Okayama Economic Review 12 no. 3 (1980) 77–95

A Characterization of the Price-Mediated Exchange Equilibrium

Masayoshi Maruyama*

Okayama University

INTRODUCTION

Until now we have two main approaches to the analysis of market adjustment process, identified here as Walrasian tâtonnement process and Edgeworthian recontracting process, which depend on whether the theory focuses on interdependent optimization for the former, or exchange mechanism for the latter. In the former the market adjustment is supposed to be left to a Walrasian auctioneer, i. e., the impersonal force of the market separated from the behaviour of active market participants. Furthermore the actual process of delivery and the redistribution of the endowment are not dealt there. In other words, the model pretends to picture a decentralized economic system, however, it presupposes a completely centralized economic system because it allows for no communication link between active market participants besides via a unique market authority. From the viewpoint which purports to explore the monetary exchange Ostroy [7] and Starr [10] suggest the necessity of explicit analysis of exchange process with the supposition that exchange is a do-it-yourself affair for market participants. In the Edgeworthian recontracting model, e.g., Debreu and Scarf [3] the role of price which plays in the exchange process has not been fully treated even if the process was described on the base of active market participant's behaviour. A recent series of papers of game-theoretic treatment of market exchange, e.g., Shapley [8] and Shapley and Shubik [9] suggest the significance of considering the effect of agent's bidding behaviour upon the market price formation in order to complement the theoretical lacuna: the process or the rules by which prices are determined may be considered to be not explicit.

In this article we follow this stream of thinking mode and formulate the exchange process with explicitly considering the role of price in the process of exchange. At first we pose a natural trading rule in the price-mediated exchange economy and define some equilibrium concept. Then we examine the relation of this newly defined equilibrium, Walras equilibrium and Pareto optimality. Next we explore the close relation between this equilibrium concept and the non-Walrasian equilibrium (fixprice equilibrium). The recent paper of Grandmont, Laroque and Younes [4] characterizes the fixprice equilibrium as a stable solution concept from the game-theoretic point of view. Our present analysis may also be viewed as giving another interpretation for the meaning of disequilibrium transaction from the viewpoint of price-mediated exchange game.

Finally we give a heuristic proof of the existence of newly defined equilibria. We employ the work of Laroque [5] as a lemma for this proof. In his paper Laroque suggests the way for the analysis of disequilibrium dynamics by considering the process generated from the outcome of game. Our present approach may accord with this line of analysis.

Let $\mathscr{E} = \{(Z_i, U_i, \omega_i)_{i \in I}, J\}$ denote an exchange economy with a finite set of consumers $I = \{i \mid i = 1, 2, \ldots, n\}$ and a set of commodities $J = \{j \mid j = 1, 2, \ldots, m\}$. $Z_i \subset R^m$ denotes consumer i's feasible set of net trade (excess demand) vector, U_i denotes his utility function, and ω_i denotes his initial endowment of commodities. Concerning to the characteristics of each agent we assume the following:

Assumption 1 : Z_i is closed, convex and has a lower bound for $\geqq$ and 0 belongs to Z_i,

Assumption 2: initial endowment vector ω_i is positive, and

Assumption 3: utility function is strictly quasi-concave, monotone increasing and continuous.

I. *Propositions and Proofs*

We will define the solution concept of the price-mediated exchange economy. In order to do so, at first we will define the concept of *p-blocking* in the following manner.

Definition 1: We will say that a coalition C can *p-block* the allocation $(x_i)_{i \in I}$ from the initial allocation $(\omega_i)_{i \in I}$ is there exists a pair of price vector p and the allocation $(p, (\bar{x}_i)_{i \in C})$, such that

[i] $\sum_{i \in C} \bar{x}_i = \sum_{i \in C} \omega_i$,

[ii] $(\forall i \in C) : (\sum_j p_j \bar{x}_{ij} = \sum_j p_j \omega_{ij})$

[iii] $(\forall i \in C) : (U_i(\bar{x}_i) \geqq U_i(x_i))$ and $(\exists i \in C) : (U_i(\bar{x}_i) > U_i(x_i))$, and

[iv] $(\forall i \in C) : (\nexists t \in [0, 1]) : (U_i(x_i(t)) > U_i(\bar{x}_i))$

, where x_i denotes agent i's consumption vector, x_{ij} and ω_{ij} denote his consumption and initial endowment of commodity j, respectively, and $x_i(t) = \omega_i + t(\bar{x}_i - \omega_i)$.

This definition differs from the usual one of cooperative market game in two points, condition [ii] and [iv] : in the price-mediated exchange economy any agent is naturally supposed to trade subject to his budget constraint and in the voluntary exchange economy it is natural to suppose that nobody can be forced to trade beyond he wishes. With this concept of p-blocking we define the following concepts.

Definition 2: We say that the allocation $(x_i)_{i\in I}$ is *p-Pareto optimal*, when it is feasible, i.e., $\sum_{i\in I} x_i = \sum_{i\in I}\omega_i$ and $x_i \in X_i = Z_i + \{\omega_i\}$ for any agent i and it is not *p-blocked* by any coalition which consists of the whole of agents. Furthermore if the allocation $(x_i)_{i\in I}$ is feasible and it is not *p-blocked* by any coalition we say that this allocation belongs to the *p-core*.

Morishima [6] gives a characterization of Walras equilibrium: the allocation $(x_i)_{i\in I}$ is a Walrasian exchange equilibrium one relative to the initial allocation $(\omega_i)_{i\in I}$ if and only if it is Pareto optimal in the usual sense and the conditions

[a] $(\exists\, p \geqq 0):(\forall\, i \in I):(\sum_j p_j x_{ij} = \sum_j p_i \omega_{ij})$, and

[b] $(\forall\, i \in I):(\nexists\, t \in [0, 1]):(U_i(x_i(t)) > U_i(x_i))$

are satisfied, where $x_i(t) = \omega_i + t(x_i - \omega_i)$. We are now considering the economy where any exchange is carried out price-mediatedly. So we will employ the following one as the solution concept.

Definition 3: We say the pair of price vector and the allocation $(p, (x_i)_{i \in I})$ a ***price-mediated exchange equilibrium*** relative to the initial allocation $(\omega_i)_{i \in I}$ when it is ***p-Pareto optimal*** and the above two conditions [a] and [b] are satisfied.

Remark: As the direct consequence of Morishima's result we can state that the pair of price vector and the allocation $(p, (x_i)_{i \in I})$ is a Walrasian equilibrium if and only if it is a price-mediated exchange equilibrium and the allocation $(x_i)_{i \in I}$ is Pareto optimal in the usual sense. As a corollary of this result we can say that any price-mediated exchange equilibrium allocation except for Walrasian exchange equilibrium one is not Pareto optimal in the usual sense. This result partially characterizes the concept of equilibrium in the price-mediated exchange.

We say that the price-mediated exchange equilibrium except for Walrasian equilibrium is a *non-trivial* price-mediated exchange equilibrium. In the following we confine our analysis to the non-trivial price-mediated exchange equilibrium. We will consider the exchange economy where there are two commodities. By examining the relation of the non-trivial price-mediated exchange equilibrium to the non-Walrasian equilibrium (fixprice equilibrium) we use a heuristic method of proof of the existence of non-trivial price-mediated exchange equilibrium instead of going to the direct way for it. We begin with defining the following concept.

Definition 4: We say that the allocation $(x_i)_{i \in I}$ is ***constrained Pareto optimal*** at a given price vector p with respect to the initial allocation $(\omega_i)_{i\in I}$ when the following conditions are satisfied:

[i] $\sum_{i\in I}x_i=\sum_{i\in I}\omega_i$,

[ii] $(\forall i\in I):(x_i\in X_i,\ \sum_j p_j x_{ij}=\sum_j p_j \omega_{ij})$,

[iii] there does not exist the allocation $(\bar{x}_i)_{i\in I}\in\gamma(p)$ such that $(\forall i\in I):(U_i(\bar{x}_i)\geqq U_i(x_i))$ and $(\exists i\in I):(U_i(\bar{x}_i)>U_i(x_i))$, where $\gamma(p)=\{(\bar{x}_i)_{i\in I}\mid\sum_{i\in I}\bar{x}_i=\sum_{i\in I}\omega_i$ and $(\forall i\in I):(\bar{x}_i\in X_i,\ \sum_j p_j\bar{x}_{ij}=\sum_j p_j\omega_{ij})\}$, and

[iv] $(\forall i\in I):(\nexists t\in[0,\ 1]):(U_i(x_i(t))>U_i(x_i))$.

It is trivial that any price-mediated exchange equilibrium is constrained Pareto optimal. Here we will have the following properties of the constrained Pareto optimal allocation.

Property 1 : The pair $(p,(x_i)_{i\in I})$ is supposed to be constrained Pareto optimal. Let $\tilde{z}_i(p)$ denotes agent i's Walrasian net trade vector at the price vector p. Then

$$(\forall i\in I):(\tilde{z}_{ij}(p)z_{ij}\geqq 0 \text{ and } |z_{ij}|\leqq|\tilde{z}_{ij}(p)|) \text{ and}$$
$$(\forall i\in I):(z_i=\alpha_i\tilde{z}_i(p))$$

, where $z_i=x_i-\omega_i$ and $0\leqq\alpha_i\leqq 1$.

This property can be easily verified by noting the condition [iv] of the constrained Pareto optimal and the strict quasi-concavity of utility function.

Property 2 : The pair $(p,(x_i)_{i\in I})$ is supposed to be constrained Pareto optimal. Let us define the set $\mathscr{J}=\{i\in I\mid z_i\neq\tilde{z}_i(p)\}$. Then

$$(\forall i,\ k\in\mathscr{J}):((\tilde{z}_{ij}(p)-z_{ij})(\tilde{z}_{kj}(p)-z_{kj})>0).$$

Proof : Suppose that $(\exists h,\ k\in\mathscr{J}):((\tilde{z}_{hj}(p)-z_{hj})(\tilde{z}_{kj}(p)-z_{kj})<0)$. Then there exist $\lambda_h,\ \lambda_k\in(0,\ 1)$ such that

$\lambda_h(\tilde{x}_{hj}(p) - x_{hj}) + \lambda_k(\tilde{x}_{kj}(p) - x_{kj}) = 0$, where $\tilde{x}_h(p) = \tilde{z}_h(p) + \omega_h$.
Here we define the new allocation $(\bar{x}_i)_{i \in I}$
such that

$$\bar{x}_h = x_h + \lambda_h(\tilde{x}_h(p) - x_h)$$
$$\bar{x}_k = x_k + \lambda_k(\tilde{x}_k(p) - x_k)$$
$$\bar{x}_i = x_i \quad \text{for any agent } i \neq h, k.$$

Then we can easily show that this new allocation $(\bar{x}_i)_{i \in I} \in \gamma(p)$. Furthermore by noting the strict quasi-concavity of utility function it follows that $U_h(\bar{x}_h) > U_h(x_h)$, $U_k(\bar{x}_k) > U_k(x_k)$, and $U_i(\bar{x}_i) = U_i(x_i)$ for any agent $i \neq h, k$. This contradicts the constrained Pareto optimality of $(p, (x_i)_{i \in I})$. This completes the proof. ||

Property 3 : The pair $(p, (x_i)_{i \in I})$ is supposed to be constrained Pareto optimal. Let us define the following sets :

$$\mathscr{I}(j) = \{i \in I \mid z_{ij} \neq \tilde{z}_{ij}(p)\},$$
$$\mathrm{I}^+(j) = \{i \in I \mid z_{ij} > 0\}, \text{ and}$$
$$\mathrm{I}^-(j) = \{i \in I \mid z_{ij} < 0\}.$$

Then it is not possible that $\mathscr{I}(j) \cap \mathrm{I}^+(j) \neq \phi$ and $\mathscr{I}(j) \cap \mathrm{I}^-(j) \neq \phi$.

Proof : Suppose that there exist some agent $h \in \mathscr{I}(j) \cap \mathrm{I}^+(j)$ and agent $k \in \mathscr{I}(j) \cap \mathrm{I}^-(j)$. Then from *Property 1* it follows that

$$0 < z_{hj} < \tilde{z}_{hj}(p) \text{ and } \tilde{z}_{kj}(p) < z_{kj} < 0,$$
$$\text{that is } (\tilde{z}_{hj}(p) - z_{hj})(\tilde{z}_{kj}(p) - z_{kj}) < 0.$$

This contradicts the *property 2*. This completes the proof. ||

Here we give the definition of a *fixprice equilibrium*, following Benassy [1] and consider its relation to the constrained Pareto optimality.

Definition 5 : We say the pair $(p,(x_i)_{i\in I})$ a *fixprice equilibrium* with respect to the initial allocation $(\omega_i)_{i\in I}$ when the following conditions are satisfied:

[i] $(\forall\, i\in S(p)):(z_i=\tilde{z}_i(p))$,

[ii] $(\forall\, i\in L(p)):(|z_{ij}|\leqq|\tilde{z}_{ij}(p)|,\ z_{ij}\tilde{z}_{ij}(p)\geqq 0$, and $pz_i=0)$, and

[iii] $\sum_{i\in I}z_i=0$

, where $S(p)=\{i\in I\,|\,\tilde{z}_{ij}(p)\sum_{i\in I}\tilde{z}_{ij}(p)\leqq 0\}$ and

$L(p)=\{i\in I\,|\,\tilde{z}_{ij}(p)\sum_{i\in I}\tilde{z}_{ij}(p)>0\}$.

That is, $S(p)$ denotes the set of agents on the *short side of the market* at p and $L(p)$ denotes the set of agents on the *long side of the market.*

Proposition 1 : The allocation $(x_i)_{i\in I}$ is constrained Pareto optimal at a given price vector p with respect to the initial allocation $(\omega_i)_{i\in I}$ if and only if the pair $(p,(x_i)_{i\in I})$ is a fixprice equilibrium.

Proof: At first we show the "only if" (necessity) part. Suppose that the pair $(p,(x_i)_{i\in I})$ is constrained Pareto optimal. Then by noting the feasibility of allocation and *Property 1* it follows that condition [ii] and [iii] of the fixprice equilibrium are satisfied. Hence it suffices to show that $z_i=\tilde{z}_i(p)$ for any $i\in S(p)$. Without loss of generality we assume that $\sum_{i\in I}\tilde{z}_{ij}(p)>0$ for some commodity j. Then $S(p)=\{i\in I\,|\,\tilde{z}_{ij}(p)\leqq 0\}$ and $L(p)=\{i\in I\,|\,\tilde{z}_{ij}(p)>0\}$. For any agent i such that $\tilde{z}_{ij}(p)=0$ we can state that $z_i=\tilde{z}_i(p)$. Therefore we must show that for any agent $i\in S^-=\{i\in I\,|\,\tilde{z}_{ij}(p)<0\}$, $z_i=\tilde{z}_i(p)$. Now suppose that there exists some agent $i\in S^-$ such that $z_i\neq\tilde{z}_i(p)$. Then from *Property 3* we know that for any agent $i\in I^+=L$, $z_i=\tilde{z}_i(p)$. Then

$$0>\sum_{i\in S}z_{ij}>\sum_{i\in S}\tilde{z}_{ij}(p) \text{ and}$$
$$\sum_{i\in L}z_{ij}=\sum_{i\in L}\tilde{z}_{ij}(p)>0.$$

From this fact it follows that $0=\sum_{i\in I}z_{ij}>\sum_{i\in I}\tilde{z}_{ij}(p)$. This contradicts the supposition that $\sum_{i\in I}\tilde{z}_{ij}(p)>0$.

The "if" (sufficiency) part proceeds as follows. Suppose that the pair $(p,(x_i)_{i\in I})$ is a fixprice equilibrium. It is trivial that conditions [i], [ii], and [iv] of the constrained Pareto optimality are satisfied by this pair. Hence we must show that the condition [iii] of the constrained Pareto optimality is satisfied. Suppose that the pair $(p,(x_i)_{i\in I})$ did not satisfy the condition [iii], that is, there existed an allocation $(\hat{x}_i)_{i\in I}\in\gamma(p)$ such that $(\forall i\in I):(U_i(\hat{x}_i)\geqq U_i(x_i))$ and $(\exists i\in I):(U_i(\hat{x}_i)>U_i(x_i))$. For any agent $i\in S(p)$, $\hat{x}_i=x_i$, because of the fact that $x_i=\tilde{x}_i(p)$ for any $i\in S(p)$. Hence, if there exists some agent i such that $U_i(\hat{x}_i)>U_i(x_i)$, then this agent $i\in L(p)$ and for him it must hold that $|\hat{z}_{ij}|>|z_{ij}|$. Then we have $\sum_{i\in L}|\hat{z}_{ij}|>\sum_{i\in L}|z_{ij}|$ and $\sum_{i\in S}|\hat{z}_{ij}|=\sum_{i\in S}|z_{ij}|$. Hence $\sum_{i\in I}\hat{z}_{ij}\neq 0$. This contradicts the supposition that $(\hat{x}_i)_{i\in I}\in\gamma(p)$. These considerations complete the proof. ||

With these preliminary analyses we will give the following characterization of the price-mediated exchange equilibrium.

Proposition 2: The pair $(p,(x_i)_{i\in I})$ is a price-mediated exchange equilibrium, if and only if it is constrained Pareto optimal and there exists no price vector $\hat{p}\neq p$ and its corresponding fixprice equilibrium allocation $(x_i(\hat{p}))_{i\in I}$ such that $(\forall i\in I):(U_i(x_i(\hat{p}))\geqq U_i(x_i))$ and $(\exists i\in I):(U_i(x_i(\hat{p}))>U_i(x_i))$.

Proof: The "only if" (necessity) part is trivial. The "if" (sufficiency) part proceeds as follows. It suffices to show that the allocation $(x_i)_{i\in I}$ is p-Pareto optimal when the 'if clause' is satisfied. We will show this by reductio ad absurdum. Suppose that the allocation $(x_i)_{i\in I}$ is not p-Pareto optimal. Then there exists some pair $(\hat{p},(\bar{x}_i)_{i\in I})$ such that the coalition I p-blocks the allocation $(x_i)_{i\in I}$ from the initial allocation $(\omega_i)_{i\in I}$ by the pair $(\hat{p},(\bar{x}_i)_{i\in I})$. If the pair $(\hat{p},(\bar{x}_i)_{i\in I})$ is a fixprice equilibrium, it contradicts the above condition of *Proposition 2*. Hence the pair $(\hat{p},(\bar{x}_i)_{i\in I})$ must not be a fixprice equilibrium, which implies that the allocation $(\bar{x}_i)_{i\in I}$ is not constrained Pareto optimal by noting *Proposition 1*. Then there exist a fixprice equilibrium allocation $(x_i(\hat{p}))_{i\in I}$ such that $(\forall\, i\in I)$: $(U_i(x_i(\hat{p}))\geqq U_i(\bar{x}_i))$ and $(\exists\, i\in I)$:$(U_i(x_i(\hat{p}))>U_i(\bar{x}_i))$. However this implies that the fixprice equilibrium allocation $(x_i(\hat{p}))_{i\in I}$ satisfies the relation that $(\forall\, i\in I)$:$(U_i(x_i(\hat{p}))\geqq U_i(x_i)$ and $(\exists\, i\in I)$:$(U_i(x_i(\hat{p}))>U_i(x_i))$. This contradicts the condition of *Proposition 2*. These considerations complete the proof. ||

Next in order to make above contentions clear we give a graphical example of the two commodity and two agent case. We consider the economy with two agents indexed i and k, and two commodities indexed 1 and 2. We will explain by using the usual Edgeworth's box-diagram. We denote the agent i's origin by O_i and the agent k's origin by O_k. The horizontal axis measures the quantity of commodity 1 and the vertical axis measures the quantity of commodity 2.

Let $\omega=(\omega_i,\omega_k)$ denote the initial allocation. We denote OC_i as the agent i's offer curve (price-consumption curve) and OC_k as the agent k's offer curve. In *Figure 1* the point $C(\omega)$ indicates the Walrasian equilibrium allocation. The set of fixprice equilibrium allocations are the set of allocations on the closed curve $\omega\ C(\omega)\omega$. For example, if the price vector is given by p, the point F indicates the fixprice equilibrium allocation relative to this price vector. In the case of *Figure 1*, however, this point F is not a price-mediated exchange equilibrium. Because there exists a price vector $\hat{p}$ such that the corresponding fixprice equilibrium allocation (i. e. point G) is mutually advantageous. Here I_i denotes agent i's indifference curve and I_k denotes agent k's indifference curve. The shaded region indicates the set of allocations which are mutually advantageous than the point F. Hence the fixprice equilibrium allocation such as the point F is not likely to be a satisfactory description of the equilibrium in the price-mediated exchange economy.

How can we describe the set of price-mediated exchange equilibrium allocation. By using *Proposition 1* and *Proposition 2*, we know that the price-mediated exchange equilibrium allocation is a fixprice equilibrium allocation such that this allocation cannot be improved by any other fixprice equilibrium allocation. If offer curves and indifference curves are drawn in *Figure 2*, the set of price-mediated exchange equilibrium allocation is the set of allocation on the curve $MC(\omega)N$, where point M denotes the tangent point of agent i's offer curve and agent k's indifference curve, and the point N denotes the tangent point of agent k's offer curve and agent i's indifference curve.

The price-mediated exchange equilibrium includes two representative concept of equilibrium. The one is the *Walrasian competitive*

Figure 1

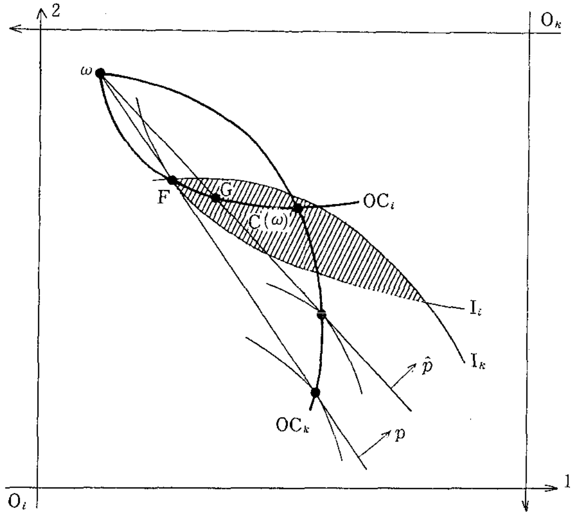
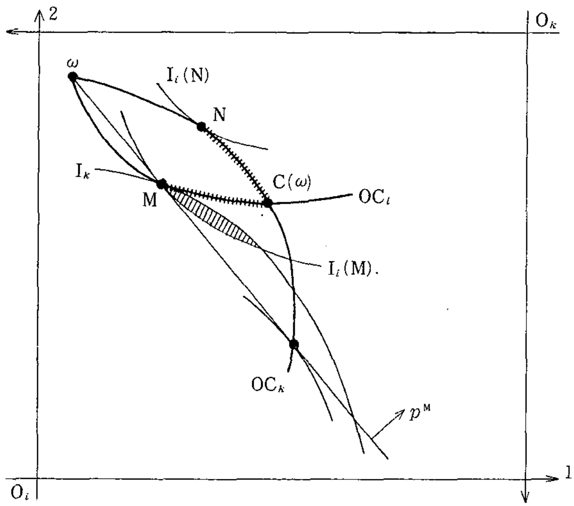


Figure 2



equilibrium allocation depicted by the point C(ω). The other is the *monopolistic equilibrium allocation* depicted by the points M and N. For example, the point M is the best point of agent k among the allocation on the reaction curve OC_i when the agent i behaves as a price-taker. Hitherto we have not dealt explicitly the number and the relative size of market participants. By considering this aspect it will be possible to characterize two representative equilibria from the price-mediated exchange game.

Next we will prove the existence of price-mediated exchange equilibria. We can show the existence of Walrasian competitive equilibria under the *Assumptions 1 - 3*. By noting *Remark*, Walrasian competitive equilibrium is a price-mediated exchange equilibrium, hence we have already known that the set of price-mediated exchange equilibrium is not empty. We named the price-mediated exchange equilibria except for Walrasian competitive equilibria *non-trivial* price-mediated exchange equilibria. Therefore in the following we will show the existence of non-trivial price-mediated exchange equilibria under some regularity conditions.

Proposition 3: Suppose that the aggregate excess demand function satisfies the weak axiom of revealed preference and the following conditions are satisfied:

[i] $(\forall i \in I):(z_i^* \in Int\ Z_i$, and $\tilde{z}_i(p)$ is differentiable), and

[ii] $(\exists i \in I):(z_i^* \neq 0)$

, where z_i^* denotes agent i's Walrasian net trade vector and Int Z_i denotes the interior of Z_i. Then there exist non-trivial price-mediated exchange equilibria.

In order to prove this proposition we use the following lemma which is due to Laroque [5]

Lemma: Let $(p^*, (z_i^*)_{i\in I})$ be locally stable competitive equilibrium under the Walrasian tâtonnement process. Suppose that the following conditions are satisfied:

[i] $(\forall\, i\in I):(z_i^* \in \mathrm{Int}\ Z_i$, and $\tilde{z}_i(p)$ is differentiable), and

[ii] $(\exists\, i\in I):(z_i^* \neq 0)$.

Then $(\exists\, N(p^*)):(\forall\, p\in N(p^*)):(p\neq p^*):((\forall\, i\in L(p)):(U_i(x_i(p)) > U_i(x_i^*)))$, where $N(p^*)$ is a closed neighborhood of p^* and $x_i(p)$ is agent i's consumption vector of fixprice equilibrium at p, and $x_i^* = z_i^* + \omega_i$.

Proof of Proposition 3: The conditions of *Proposition 3* suffice to apply *Lemma*. Hence by using *Lemma*, we have

$(\forall\, i\in L(p)):(p\in N(p^*)$ and $p \neq p^*):(U_i(x_i(p)) > U_i(x_i(p^*)).$

Choose any p, where $p \neq p^*$ and $p\in N(p^*)\cap \{p \mid p=(p_1, 1),\ 0\leqq p_1 \leqq p_1^*\}$

For any chosen agent $k\in L(p)$ we consider the following problem:

$$\begin{array}{ll} \mathrm{Max} & U_k(x_k(p)) \\ p\in N(p^*)\cap Q & \end{array}$$

, where $Q=\{p \mid p=(p_1, 1),\ 0\leqq p_1 \leqq p_1^*\}$ and $x_k(p)$ is agent k's consumption vector of fixprice equilibrium at p. Denote the set of solutions by $\bar{Q}$ and define $\hat{p}=(\max_{p\in\bar{Q}} p_1,\ 1)$. Then in the following we will show that the pair $(\hat{p}, (x_i(\hat{p}))_{i\in I})$ is a non-trivial price-mediated exchange equilibrium.

From the definition of $\hat{p}\in\bar{Q}\subset N(p^*)$ it follows that $\hat{p} \neq p^*$. And it is obvious from the definition of constrained Pareto optimality and *Proposition 1* that the pair $(\hat{p}, (x_i(\hat{p}))_{i\in I})$ satisfies the conditions [a] and [b] of price-mediated exchange equilibrium. Hence, by noting

Proposition 2 it suffices to show that

$(\forall p \geqq 0):(p \neq \hat{p}):((\exists\, i \in I):(U_i(x_i(\hat{p})) > U_i(x_i(p)))$ or

$(\forall\, i \in I):(U_i(x_i(\hat{p})) \geqq U_i(x_i(p))))$.

Step 1: $(\forall p \in (N(p^*) \cap Q) \setminus \bar{Q}):(\exists\, i \in I):(U_i(x_i(\hat{p})) > U_i(x_i(p)))$. From the definition of $\bar{Q}$ we can say that

$(\forall p \in (N(p^*) \cap Q) \setminus \bar{Q}):(U_k(x_k(\hat{p})) > U_k(x_k(p)))$.

Step 2: $(\forall p \geqq 0$ and $p \notin Q):((\exists\, i \in I):(U_i(x_i(\hat{p})) > U_i(x_i(p)))$ or $(\forall\, i \in I):(U_i(x_i(\hat{p})) \geqq U_i(x_i(p))))$.

We will show this by reductio ad absurdum. Suppose that $(\exists\, p \geqq 0$ and $p \notin Q):((\forall\, i \in I):(U_i(x_i(\hat{p})) \leqq U_i(x_i(p)))$ and $(\exists\, i \in I):(U_i(x_i(\hat{p})) < U_i(x_i(p))))$.

Then we can show that the following statements are true;

(a) $(\forall i \in S(\hat{p})):((p_1 > \hat{p}_1$ and $U_i(x_i(\hat{p})) \leqq U_i(x_i(p)))$ imply $z_{i1}(p) < 0)$, and

(b) $(\forall\, i \in L(\hat{p})):((p_1 > \hat{p}_1$ and $U_i(x_i(\hat{p})) \leqq U_i(x_i(p)))$ imply $z_{i1}(p) < 0)$.

First we will show the statement (a) by reductio ad absurdum. Suppose that $(\exists\, i \in S(\hat{p})):(p_1 > \hat{p}_1$ and $U_i(x_i(\hat{p})) \leqq U_i(x_i(p))$ and $z_{i1}(p) \geqq 0)$. Define $\bar{z}_i$ by $\bar{z}_i = \alpha z_i(p) + (1-\alpha) z_i(\hat{p})$, where $\alpha \in (0, 1)$.

Then from the strict quasi-convavity of U_i we have $U_i(\bar{z}_i + \omega_i) > U_i(z_i(\hat{p}) + \omega_i)$. When $z_{i1}(p) \geqq 0$, it follows that $\hat{p}\bar{z}_i = \alpha \hat{p} z_i(p) \leqq \alpha p z_i(p) = 0$. By noting the fact that $(\forall\, i \in S(\hat{p})):(z_i(\hat{p}) = \tilde{z}_i(\hat{p}))$, this contradicts the definition of $\tilde{z}_i(\hat{p})$. Hence the statement (a) was shown to be true.

Next we will show the statement (b) by reductio ad absurdum. Suppose that $(\exists\, i \in L(\hat{p})):(p_1 > \hat{p}_1$ and $U_i(x_i(\hat{p})) \leqq U_i(x_i(p))$ and $z_{i1}(p) \geqq 0)$. From the definition of $\hat{p} \in N(p^*)$ we have already shown that for any $i \in L(\hat{p})$, $U_i(x_i(\hat{p})) > U_i(x_i^*)$.

Hence $(\forall\, i\in L(\hat{p})):(U_i(x_i(p))>U_i(x_i^*))$. Define $\bar{\bar{z}}_i$

$$\bar{\bar{z}}_i=\alpha z_i(p)+(1-\alpha)\,\tilde{z}_i(p^*)\quad ,\text{where } \alpha\in(0,1).$$

Then from the strict quasi-concavity of U_i we have $U_i(\bar{\bar{z}}_i(+\omega_i)>U_i(z_i^*+\omega_i)$. When $z_{i1}(p)\geqq 0$, it follows that $p^*\bar{\bar{z}}_i=\alpha p^* z_i(p)\leqq pz_i(p)=0$ because $p_1>p_1^*$. This fact contradicts the definition of $\tilde{z}_i(p^*)$. Hence the statement (b) was shown to be true.

From the above statements (a) and (b) we can say that
$((\exists\, p\geqq 0$ and $p\notin Q):(\forall\, i\in I):(U_i(x_i(p))\geqq U_i(x_i(\hat{p})))$ implies
$(\forall\, i\in I):(z_{i1}(p)<0)$.
This contradicts the feasibility of the allocation $(x_i(p))_{i\in I}$, i.e., $\sum_{i\in I}z_i(p)=0$. Hence it was shown to be true that
$(\forall p\geqq 0$ and $p\notin Q):((\exists\, i\in I):(U_i(x_i(\hat{p}))>U_i(x_i(p)))$ or
$(\forall\, i\in I):(U_i(x_i(\hat{p}))\geqq U_i(x_i(p))))$.

Step 3 : $(\forall p\in\bar{Q}\cup(Q\setminus N(p^*))):(p\neq\hat{p}):((\exists\, i\in I):(U_i(x_i(\hat{p}))>U_i(x_i(p)))$or $(\forall\, i\in I):(U_i(x_i(\hat{p}))\geqq U_i(x_i(p))))$. We will show this by reductio ad absurdum.

Suppose that $(\exists\, p\in\bar{Q}\cup(Q\setminus N(p^*))):(p\neq\hat{p}):((\forall\, i\in I):(U_i(x_i(\hat{p}))\leqq U_i(x_i(p)))$ and $(\exists\, i\in I):(U_i(x_i(\hat{p}))<U_i(x_i(p))))$. Then we can show that the following statements are true;

(c) $(\forall\, i\in S(\hat{p})):((p_1<\hat{p}_1$ and $U_i(x_i(\hat{p}))\leqq U_i(x_i(p)))$ imply $z_{i1}(p)>0)$,

and

(d) $(\forall\, i\in L(\hat{p})):((p_1<\hat{p}_1$ and $U_i(x_i(\hat{p}))\leqq U_i(x_i(p)))$ imply $z_{i1}(p)>0)$.

First we will show the statement (c) by reductio ad absurdum. Suppose that $(\exists\, i\in S(\hat{p})):(p_1<\hat{p}_1$ and $U_i(x_i(\hat{p}))\leqq U_i(x_i(p))$ and $z_{i1}(p)\leqq 0)$. Define $\bar{z}_i$ by $\bar{z}_i=\alpha z_i(p)+(1-\alpha)z_i(\hat{p})$,where $\alpha\in(0,1)$. Then from the strict quasi-concavity of U_i we have $U_i(\bar{z}_i+\omega_i)>U_i(x_i(\hat{p}))$. When $z_{i1}(p)\leqq 0$, it follows that $\hat{p}\bar{z}_i=\alpha\hat{p}z_i(p)\leqq\alpha pz_i(p)=0$. By noting the fact that $(\forall\, i \in S(\hat{p})):(z_i(\hat{p})=\tilde{z}_i(\hat{p}))$, this contradicts the definition of $\tilde{z}_i(\hat{p})$. Hence the statement (c) was shown to be true.

Next we will show the statement (d) by reductio ad absurdum. Suppose that $(\forall\, i \in L(\hat{p})):(p_1 < \hat{p}_1$ and $U_i(x_i(\hat{p})) \leqq U_i(x_i(p))$ and $z_{i1}(p) \leqq 0)$. Define $\bar{\bar{z}}_i$ by $\bar{\bar{z}}_i = \alpha z_i(p) + (1-\alpha) z_i(\hat{p})$, where $\alpha \in (0, 1)$. Then from the strict quasi-concavity of U_i we have $U_i(\bar{\bar{z}}_i + \omega_i) > U_i(x_i(\hat{p}))$. In the meanwhile, from the weak axiom of revealed preference and Walras' Law we have $(\forall\, p \neq p^*):((p_1 - p_1^*)\sum_{i \in I} \tilde{z}_{i1}(p) < 0)$. Hence it follows that $(\forall\, i \in L(\hat{p})):(\hat{p}_1 < p_1^*):(z_{i1}(\hat{p}) > 0)$. By noting the definition of fixprice equilibrium and the supposition that $U_i(x_i(p)) \geqq U_i(x_i(\hat{p}))$ it can be seen that $z_{i1}(p) \neq 0$ for any $i \in L(\hat{p})$. Hence, when $z_{i1}(p) < 0$ and $p_1 < \hat{p}_1$, it follows that $(\exists\, \bar{\alpha} \in (0, 1)):(\bar{\bar{z}}_i(\bar{\alpha}) \leqq (0, 0))$, where $\bar{\bar{z}}_i(\bar{\alpha}) = \bar{\alpha} z_i(p) + (1-\bar{\alpha}) z_i(\hat{p})$. From the monotonicity of U_i we have $U_i(\bar{\bar{z}}_i(\bar{\alpha}) + \omega_i) \leqq U_i(\omega_i)$. From the definition of fixprice equilibrium we have $U_i(x_i(\hat{p})) \geqq U_i(\omega_i)$. Hence it follows that $U_i(x_i(\hat{p})) \geqq U_i(\bar{\bar{z}}_i(\bar{\alpha}) + \omega_i)$. This is a contradiction. Accordingly the statement (d) was shown to be true. From the above two statements (c) and (d) we can say that $((\exists\, p \in \bar{Q} \cup (Q \setminus N(p^*))):(p \neq \hat{p}):(\forall\, i \in I):(U_i(x_i(p)) \geqq U_i(x_i(\hat{p}))))$ implies $(\forall\, i \in I):(z_{i1}(p) > 0)$. This contradicts the feasibility of $(x_i(p))_{i \in I}$, i.e. $\sum_{i \in I} z_i(p) = 0$. Hence it was shown to be true that $(\forall\, p \in \bar{Q} \cup (Q \setminus N(p^*))):(p \neq \hat{p}):((\exists\, i \in I):(U_i(x_i(\hat{p})) > U_i(x_i(p)))$ or $(\forall\, i \in I):(U_i(x_i(\hat{p})) \geqq U_i(x_i(p))))$. This consideration completes the proof. ||

II. *Concluding Remark*

This article intends to analyze the exchange process mediated by prices. We fundamentally follow the stream of thinking mode emphasizing the need of analysis of so-called 'market coodination problem' by Clower and Leijonhufvud [2]. We provide a natural trading rule and a new equilibrium concept (price-mediated exchange equilibrium) and consider its relation to the Walras equilibrium and non-Walrasian equilibrium (fixprice equilibrium). When we intend to analyze the exchange process systematically, at least following problems may be worthy to be considered. First, the dynamic analysis of the exchange process. Our heuristic proof of existence of equilibria suggests the way for this analysis. Second problem is a specification of equilibrium under the possible conditions of information. The exchange process can be viewed as a communication process. The contractual form of exchange may naturally reflect the information condition: the dispersion of information, the degree of freedom of communication, the cost of communication etc. So that the second problem may be suit to the game-theoretic representation in strategic form (or normal form): we explicitly consider the reaction or interaction of other agents or groups when we specify some agent's behaviour. Furthermore we must treat this problem in relation to the number and the relative size of market participants. These are worthy of further investigations.

* This research was partly financially supported by the Japan Society for the Promotion of Science and the Tokyo Center for Economic Research.

References

[1] Benassy, J. P., "Neokeynesian Disequilibrium Theory in a Monetary Economy," *Review of Economic Studies* (1975), pp. 503-523.

[2] Clower, R.W. and A. Leijonhufvud, "The Coordination of Economic Activities: A Keynesian Perspective," *American Economic Review* (1975), pp. 182-194.

[3] Debreu, G. and H. Scarf, "A Limit Theorem on the Core of an Economy," *International Economic Review* (1963), pp. 235-246.

[4] Grandmont, J. M., G. Laroque, and Y. Younes, "Equilibrium with Quantity Rationing and Recontracting,", *Journal of Economic Theory* (1978), pp. 84-102.

[5] Laroque, G., "On the Dynamics of Disequilibrium: A Simple Remark," *Review of Economic Studies* (1978), pp. 273-278.

[6] Morishima, M., *Walras' Economics,* Cambridge University Press (1977).

[7] Ostroy, J. M., "The Informational Efficiency of Monetary Exchange," *American Economic Review* (1973), pp. 597-610.

[8] Shapley, L. S., "Noncooperative General Exchange," in *Theory and Measurement of Economic Externalities,* ed. by S. A. Y. Lin Academic Press (1976).

[9] Shapley, L. S. and M. Shubik, "Trade Using One Commodity as a Means of Payment," *Journal of Political Economy* (1977), pp. 937-968.

[10] Starr, R. M., "The Structure of Exchange in Barter and Monetary Economies," *Quarterly Journal of Economics* (1972), pp. 290-302.