be points having the properties in the theorem. If we further set $w_p(x) = \chi_0(|x - p|)$, then $w_p v_0 = w_p$ and $w_p v_1 = w_p$; similarly for $w_q = \chi_0(|x - q|)$. The operator K_α coincides with $K_{\rm AB}$ on the support of v_1. We compute

$$\begin{aligned} w_p G_\alpha w_q &= w_p G_{\rm AB} w_q + w_p G_{\rm AB} \left(K_{\rm AB} v_1 - v_1 K_\alpha\right) G_\alpha w_q \\ &= w_p G_{\rm AB} w_q + w_p G_{\rm AB} [u_1, K_{\rm AB}] G_\alpha w_q. \end{aligned}$$

Since $v_0 = 1$ on the support of ∇u_1 and since $K_\alpha = K_{\rm AB}$ on the support of v_0, we repeat the above argument to get

$$w_p G_\alpha w_q = w_p G_{\rm AB} w_q + w_p G_{\rm AB} [u_1, K_{\rm AB}] \left(G_{\rm AB} + G_\alpha [K_{\rm AB}, u_0] G_{\rm AB}\right) w_q.$$

Note that

$$w_p G_{\rm AB} [u_1, K_{\rm AB}] G_{\rm AB} w_q = w_p G_{\rm AB} u_1 w_q - w_p u_1 G_{\rm AB} w_q = 0$$

and hence we have

$$w_p G_\alpha w_q = w_p G_{\rm AB} w_q + w_p G_{\rm AB} [u_1, K_{\rm AB}] G_\alpha [K_{\rm AB}, u_0] G_{\rm AB} w_q.$$

If we insert $G_\alpha(\zeta) = G_\alpha(E) + (G_\alpha(\zeta) - G_\alpha(E))$ into the operator on the right side and if we apply Theorems 1.1 and 3.1 to the operator

$$w_p G_{\rm AB} w_q + w_p G_{\rm AB} [u_1, K_{\rm AB}] G_\alpha(E) [K_{\rm AB}, u_0] G_{\rm AB} w_q,$$

then it follows from Lemma 4.2 that the integral kernel of this operator has the desired asymptotic form at points p and q fixed arbitrarily. Since

$$[u_1, K_{\rm AB}] \left(G_\alpha(\zeta) - G_\alpha(E)\right) [K_{\rm AB}, u_0] : L^2 \to L^2, \quad \zeta = E - i\eta,$$

is bounded by elliptic estimates and since its norm obeys $O\left((\log \lambda)/\lambda\right)$ by continuity (Lemma 4.1), the integral kernel of the operator

$$w_p G_{\rm AB} [u_1, K_{\rm AB}] \left(G_\alpha(\zeta) - G_\alpha(E)\right) [K_{\rm AB}, u_0] G_{\rm AB} w_q$$

is dealt with as a remainder term. Thus statement (i) is proved.

(ii) Let $p \in \boldsymbol{R}^2$, $|p| \gg 1$, be again fixed. We use the notation $u_0,\ u_1,\ v_0,$ v_1 and w_p with the same meaning as ascribed above. We further set $u_2 = \chi_0(|x|/8)$ and $v_2 = 1 - u_2$. For brevity, we assume that $|y| < 2$. Then we compute

$$\begin{aligned} w_pG_\alpha u_0 = w_pv_2G_\alpha u_0 = w_p[v_2, G_\alpha]u_0 &= w_pG_\alpha[u_2, K_\alpha]G_\alpha u_0 \\ &= w_p\left(G_{\rm AB} + G_{\rm AB}[K_\alpha, v_1]G_\alpha\right)[u_2, K_\alpha]G_\alpha u_0 \\ &= w_pG_{\rm AB}\left(v_1 + [K_\alpha, v_1]G_\alpha\right)[u_2, K_\alpha]G_\alpha u_0. \end{aligned}$$

We replace $G_\alpha = G_\alpha(\zeta)$ on the right side by $G_\alpha(E)$ to derive the leading term. As is easily seen,

$$(v_1 + G_\alpha^*(E)[v_1, K_\alpha])\,\varphi_{-{\rm AB}} = \varphi_{-\alpha} \tag{4.6}$$

and $u_0G_\alpha^*(E)[K_\alpha, u_2]\varphi_{-\alpha} = u_0\varphi_{-\alpha}$. These relations, together with Theorem 3.1 (i), yield the desired asymptotic form for the integral kernel of the operator $w_pG_\alpha u_0$, and the proof of (ii) is complete. □

We end the section by deriving the representation for the backward scattering amplitude $f_\alpha(\omega \to -\omega; E)$ with integer flux α as a special case of Lemma 4.2. This is used in proving Theorem 1.2.

Lemma 4.3. Keep the same notation as in Lemma 4.2. Assume that the flux α is an integer. Then one has

$$f_\alpha(\omega \to -\omega; E) = \\ -c(E)e^{-i\alpha\gamma(-\omega)}\left([u_0, H_0]\varphi_0(\cdot;\omega, E), e^{-i\alpha\gamma(\cdot)}\varphi_{-\alpha}(\cdot; -\omega, E)\right),$$

where $\varphi_0(x;\omega,E) = \exp(iE^{1/2}x\cdot\omega)$ and $\gamma(x)$ denotes the azimuth angle from the positive x_1 axis.

Proof. We make use of relation (2.6). If α is an integer, then we have

$$K_{\rm AB} = H(\alpha\Phi) = \exp(i\alpha\gamma(x;-\omega))H_0\exp(-i\alpha\gamma(x;-\omega))$$

and $f_{\rm AB}(\omega \to \theta; E) = 0$. The outgoing eigenfunction $\varphi_{+{\rm AB}}(x;\omega,E)$ is written as

$$\varphi_{+{\rm AB}}(x;\omega,E) = e^{i\alpha\gamma(x;-\omega)}\varphi_0(x;\omega,E).$$

In fact, $\varphi_{+{\rm AB}}(x;\omega,E)$ behaves like $\varphi_0(x;\omega,E)$ at infinity in the direction $-\omega$. We represent the commutator $[K_{\rm AB}, u_0]$ in Lemma 4.2 as

$$[K_{\rm AB}, u_0] = -\exp(i\alpha\gamma(x;-\omega))[u_0, H_0]\exp(-i\alpha\gamma(x;-\omega))$$

and the incoming eigenfunction $\varphi_{-\alpha}(x;-\omega,E)$ as (4.6). Hence

$$G_\alpha^*(E)[K_{\rm AB}, u_1]\varphi_{-{\rm AB}} = \varphi_{-\alpha} - v_1\varphi_{-{\rm AB}}, \quad u_1 = 1 - v_1.$$

Since $e^{i\alpha\gamma(x;-\omega)} = e^{i\alpha\gamma(x)}e^{-i\alpha\gamma(-\omega)}$ for α integer and since $[H_0, u_0]v_1 = 0$ as an operator, the proof is completed by inserting these relations into the representation in Lemma 4.2. □

5. Schrödinger operators with two fields at large separation

We prove Theorem 1.2 in the present and next sections. We first set up the problem in a more convenient form. Let

$$b_\alpha,\ b_\mu \in C_0^\infty\left(\boldsymbol{R}^2 \to \boldsymbol{R}\right), \qquad \operatorname{supp} b_\alpha,\ \operatorname{supp} b_\mu \subset \{|x| < 1\},$$

be given two magnetic fields with fluxes α and μ, respectively. Then we make the main assumption that

$$\mu \in \boldsymbol{Z} \ \text{ is an integer.} \tag{5.1}$$

As stated in the previous section, we can construct the corresponding potentials

$$A_\alpha,\ A_\mu \in C^\infty\left(\boldsymbol{R}^2 \to \boldsymbol{R}^2\right), \qquad \nabla \times A_\alpha = b_\alpha, \quad \nabla \times A_\mu = b_\mu,$$

in such a way that

$$A_\alpha(x) = \alpha\Phi(x), \qquad A_\mu(x) = \mu\Phi(x) \tag{5.2}$$

over $\{|x| > 2\}$, where $\Phi(x)$ is defined by (1.3). If we set $A_{d\mu}(x) = A_\mu(x-d)$ for $d \in \boldsymbol{R}^2$ with $|d| \gg 1$, then the operator H_d in (1.8) takes the form

$$H_d = H(A_\alpha + A_{d\mu}), \tag{5.3}$$

and it has the magnetic field $b_\alpha(x) + b_\mu(x-d)$ with two compact supports at large separation. The main result here is formulated at the end of the section as Proposition 5.1, which gives a sufficient condition for a region in the lower half of the complex plane to be free of resonances of H_d.

We write $R(\zeta; H)$ for the resolvent $(H-\zeta)^{-1}$, $\operatorname{Im}\zeta > 0$, of a self–adjoint operator H. We also use the same notation $R(\zeta; H)$ to denote the operator obtained by analytic continuation over the lower half plane. According to notation (1.1), we define the following operators

$$K_\alpha = H(A_\alpha), \qquad K_\mu = H(A_\mu), \qquad K_{d\mu} = H(A_{d\mu}) \tag{5.4}$$

and denote their resolvents by

$$G_\alpha(\zeta) = R(\zeta; K_\alpha), \quad G_\mu(\zeta) = R(\zeta; K_\mu), \quad G_{d\mu}(\zeta) = R(\zeta; K_{d\mu}).$$

These operators are all self–adjoint with the same domain $H^2(\boldsymbol{R}^2)$, and it follows from Lemma 4.1 that $G_\alpha(\zeta)$ is analytically continued over the lower half plane as a meromorphic function with values in operators from $L^2_{\rm com}$ to $L^2_{\rm loc}$; similarly for $G_\mu(\zeta)$ and $G_{d\mu}(\zeta)$.

We now recall that $\gamma(x)$ denotes the azimuth angle from the positive x_1 axis. By assumption (5.1), $\exp(ig_\mu)$ with $g_\mu(x) = \mu\gamma(x)$ is well defined as a single valued function, although $g_\mu(x)$ itself is not necessarily a single valued function. We set $g_{d\mu} = g_\mu(x - d)$ and define the operator

$$H_{d\alpha} = H(A_\alpha + \nabla g_{d\mu}) = \exp(ig_{d\mu})K_\alpha \exp(-ig_{d\mu}), \tag{5.5}$$

which is self–adjoint in L^2 under the boundary condition $\lim_{|x-d|\to 0} |u(x)| < \infty$ at point $x = d$. If we take account of relation (2.6), then

$$H_{d\alpha} = H_d \qquad \text{on} \ \ \{|x - d| > 2\}\,, \tag{5.6}$$

because $\nabla g_{d\mu} = A_{d\mu}$ there by (5.2). We further introduce another auxiliary operator. Let $g_\alpha \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ be a bounded function such that

$$g_\alpha(x) = \alpha\gamma(x; -\hat{d}) \qquad \text{on} \ \ \{|x - d| < |d|/2\}\,. \tag{5.7}$$

Then we define the self–adjoint operator

$$H_{d\mu} = H(\nabla g_\alpha + A_{d\mu}) = \exp(ig_\alpha)K_{d\mu} \exp(-ig_\alpha) \tag{5.8}$$

with domain $H^2(\boldsymbol{R}^2)$. By (5.2) again, we have

$$H_{d\mu} = H_d \qquad \text{on} \ \ \{|x - d| < |d|/2\}\,. \tag{5.9}$$

It follows from (5.5) that

$$R_{d\alpha}(\zeta) = R(\zeta; H_{d\alpha}) = \exp(ig_{d\mu})G_\alpha(\zeta) \exp(-ig_{d\mu}), \tag{5.10}$$

and similarly we have

$$R_{d\mu}(\zeta) = R(\zeta; H_{d\mu}) = \exp(ig_\alpha)G_{d\mu}(\zeta) \exp(-ig_\alpha) \tag{5.11}$$

for the resolvent of $H_{d\mu}$. Thus $R_{d\alpha}(\zeta)$ and $R_{d\mu}(\zeta)$ are also analytically continued over the lower half plane as a meromorphic function with values in operators from L^2_{com} to L^2_{loc} by Lemma 4.1.

We take the spectral parameter ζ as

$$\zeta = E - i\eta, \quad 0 \le \eta \le \eta_0 \left(\log |d|\right)/|d|,$$

in the lower half plane, $E > 0$ and $\eta_0 > 0$ being fixed, and we sometimes write G_α for $G_\alpha(\zeta)$, skipping the dependence on ζ throughout the discussion in the sequel. We set

$$u_d = \chi_0(|x - d|/2), \qquad v_d = 1 - u_d$$

for the cut–off function χ_0 with property (3.13). Then (5.6) and (5.9) enable us to compute

$$(H_d - \zeta)\left(u_d R_{d\mu} + v_d R_{d\alpha}\right) = Id + X_d,$$

where

$$X_d(\zeta) = [H_d, u_d]R_{d\mu} + [H_d, v_d]R_{d\alpha}.$$

Since ∇u_d and ∇v_d have the same support in $\Omega_d = \{|x-d| < 4\}$, we can regard $X_d(\zeta)$ as an operator acting on $L^2_{\rm com}(\Omega_d)$. If

$$Id + X_d(\zeta) : L^2_{\rm com}(\Omega_d) \to L^2_{\rm com}(\Omega_d)$$

is shown to be invertible, then $R(\zeta; H_d)$ is represented as

$$R(\zeta; H_d) = (u_d R_{d\mu}(\zeta) + v_d R_{d\alpha}(\zeta))\,(Id + X_d(\zeta))^{-1} : L^2_{\rm com}(\Omega_d) \to L^2_{\rm loc}.$$

Thus we see that the resolvent $R(\zeta; H_d)$ in question is continued over the lower half plane as a meromorphic function with values in operators from $L^2_{\rm com}$ to $L^2_{\rm loc}$. This is shown in the same way as in the proof of Lemma 4.1 by taking the difference between $R(\zeta; H_d)$ and $R_{d\alpha}(\zeta)$.

We represent $X_d(\zeta)$ as a sum of two operators. To do this, we define

$$H_{d0} = H(\nabla g_d) = \exp(ig_d) H_0 \exp(-ig_d), \tag{5.12}$$

with $g_d = g_\alpha + g_{d\mu}$, and we denote by $R_{d0}(\zeta) = R(\zeta; H_{d0})$ the resolvent of the self–adjoint operator H_{d0} with the same domain as $H_{d\alpha}$. By (5.7), H_{d0} equals

$$H_{d0} = H(\nabla g_\alpha + \nabla g_{d\mu}) = H(A_\alpha + \nabla g_{d\mu}) = H_{d\alpha}$$

over $\{|x-d| < |d|/2\}$. If we set

$$w_d(x) = \chi_0(4|x-d|/|d|),$$

then $w_d = 1$ on $\Omega_d = \{|x-d| < 4\}$, and we have

$$\begin{aligned} [H_d, v_d] R_{d\alpha} &= [H_d, v_d] R_{d0} + [H_d, v_d] R_{d\alpha}\,(w_d H_{d0} - H_{d\alpha} w_d)\, R_{d0} \\ &= [H_d, v_d] R_{d0} + [H_d, v_d] R_{d\alpha} [w_d, H_{d0}] R_{d0} \end{aligned}$$

as an operator acting on $L^2_{\rm com}(\Omega_d)$. Thus $X_d(\zeta)$ is decomposed into the sum $X_d(\zeta) = Y_d(\zeta) + Z_d(\zeta)$, and we have the relation

$$(H_d - \zeta)\,(u_d R_{d\mu} + v_d R_{d\alpha}) = Id + Y_d + Z_d,$$

where the two operators

$$Y_d(\zeta) = [H_d, u_d] R_{d\mu} + [H_d, v_d] R_{d0}, \quad Z_d(\zeta) = [H_d, v_d] R_{d\alpha} [w_d, H_{d0}] R_{d0}$$

act on $L^2_{\rm com}(\Omega_d)$.

We shall show that

$$Id + Y_d(\zeta) : L^2_{\rm com}(\Omega_d) \to L^2_{\rm com}(\Omega_d)$$

is invertible. We make use of (5.8) and (5.9) to calculate the two commutators in the operator $Y_d(\zeta)$ as follows :

$$\begin{aligned} [H_d, u_d] &= \exp(ig_\alpha)[K_{d\mu}, u_d] \exp(-ig_\alpha) \\ [H_d, v_d] &= [u_d, H_d] = \exp(ig_\alpha)[K_{d\mu}, v_d] \exp(-ig_\alpha). \end{aligned}$$

We denote by $K_{d0} = H(\nabla g_{d\mu})$ the self–adjoint operator with the same domain as $H_{d0} = H(\nabla g_d)$. Then the resolvent $R_{d0}(\zeta) = R(\zeta; H_{d0})$ is represented as

$$R_{d0}(\zeta) = \exp(ig_\alpha)G_{d0}(\zeta)\exp(-ig_\alpha) \tag{5.13}$$

with $G_{d0}(\zeta) = R(\zeta; K_{d0})$, and hence it follows from (5.11) that

$$Id + Y_d(\zeta) = \exp(ig_\alpha)\left(Id + [K_{d\mu}, u_d]G_{d\mu} + [K_{d\mu}, v_d]G_{d0}\right)\exp(-ig_\alpha).$$

Since $K_{d\mu} = H(A_{d\mu}) = H(\nabla g_{d\mu}) = K_{d0}$ on the support of v_d, we have

$$(K_{d\mu} - \zeta)\left(u_d G_{d\mu} + v_d G_{d0}\right) = Id + [K_{d\mu}, u_d]G_{d\mu} + [K_{d\mu}, v_d]G_{d0}.$$

This relation implies that the resolvent $G_{d\mu}(\zeta) = R(\zeta; K_{d\mu})$ is represented as

$$G_{d\mu} = \left(u_d G_{d\mu} + v_d G_{d0}\right)\left(Id + [K_{d\mu}, u_d]G_{d\mu} + [K_{d\mu}, v_d]G_{d0}\right)^{-1} \tag{5.14}$$

and the inverse $(Id + Y_d)^{-1}$ takes the form

$$(Id + Y_d)^{-1} = e^{ig_\alpha}\left(Id + [K_{d\mu}, u_d]G_{d\mu} + [K_{d\mu}, v_d]G_{d0}\right)^{-1} e^{-ig_\alpha}. \tag{5.15}$$

Thus we have the relation

$$Id + X_d = Id + Y_d + Z_d = \left(Id + Z_d\,(Id + Y_d)^{-1}\right)(Id + Y_d).$$

We now recall the representation for $Z_d(\zeta)$ and calculate the operator

$$Z_d\,(Id + Y_d)^{-1} = [H_d, v_d]R_{d\alpha}[w_d, H_{d0}]R_{d0}\,(Id + Y_d)^{-1}$$

acting on $L^2_{\mathrm{com}}(\Omega_d)$. We see by (5.13) and (5.15) that

$$R_{d0}(Id + Y_d)^{-1} = e^{ig_\alpha}G_{d0}\left(Id + [K_{d\mu}, u_d]G_{d\mu} + [K_{d\mu}, v_d]G_{d0}\right)^{-1} e^{-ig_\alpha}.$$

If we write

$$\exp(ig_\alpha)G_{d0} = \exp(ig_\alpha)\,(u_d + v_d)\,G_{d0},$$

then $[w_d, H_{d0}]v_d = [w_d, H_{d0}]$ and $[w_d, H_{d0}]u_d = 0$, and hence it follows from (5.14) that

$$[w_d, H_{d0}]R_{d0}\,(Id + Y_d)^{-1} = [w_d, H_{d0}]\exp(ig_\alpha)G_{d\mu}\exp(-ig_\alpha).$$

Thus we have

$$Z_d\,(Id + Y_d)^{-1} = [H_d, v_d]R_{d\alpha}[w_d, H_{d0}]\exp(ig_\alpha)G_{d\mu}\exp(-ig_\alpha). \tag{5.16}$$

We calculate the two commutators in the above representation. The operator H_d coincides with

$$H_d = H(A_\alpha + A_{d\mu}) = H(A_\alpha + \nabla g_{d\mu}) = \exp(ig_{d\mu})K_\alpha\exp(-ig_{d\mu})$$

on the support of v_d, and also K_α equals

$$K_\alpha = H(A_\alpha) = H(\nabla g_\alpha) = \exp(ig_\alpha)H_0\exp(-ig_\alpha)$$

on the support of u_d. Thus we have

$$\begin{aligned}[H_d, v_d] &= \exp(ig_{d\mu})[K_\alpha, v_d]\exp(-ig_{d\mu}) \\ &= \exp(ig_{d\mu})[u_d, K_\alpha]\exp(-ig_{d\mu}) = \exp(ig_d)[u_d, H_0]\exp(-ig_d),\end{aligned}$$

where g_d is defined in (5.12). The other commutator $[w_d, H_{d0}]$ equals

$$[w_d, H_{d0}] = [w_d, H(\nabla g_d)] = \exp(ig_d)[w_d, H_0]\exp(-ig_d).$$

We insert the expression (5.10) for $R_{d\alpha}(\zeta)$ into (5.16) to obtain

$$Id + Z_d\,(Id + Y_d)^{-1} = \exp(ig_d)\,(Id + T_d)\exp(-ig_d),$$

where

$$T_d(\zeta) = [u_d, H_0]\exp(-ig_\alpha)G_\alpha\exp(ig_\alpha)[w_d, H_0]\exp(-ig_{d\mu})G_{d\mu}\exp(ig_{d\mu}).$$

Summing up the result obtained above, we can get the following proposition.

Proposition 5.1. Let $\zeta = E - i\eta$ with $0 \le \eta < \eta_0\,(\log|d|)\,/|d|$ and let

$$T_d(\zeta) : L^2_{\mathrm{com}}(\Omega_d) \to L^2_{\mathrm{com}}(\Omega_d), \quad \Omega_d = \{|x-d| < 4\},$$

be as above. If $Id + T_d(\zeta)$ has a bounded inverse, then ζ is not a resonance of H_d.

6. Resonance widths in scattering by two separated fields

In this section we prove Theorem 1.2 after reformulating it under the setting in the previous section.

Theorem 6.1. Let H_d, K_α and K_μ be as in (5.3) and (5.4), and let $f_\alpha(\omega \to \theta; E)$ and $f_\mu(\omega \to \theta; E)$ denote the scattering amplitude at energy $E > 0$ for K_α and K_μ, respectively. Assume that the flux μ of K_μ is an integer. Then, for any $\varepsilon > 0$ small enough, there exists $d_\varepsilon(E) \gg 1$ such that $\zeta = E - i\eta$ with

$$0 < \eta < \frac{E^{1/2}}{|d|}\left\{\log|d| - \log\left|f_\alpha(-\hat d \to \hat d; E)f_\mu(\hat d \to -\hat d; E)\right| - \varepsilon\right\}$$

is not a resonance of H_d for $d > d_\varepsilon(E)$.

Proof. We assume throughout the proof that $\zeta = E - i\eta$ is taken as in the theorem for $\varepsilon > 0$ fixed arbitrarily. Then it follows that

$$\left|\frac{e^{i2k|d|}}{|d|}\,f_\alpha(-\hat d \to \hat d; E)f_\mu(\hat d \to -\hat d; E)\right| < 1 \tag{6.1}$$

strictly for $\operatorname{Im} k = \operatorname{Im}\zeta^{1/2} < 0$. By Proposition 5.1, it suffices to show that

$$Id + T_d(\zeta) : L^2_{\mathrm{com}}(\Omega_d) \to L^2_{\mathrm{com}}(\Omega_d), \quad \Omega_d = \{|x-d| < 4\},$$

has a bounded inverse. The proof is done by dividing it into four steps.

(1) We begin by recalling that $T_d(\zeta)$ is defined by

$$T_d(\zeta) = [u_d, H_0]\exp(-ig_\alpha)G_\alpha \exp(ig_\alpha)[w_d, H_0]\exp(-ig_{d\mu})G_{d\mu}\exp(ig_{d\mu}),$$

where $u_d(x) = \chi_0(|x-d|/2)$. We analyze the behavior as $|d| \to \infty$ of the integral kernel $T_d(x, y; \zeta)$ of $T_d(\zeta)$ for $(x, y) \in \Omega_d \times \Omega_d$. We denote by $G_\alpha(x, y; \zeta)$, $G_\mu(x, y; \zeta)$ and $G_{d\mu}(x, y; \zeta)$ the integral kernels of the resolvents $G_\alpha(\zeta)$, $G_\mu(\zeta)$ and $G_{d\mu}(\zeta)$ for the self–adjoint operators defined in (5.4), respectively. Since $w_d(x) = \chi_0(4|x-d|/|d|)$, ∇w_d has support in

$$Q_d = \{|d|/4 < |x-d| < |d|/2\}.$$

Thus $T_d(x, y; \zeta)$ takes the form

$$T_d(x, y; \zeta) = [u_d, H_0]\{\exp(-ig_\alpha(x))F_d(x, y; \zeta)\}\exp(ig_{d\mu}(y)),$$

where

$$F_d = \int_{Q_d} G_\alpha(x, q; \zeta)e^{ig_\alpha(q)}[w_d, H_0]\left\{e^{-ig_{d\mu}(q)}G_{d\mu}(q, y; \zeta)\right\} dq.$$

We only consider the leading term in the asymptotic behavior of $T_d(x, y; \zeta)$ without referring to any contribution from remainder terms, which is seen to be at most of order $O\left((\log|d|)^2/|d|\right)$. If $x \in \Omega_d$ and $q \in Q_d$, then it follows from Theorem 4.1 (i) with $\lambda = |d|$ that $G_\alpha(x, q; \zeta)$ behaves like

$$\begin{aligned} G_\alpha \sim\ & c(E)e^{i\alpha(\gamma(\hat{x};-\hat{q})-\pi)}|x-q|^{-1/2}e^{ik|x-q|} \\ & + c(E)\left(|x||q|\right)^{-1/2}e^{ik(|x|+|q|)}f_\alpha(-\hat{q} \to \hat{x}; E) \end{aligned}$$

Since $G_{d\mu}(q, y; \zeta) = G_\mu(q-d, y-d; \zeta)$, we also have

$$G_{d\mu} \sim c(E)e^{ik|q-d|}|q-d|^{-1/2}\overline{\varphi}_{-\mu}(y-d; \hat{q}_d, E), \quad q \in Q_d, \quad y \in \Omega_d,$$

by Theorem 4.1 (ii), where $\hat{q}_d = \widehat{q-d} = (q-d)/|q-d|$ and $\varphi_{-\mu}$ is the incoming eigenfunction of $K_\mu = H(A_\mu)$. Thus $F_d(x, y; z)$ behaves like

$$F_d(x, y; \zeta) \sim F_{d0}(x, y; \zeta) + F_{d1}(x, y; \zeta), \quad |d| \to \infty,$$

where

$$F_{d0} = c(E)^2 \int_{Q_d} e^{ik|x-q|}e^{ik|q-d|}e^{i\alpha\gamma(\hat{x};-\hat{q})}|q-x|^{-1/2}h_d(q, y)\, dq$$

$$F_{d1} = c(E)^2|x|^{-1/2}e^{ik|x|}\int_{Q_d} e^{ik|q|}e^{ik|q-d|}|q|^{-1/2}f_\alpha(-\hat{q} \to \hat{x}; E)h_d(q, y)\, dq,$$

while $h_d(q, y)$ is defined by

$$h_d(q, y) = e^{ig_\alpha(q)}e^{-ik|q-d|}[w_d, H_0]e^{ik|q-d|}\tau_d(q, y) \tag{6.2}$$

with

$$\tau_d(q, y) = e^{-ig_{d\mu}(q)}|q-d|^{-1/2}\overline{\varphi}_{-\mu}(y-d; \hat{q}_d, E).$$

We assert that F_{d0} obeys

$$F_{d0}(x,y;\zeta) = O(|d|^{-N}) \tag{6.3}$$

for any $N \gg 1$ and that F_{d1} behaves like

$$F_{d1}(x,y;\zeta) \sim \sigma_d \varphi_0(x-d;-\hat{d},E)\overline{\varphi}_{-\mu}(y-d;-\hat{d};E) \tag{6.4}$$

uniformly in ζ and in $x,\ y \in Q_d$, where

$$\sigma_d = \frac{e^{2ik|d|}}{|d|}\, c(E) f_\alpha(-\hat{d} \to \hat{d};E) e^{i\alpha\pi} e^{-i\mu\gamma(-\hat{d})} \tag{6.5}$$

is bounded uniformly in $|d| \gg 1$.

(2) To prove (6.3), we consider the oscillatory integral

$$I_{d0}(x,y;\zeta) = \int_{Q_d} e^{ik|q-x|} e^{ik|q-d|} e^{i\alpha\gamma(\hat{x};-\hat{q})} |q-x|^{-1/2} h_d(q,y)\, dq$$

and show that

$$I_{d0}(x,y;\zeta) = \left(|d|^{-N}\right), \tag{6.6}$$

which implies (6.3) immediately. Since $w_d(q) = \chi_0(4|q-d|/|d|)$, the commutator $[w_d, H_0]$ takes the form

$$[w_d, H_0] = 2\nabla w_d \cdot \nabla + O(|d|^{-2}) \sim 8|d|^{-1}\chi_0'(4|q-d|/|d|)\,(\hat{q}_d \cdot \nabla)\,.$$

Hence $h_d(q,y)$ fulfills $\partial_q^m h_d(q,y) = O\left(|d|^{-1-|m|}\right)$ and behaves like

$$h_d(q,y) \sim 8iE^{1/2}|d|^{-1}\chi_0'(4|q-d|/|d|) e^{ig_\alpha(q)} \tau_d(q,y) \tag{6.7}$$

uniformly in $y \in \Omega_d$. Since $|q-x| = O(|d|)$ and $|q-d| = O(|d|)$, we also have that $\left|e^{ik|q-x|}e^{ik|q-d|}\right| = O\left(|d|^L\right)$ for some $L > 1$. Note that

$$\left|\nabla\Big(|q-x| + |q-d|\Big)\right| > c > 0$$

for $x \in \Omega_d$. Thus (6.6) follows by repeated use of partial integration.

(3) The main body of the proof is occupied by showing (6.4). We analyze the behavior of the oscillatory integral

$$I_{d1}(x,y;\zeta) = \int_{Q_d} e^{ik|q|} e^{ik|q-d|} |q|^{-1/2} f_\alpha(-\hat{q} \to \hat{x};E) h_d(q,y)\, dq.$$

To do this, we work in the polar coordinate system (r,θ) with point d at center, where $r = |q-d|$ and $\theta = \gamma(q-d;-\hat{d})$ denotes the azimuth angle from $-\hat{d}$ to $\hat{q}_d$. Then

$$|q| = \left(|d|^2 + r^2 - 2|d|r\cos\theta\right)^{1/2} = |d|\left(1+\rho^2 - 2\rho\cos\theta\right)^{1/2}$$

by making the change of variable $r = |d|\rho$, and the integral takes the form

$$I_{d1} = |d|^2 \int_0^\infty \left\{ \int_0^{2\pi} e^{i|d|\psi(\rho,\theta)} e^{ia(\rho,\theta)} g(\rho,\theta)\, d\theta \right\} \rho d\rho,$$

where $\psi(\rho,\theta) = E^{1/2}\left(\left(1+\rho^2 - 2\rho\cos\theta\right)^{1/2} + \rho \right)$ and

$$a(\rho,\theta) = i(k - E^{1/2})|d| \left((1+\rho^2 - 2\rho\cos\theta)^{1/2} + \rho \right)$$

$$g(\rho,\theta) = |q|^{-1/2} f_\alpha(-\hat{q} \to \hat{x}; E) h_d(q,y) \tag{6.8}$$

with $q = (q-d)+d = |d|\rho\hat{q}_d + d$. Since $q \in Q_d$, ρ ranges over the interval $1/4 < \rho < 1/2$, and $a(\rho,\theta)$ obeys $|a(\rho,\theta)| = O(\log|d|)$. We apply the stationary phase method to the integral with respect to θ in the brackets. There are two stationary points $\theta = 0$ and $\theta = \pi$. If $\theta = \pi$, then

$$\psi(\rho,\pi) = E^{1/2}(2\rho+1), \qquad (\partial/\partial\rho)\,\psi(\rho,\theta) = 2E^{1/2} > 0,$$

and hence we see by repeated use of partial integral in variable ρ that the contribution from the stationary point $\theta = \pi$ is negligible. Thus we have only to consider the case when the stationary point is attained at $\theta = 0$. At that point, ψ satisfies

$$e^{i|d|\psi(\rho,0)} e^{a(\rho,0)} = e^{ik|d|}, \quad \psi''(\rho,0) = (\partial/\partial\theta)^2\psi(\rho,0) = E^{1/2}\left(\rho/(1-\rho)\right).$$

The second relation implies

$$\left(|d|\psi''(\rho,0)/2\pi i\right)^{-1/2} = (2\pi)^{1/2}\exp(i\pi/4)E^{-1/4}\left((1-\rho)/\rho\right)^{1/2}|d|^{-1/2}.$$

We look at the value at $\theta = 0$ of $g(\rho,\theta)$ defined by (6.8). We see that $\hat{q} = \hat{d}$ and $\hat{q}_d = \widehat{q-d} = -\hat{d}$ at $\theta = 0$, so that we have $\exp(ig_\alpha(q)) = \exp(i\alpha\pi)$ and

$$\exp(-ig_{d\mu}(q)) = \exp(-i\mu\gamma(q-d)) = \exp(-i\mu\gamma(-d))$$

$$f_\alpha(-\hat{q} \to \hat{x}; E) = f_\alpha(-\hat{d} \to \hat{x}; E), \quad \varphi_{-\mu}(y-d; \hat{q}_d, E) = \varphi_{-\mu}(y-d; -\hat{d}, E).$$

We also have

$$|q|^{-1/2} = |d|^{-1/2}(1-\rho)^{-1/2}, \quad |q-d|^{-1/2} = |d|^{-1/2}\rho^{-1/2}$$

at $\theta = 0$. We further note that

$$4\chi_0'(4|q-d|/|d|) = 4\chi_0'(4\rho) = (d/d\rho)\,\chi_0(4\rho)$$

and hence $\int_0^\infty 4\chi_0'(4\rho)\,d\rho = -1 = e^{-i\pi}$. Thus it follows from (6.7) that the leading term of the integral $I_{d1}(x,y;\zeta)$ under consideration takes the form

$$\frac{e^{ik|d|}}{|d|^{1/2}}\,(8\pi)^{1/2}\, e^{-i\pi/4} E^{1/4} e^{i\alpha\pi} e^{-i\mu\gamma(-d)} f_\alpha(-\hat{d} \to \hat{x}; E)\overline{\varphi}_{-\mu}(y-d; -\hat{d}, E),$$

which, together with (1.6), implies that $I_{d1}(x, y; \zeta)$ behaves like

$$I_{d1} \sim \frac{e^{ik|d|}}{|d|^{1/2}} \frac{f_\alpha(-\hat{d} \to \hat{x}; E)}{c(E)} e^{i\alpha\pi} e^{-i\mu\gamma(-\hat{d})} \overline{\varphi}_{-\mu}(y - d; -\hat{d}, E).$$

If $x \in Q_d$, then

$$\hat{x} = \hat{d} + O(|d|^{-1}), \qquad |x|^{-1/2} = |d|^{-1/2} \left(1 + O(|d|^{-1})\right)$$

and

$$\begin{aligned} e^{ik|x|} &= e^{ik|d|} e^{ik(x-d)\cdot\hat{d}} \left(1 + O(|d|^{-1})\right) \\ &= e^{ik|d|} \varphi_0(x - d; \hat{d}, E) \left(1 + O\left((\log|d|)/|d|\right)\right). \end{aligned}$$

Hence (6.4) follows at once.

(4) The proof is completed in this step. We combine the results obtained in steps (2) and (3) to see that $T_d(x, y; \zeta)$ takes the asymptotic form

$$T_d(x, y; \zeta) = T_{d0}(x, y; \zeta) + O(|e^{i2k|d|}|/|d|) O((\log|d|)^2/|d|) + O(|d|^{-N})$$

as $|d| \to \infty$, where

$$\begin{aligned} T_{d0}(x, y; \zeta) = & \frac{e^{i2k|d|}}{|d|} f_\alpha(-\hat{d} \to \hat{d}; E) c(E) e^{-i\mu\gamma(-\hat{d})} \times \\ & \times \left([u_d, H_0]\varphi_0(x - d; \hat{d}, E)\right) \times \left(e^{i\mu\gamma(y-d)} \overline{\varphi}_{-\mu}(y - d; -\hat{d}; E)\right). \end{aligned}$$

If we write $u_d(x) = u_0(x - d)$ with $u_0(x) = \chi_0(|x|/2)$, then the integral

$$\int \left([u_d, H_0]\varphi_0(x - d; \hat{d}, E)\right) e^{i\mu\gamma(x-d)} \overline{\varphi}_{-\mu}(x - d; -\hat{d}; E)\, dx$$

coincides with the L^2 scalar product

$$\left([u_0, H_0]\varphi_0(\cdot; \hat{d}, E), e^{-i\mu\gamma(\cdot)} \varphi_{-\mu}(\cdot; -\hat{d}; E)\right)$$

after making change of variable $x - d \to x$. We now apply Lemma 4.3. Then the integral operator $T_{d0}(\zeta) : L^2_{\mathrm{com}}(\Omega_d) \to L^2_{\mathrm{com}}(\Omega_d)$ with the kernel $T_{d0}(x, y; \zeta)$ above is an operator of rank one and has

$$-\frac{e^{i2k|d|}}{|d|} f_\alpha(-\hat{d} \to \hat{d}; E) f_\mu(\hat{d} \to -\hat{d}; E)$$

as a nontrivial eigenvalue. Hence we see from (6.1) that $Id + T_d(\zeta)$ has a bounded inverse. Thus the proof is now complete. □

7. Semiclassical theory in scattering by two solenoidal fields

In this last section, we discuss the relation to the semiclassical theory for quantum resonances in scattering by two solenoidal fields. Let H_h, $0 < h \ll 1$, be the operator defined by (1.10), which is self–adjoint in $L^2 = L^2(\boldsymbol{R}^2)$

under the boundary conditions $\lim_{|x-p_j|\to 0}|u(x)| < \infty$ at centers p_1 and p_2 of two solenoidal fields $2\pi\alpha_1\delta(x-p_1)$ and $2\pi\alpha_2\delta(x-p_2)$. We define the two unitary operators

$$(U_1 f)(x) = h^{-1} f(h^{-1}x), \qquad (U_2 f)(x) = \exp(ig_h(x)) f(x)$$

acting on L^2, where

$$g_h(x) = [\alpha_1/h]\gamma(x-d_1) + [\alpha_2/h]\gamma(x-d_2), \quad d_j = p_j/h.$$

The function $g_h(x)$ satisfies

$$\nabla g_h = [\alpha_1/h]\Phi(x-d_1) + [\alpha_2/h]\Phi(x-d_2)$$

by (2.6), and $\exp(ig_h(x))$ is well defined as a single valued function. Hence H_h is unitarily transformed to the operator $\tilde{H}_d = (U_1U_2)^* H_h (U_1U_2)$, which takes the form $\tilde{H}_d = H(\Psi_d)$ defined by (1.11), where $\beta_j = \alpha_j/h - [\alpha_j/h]$. Thus the semiclassical resonance theory in scattering by two solenoidal fields is reduced to the resonance problem for the magnetic Schrödinger operator with two solenoidal fields with centers at large separation. If, in particular, at least one of α_1/h and α_2/h is an integer, then $\tilde{H}_d$ has a single solenoidal field, and hence H_h has no resonance. The interesting case is when β_1 and β_2 satisfy $0 < \beta_1, \ \beta_2 < 1$ strictly.

We now assume that neither α_1/h nor α_2/h is an integer. By (1.5), the backward scattering amplitude by the field $2\pi\beta_j\delta(x)$ at energy $E > 0$ is calculated as

$$f_j(\omega \to -\omega) = -(2\pi)^{-1/2} e^{i\pi/4}(-1)^{[\alpha_j/h]} \sin(\alpha_j\pi/h) E^{-1/4}$$

for $j = 1, \ 2$, because $[\beta_j] = 0$ for $\beta_j = \alpha_j/h - [\alpha_j/h]$ and

$$e^{i(\theta-\omega)} = e^{i\pi} = -1, \quad e^{i(\theta-\omega)}/\left(1 - e^{i(\theta-\omega)}\right) = -1/2$$

at $\theta = -\omega$. Thus the product of two backward amplitudes $f_1(-\hat{d} \to \hat{d})$ and $f_2(\hat{d} \to -\hat{d})$ takes the explicit form

$$f_1 \times f_2 = (2\pi)^{-1} i(-1)^{[\alpha_1/h]+[\alpha_2/h]} \sin(\alpha_1\pi/h)\sin(\alpha_2\pi/h) E^{-1/2}$$

for $\hat{d} = d/|d|$ with $d = d_2 - d_1$. Then Theorem 1.2 with $d = p/h$, $p = p_2 - p_1$, suggests that : For any $\varepsilon > 0$ small enough, there exists $h_\varepsilon(E) \ll 1$ such that $\zeta = E - i\eta$ with

$$0 < \eta < \frac{E^{1/2}h}{|p|}\left\{\log\frac{|p|}{h} - \log\frac{|\sin(\alpha_1\pi/h)\sin(\alpha_2\pi/h)|}{2\pi E^{1/2}} - \varepsilon\right\}. \tag{7.1}$$

is not a resonance of H_h for $h < h_\varepsilon(E)$. The argument used in the proof suggests that the location of resonance ζ in the unphysical sheet is approximately determined by the relation (see (1.9))

$$\frac{i}{2\pi}(-1)^{[\alpha_1/h]+[\alpha_2/h]}\sin(\alpha_1\pi/h)\sin(\alpha_2\pi/h)E^{-1/2}h\,\frac{e^{2ik|p|/h}}{|p|} = 1, \quad k = \zeta^{1/2}.$$

This makes a complement to the general result due to Martinez [18], which asserts that for any $M \gg 1$, there exists $h_M(E)$ such that $\zeta = E - i\eta$ with $\eta < -Mh\log h$ is not a resonance for $0 < h < h_M(E)$, provided that E is in the nontrapping energy range.

We end the paper by considering what is difficult in proving the semiclassical bound (7.1) on resonance widths in the general case. To see this, we return to the proof of Theorem 6.1. The essence of the proof lies in the fact that the coefficients of the difference between the original operator $H_d = H(A_\alpha + A_{d\mu})$ and the approximate operator $H_{d\alpha} = H(A_\alpha + \nabla g_{d\mu})$ with $g_{d\mu} = \mu\gamma(x - d)$ have compact support. If μ is not an integer, this is not the case. In fact, the gauge transformation does not work, because $\exp(i\mu\gamma(x-d))$ is not a single valued function. We now introduce a function $\tilde{g}_\mu \in C^\infty(\boldsymbol{R}^2 \to \boldsymbol{R})$ in such a way that

$$\tilde{g}_\mu(x) = \mu\gamma(x;\hat{d}) \qquad \text{on} \quad \{|x| > 1\} \cap \left\{|\hat{x} - \hat{d}| > \delta\right\}$$

for $\delta > 0$ small enough, and we set $\tilde{g}_{d\mu} = \tilde{g}_\mu(x - d)$. If, for example, we take $\tilde{H}_{d\alpha} = H(A_\alpha + \nabla\tilde{g}_{d\mu})$ as an approximate operator, then the difference between H_d and $\tilde{H}_{d\alpha}$ has support in a conic neighborhood of direction $\hat{d}$ with point d at center, so that the support extends at infinity. Since the Green function of resolvent $G_\alpha(\zeta) = R(\zeta; K_\alpha)$, $K_\alpha = H(A_\alpha)$, with spectral parameters in the unphysical sheet grows exponentially at infinity, we can not control the operator

$$\left(H_d - \tilde{H}_{d\alpha}\right)R(\zeta;\tilde{H}_{d\alpha}) = \left(H_d - \tilde{H}_{d\alpha}\right)\exp(i\tilde{g}_{d\mu})G_\alpha(\zeta)\exp(-i\tilde{g}_{d\mu}).$$

This is an essential difficulty in the case that μ is not an integer. However, this difficulty seems to be overcome by the complex scaling method initiated by [3, 5] and further developed by [23] (see [12] also). We intend to discuss the matter in details elsewhere. As stated in section 1, our purpose here is to make an elementary study of resonances without such a method.

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(*Received August 23, 2009*)
(*Revised November 6, 2009*)