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A GENERALIZATION OF THE DADE'S THEOREM ON LOCALIZATION OF INJECTIVE MODULES

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In general the localization does not preserve the injectivity. E. C. Dade gave a necessary and sufficient condition which assures the preservation of the injectivity for commutative rings in [1] Theorem 13. Recently one of the authors tried to generalize it to non commutative rings in [3]. But it does not seem to be enough sufficient. So we retried to solve this problem. We refer to [2] on the terminologies and notations mainly.

Let A be a ring not necessary commutative, $\mathcal{F}$ a class of right ideals of A which defines a hereditary torsion theory on the category of right A-modules. First, we refer to [2] Chapter IX, Proposition 2.7, namely,

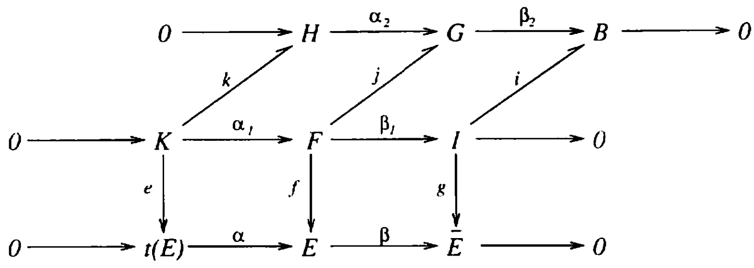
Let M be a torsion free A-module. Then the localization $M_{\mathcal{F}}$ of M is $A_{\mathcal{F}}$-injective if and only if the following holds:

For any right ideal I of A and any homomorphism $g: I \to M$ there exists $B \in \mathcal{F}$ containing I and a homomorphism $p: B \to M$ such that $p|_I = g$.

We start with the following diagram:

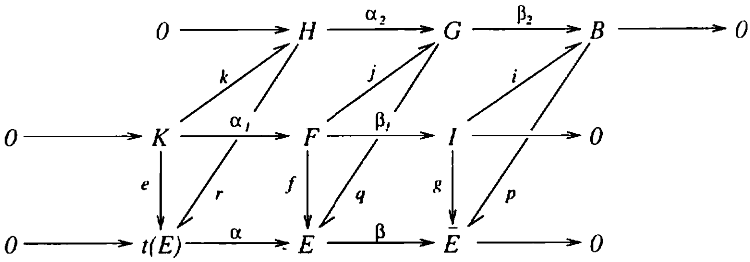
$$\begin{array}{ccccccc} & & & & & B & \\ & & & & & i \nearrow & \\ & & & & I & & \\ & & & & {\scriptstyle g}\downarrow & & \\ 0 \longrightarrow & t(E) & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & \bar{E} & \longrightarrow 0 \end{array}$$

where E is an injective module, $t(E)$ the $\mathcal{F}$-torsion submodule of E, $\bar{E} = E/t(E)$ and I and B are right ideals of A. From this we can construct the following row exact commutative diagram:



where F and G are projective modules. Furthermore if there was given a homomorphism $p : B \to \bar{E}$ then we obtain the following:

(1)

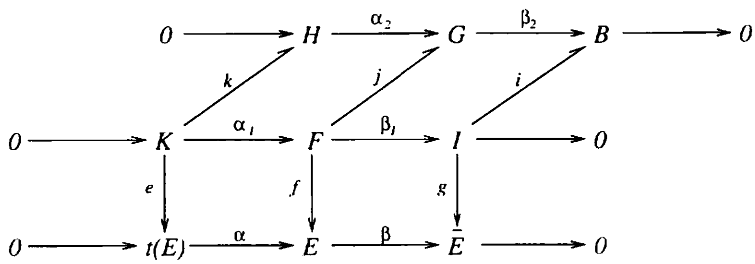


in which the rectangular parts are commutative.

Lemma 1. *In the above diagram if $g = pi$ holds then there exists a homomorphism $u : F \to t(E)$ such that $e = rk + u\alpha_1$.*

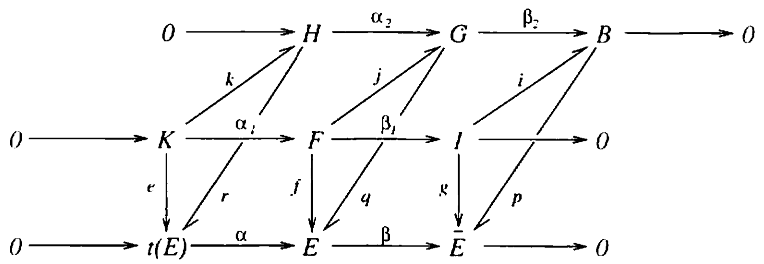
As the proof is an easy exercise we shall omit it.

Next we start from the following row exact commutative diagram:



where E is injective and i is a monomorphism. Furthermore if there was given a homomorphism $r : H \to t(E)$, then we obtain the following diagram:

(2)



Lemma 2. *In the above diagram, if there exists a map $u : F \to t(E)$ such that $e = rk + u\alpha_1$, then there exists a homomorphism $p_0 : B \to \bar{E}$ such that $g = p_0 i$.*

Proof. By the commutativity of the diagram there holds

$$0 = \alpha(e - rk - u\alpha_1) = f\alpha_1 - q\alpha_2 k - \alpha u\alpha_1 = (f - qj - \alpha u)\alpha_1.$$

Therefore $f - qj - \alpha u$ induces a homomorphism $v : I \to E$ such that $f - qj - \alpha u = v\beta_1$ holds. Then there holds

$$0 = \beta(f - qj - \alpha u - v\beta_1) = g\beta_1 - p\beta_2 j - \beta v\beta_1 = (g - pi - \beta v)\beta_1.$$

As β_1 is an epimorphism there holds $g = pi + \beta v$. Last, since i is a monomorphism and E is injective v is extended to a homomorphism $v_1 : B \to E$ such that $v_1 i = v$. Set $p_0 = p + \beta v_1$ then there holds $g = p_0 i$. □

Now we are in a position to treat the main theorem. We remark that if $e = rk + u\alpha_1$ holds then by setting $M = \ker u$ and $N = \ker r$ there holds $k^{-1}(N) \cap \alpha_1^{-1}(M) \subseteq \ker e$.

For a right ideal I we fix a presentation of I:

(3) $$0 \longrightarrow K \longrightarrow F \longrightarrow I \longrightarrow 0$$

where F is a projective module. Consider the following condition for (3).

(4) Let L be a submodule of K such that K/L is a $\mathcal{F}$-torsion module. Then there exists B in $\mathcal{F}$ containing I and if we construct the row exact

commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H & \xrightarrow{\alpha_2} & G & \xrightarrow{\beta_2} & B & \longrightarrow & 0 \\ & & {\scriptstyle k}\uparrow & & {\scriptstyle j}\uparrow & & {\scriptstyle i}\uparrow & & \\ 0 & \longrightarrow & K & \xrightarrow{\alpha_1} & F & \xrightarrow{\beta_1} & I & \longrightarrow & 0 \end{array}$$

where G is projective and i is the inclusion, there exist submodules M and N of F and H respectively such that

(4a) F/M and H/N are $\mathcal{F}$-torsion modules,

(4b) $k^{-1}(N) \cap \alpha_1^{-1}(M) \subseteq L$.

THEOREM. *Let A be a ring, $\mathcal{F}$ a class of right ideals of A which defines a hereditary torsion theory on the category of right A-modules. Then the localization $E_{\mathcal{F}}$ of any injective module E is an injective $A_{\mathcal{F}}$-module if and only if each right ideal I of A has a presentation (3) satisfying (4). In that case any presentation (3) of any such right ideal satisfies (4).*

Proof. Assume that for any injective module E the localization $E_{\mathcal{F}}$ is $A_{\mathcal{F}}$-injective. In the presentation (3) of I, let L be a submodule of K such that K/L is a torsion module. Now take an injective module E containing K/L, then K/L is in $t(E)$ and letting e be the composition map $K \to K/L \hookrightarrow t(E)$, we can construct the following row exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\alpha_1} & F & \xrightarrow{\beta_1} & I & \longrightarrow & 0 \\ & & {\scriptstyle e}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \\ 0 & \longrightarrow & t(E) & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & \bar{E} & \longrightarrow & 0. \end{array}$$

By [2] Chapter IX, Proposition 2.7 there exists $B \in \mathcal{F}$ containing I and a homomorphism $p : B \to \bar{E}$ such that $g = pi$ where i is the inclusion $I \subseteq B$. From these we obtain the diagram (1). By Lemma 1, there exists a homomorphism $u : F \to t(E)$ and there holds $e = rk + u\alpha_1$. Let $M = \ker u$ and $N = \ker r$ then M and N satisfy (4a) and (4b).

Conversely, suppose that for any right ideal I of A the presentation (3) of I satisfies (4). Let E be an injective module and a homomorphism $g : I \to \bar{E}$ was given. Then we have the following row exact commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\alpha_1} & F & \xrightarrow{\beta_1} & I & \longrightarrow & 0 \\ & & {\scriptstyle e}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \\ 0 & \longrightarrow & t(E) & \xrightarrow{\alpha} & E & \xrightarrow{\beta} & \bar{E} & \longrightarrow & 0. \end{array}$$

Let $L = \ker e$, then K/L is a torsion module. By (4) there exists $B \in \mathcal{F}$ containing I and there are submodules M and N of F and H respectively satisfying (4a) and (4b). From these we have the following diagram:

$$\begin{array}{ccccc} K/k^{-1}(N) \cap \alpha_1^{-1}(M) & \longrightarrow & K/k^{-1}(N) \oplus K/\alpha_1^{-1}(M) & \longrightarrow & H/N \oplus F/M \\ {\scriptstyle \bar{e}}\downarrow & & & & \\ K/L \subseteq t(E) & \xrightarrow{\alpha} & E & & \end{array}$$

where the composition of the upper row is a monomorphism and $\bar{e}$ is the natural map obtained from the condition (4b). As E is injective $\bar{e}$ is extended to $e_0 : H/N \oplus F/M \to E$ and its image is really in $t(E)$ since $H/N \oplus F/M$ is a $\mathcal{F}$-torsion module. The natural maps $H \to H/N$ and $F \to F/M$ composed with e_0 induce homomorphism $r : H \to t(E)$ and $u : F \to t(E)$. It is easily seen that there holds $e = rk + u\alpha_1$ and from these we can construct the diagram (2). By Lemma 2 there exists $p_0 : B \to \bar{E}$ such that $g = p_0 i$. Therefore $E_{\mathcal{F}}$ is an $A_{\mathcal{F}}$-injective module by [2] Chapter IX, Proposition 2.7. This completes the proof. □

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