CUT LOCI AND CONJUGATE LOCI ON LIOUVILLE SURFACES

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Abstract. In the earlier paper [4], we showed that the cut locus of a general point on two-dimensional ellipsoid is a segment of a curvature line and proved "Jacobi's last geometric statement" on the singularities of the conjugate locus. In the present paper, we show that a wider class of Liouville surfaces possess such simple cut loci and conjugate loci. The results include the determination of cut loci and the set of poles on two-sheeted hyperboloids and elliptic paraboloids.

1. Introduction

On a riemannian manifold, any geodesic $\gamma(t)$ starting at a point $\gamma(0) = p$ has the minimal length on each interval $[0, T]$, i.e., the length is equal to the distance between the end points, provided $T > 0$ being small. If the supremum t_0 of the set of such T is finite, then the point $\gamma(t_0)$ is called the *cut point* of p along the geodesic $\gamma(t)$ $(t \geq 0)$. If one considers the minimality of the geodesic segment $\gamma|_{[0,T]}$ only among the curves which are infinitesimally close to $\gamma(t)$, then one obtains the notion of (first) conjugate point of p along the geodesic $\gamma(t)$. The *cut locus* (resp. the conjugate locus) of a point p is then defined as the set of all cut points (resp. conjugate points) of p along the geodesics starting at p. For the general properties of cut loci and conjugate loci, we refer to [8], [10] (see also §2).

In the paper [4], we investigated the cut loci and the conjugate loci of general points on two-dimensional (tri-axial) ellipsoids. In particular, we showed that: 1) the cut locus of a general point is a segment of the curvature line which passes through the antipodal point of the initial point; 2) the conjugate locus of a general point contains just four singularities, which are cusps and located on the curvature lines

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passing through the antipodal point of the initial point. (The second result is known as the last geometric statement of Jacobi; [1], [2], [11]. For rotational ellipsoids, see [14].)

Those properties look quite restrictive, so it would be natural to ask whether there exist other surfaces whose cut loci and conjugate loci possess similar properties as ellipsoids. The aim of this paper is to show that certain Liouville surfaces, both compact and noncompact ones, have such simple cut loci and have similar singularities on conjugate loci for general points. In the noncompact case, our results especially include the determination of cut loci and the set of poles on two-sheeted hyperboloids and elliptic paraboloids. Here, *pole* is a point whose cut locus is the empty set. One can find in [12] some numerical experiment on the set of poles of those surfaces with beautiful graphics.

For noncompact surfaces of revolution, their cut loci and the distribution of poles have been studied in [14], [15], [16] in detail. Also, there are some numerical results for cut loci on compact Liouville surfaces [13], in which the authors conjectured that cut loci of some compact Liouville surfaces have similar properties to those of ellipsoids. Unfortunately, the relation between the extent of Liouville surfaces they considered and those treated in this paper is not clear. It should be noted that our conditions for Liouville surfaces given in this paper would be merely a part of the possible sufficient conditions for cut loci and conjugate loci being simple. A main advantage of our conditions is the simplicity of their expressions. Also, since our model space here is the ellipsoid, we do not consider Liouville surfaces diffeomorphic to tori in this paper.

This paper is organized as follows. In §2 we first review general properties of cut point (locus) and conjugate point (locus), as well as the rigorous definition of conjugate point. They are well-known facts for readers who are familiar with riemannian geometry. Next, in §3 we review general nature of Liouville surfaces which are considered in this paper. They are parametrized with a positive function in one variable and one or two constants. In §4 global behaviors of geodesics on Liouville surfaces are illustrated.

In §5 we consider compact Liouville surfaces and prove that the cut locus of a general point becomes a curve segment under a certain monotonicity condition for the defining function. The proof is divided into two steps: First we assume a stronger condition, under which the proof goes completely parallel to the case of ellipsoid [4], and we give only an outline; secondly we use a technique of “projectively equivalent metrics” to weaken the condition. Although projective equivalence does not preserve distance in general, it preserves natural coordinate lines as well as the geodesic orbits in the present case, which is enough for the determination of cut loci.

In §6 cut loci on noncompact Liouville surfaces are considered. Under a similar condition to the one in §5, we show that cut loci of general points become "simple". More precisely, the possible shapes of cut loci in this case are shown to be the following three types; the empty set, a curve segment, and a disjoint union of two curve segments. It looks strange, but disconnected cut loci appear only when the set of poles is disconnected. For two-sheeted hyperboloids the sets of poles are connected in some cases and are disconnected in others; both cases can arise. Also, it turns out that the set of poles of elliptic paraboloid comprises only two points.

In §7 we take a slightly different approach to the noncompact case and obtain similar results to those in §6. The key idea here is to show the following fact: Some noncompact Liouville surfaces can be projectively embedded onto a half of certain compact Liouville surfaces. For example, using the fact that two-sheeted hyperboloids are projectively embedded into ellipsoids, one can see how the cut loci on hyperboloids are.

Finally, in §8, we consider conjugate loci on compact and noncompact Liouville surfaces. In both cases we assume a bit stronger condition than in §5 and §6 respectively. For the compact case, we show that the conjugate locus of a general point contains just four singular points, which are cusps and located on the natural coordinate lines passing through the antipodal point of the original one. For the noncompact case, we show that: 1) the conjugate locus of a point is not empty if and only if its cut locus is not empty; 2) the singular points of the conjugate locus coincide with the end points of the cut locus (therefore they are at most two points), and they are cusps.

2. Cut locus and conjugate locus

In this section we collect standard properties of cut point (locus) and conjugate point (locus) without proofs, which seem to be more or less well-known. For the proofs and the details, see [8], [10].

Let S be a two-dimensional complete riemannian manifold, and let $p \in S$. Let $\mathrm{Exp}_p : T_pS \to S$ be the exponential mapping at p, which is defined by

$$\gamma(t) = \mathrm{Exp}_p(tv) \quad \text{is the geodesic with} \quad \gamma(0) = p, \quad \dot\gamma(0) = v \in T_pS\,.$$

Then Exp_p is a C^∞ mapping and is a diffeomorphism on a neighborhood of $0_p \in T_pS$. The rigorous definition of conjugate points are as follows: Let $\gamma(t) = \mathrm{Exp}_p(tv)$ be a geodesic, where $v \in U_pS$ (the circle of unit tangent vectors at p). Then $\gamma(T)$ is called a *conjugate point* of $p = \gamma(0)$ along the geodesic $\gamma(t)$, if $d\,\mathrm{Exp}_p$, the differential of Exp_p, at Tv

$$d\,\mathrm{Exp}_p : T_{Tv}(T_pS) = T_pS \to T_{\gamma(T)}S$$

is not linearly isomorphic. The following property is well-known.

Proposition 2.1. *$\gamma(T)$ is a conjugate point of $\gamma(0)$ along $\gamma(t)$ if and only if there is a non-zero, normal Jacobi field $Y(t)$ along $\gamma(t)$ such that $Y(0) = 0$ and $Y(T) = 0$.*

Here, normal Jacobi field $Y(t)$ is, by definition, a vector field along $\gamma(t)$ of the form $y(t)V(t)$, where $V(t)$ is the unit normal (parallel) vector field along $\gamma(t)$ and the function $y(t)$ satisfies the linear differential equation

$$y''(t) = -\sigma(\gamma(t))y(t)\,,$$

σ being the Gauss curvature of S. From this proposition we know that the conjugate points along a geodesic appear discretely. If $0 < T_1 < T_2 < \cdots$ is the set of times representing the conjugate points of $\gamma(0)$ along $\gamma(t)$, then $\gamma(T_1), \gamma(T_2), \ldots$ are called the first conjugate point, the second conjugate point, etc., respectively. The conjugate locus of a point $p \in S$ is the set of first conjugate points along the geodesics starting at p.

For each $p \in S$ and $v \in U_pS$, we define $r_0(v) > 0$ and $r_1(v) > 0$ so that $t = r_0(v)$ (resp. $t = r_1(v)$) is the time representing the cut point (resp. the first conjugate point) of p along the geodesic $\gamma(t)$ defined by $\dot\gamma(0) = v$. We have

Proposition 2.2. *Let $\gamma(t)$ be a geodesic and v, $r_0(v)$, and $r_1(v)$ be as above.*

1. *If $0 < t_0 < r_0(v)$, then the geodesic segment $\gamma|_{[0,t_0]}$ is the unique minimal curve joining the end points p and $\gamma(t_0)$.*
2. *If $t_0 > r_0(v)$, then the curve segment $\gamma|_{[0,t_0]}$ is not minimal.*
3. *$r_0(v) \le r_1(v)$.*
4. *If $r_0(v) < r_1(v)$, then there is another minimal geodesic $\bar\gamma(t)$ joining p and $\gamma(r_0(v))$.*

Proposition 2.3. *Put $V = \{tv \mid v \in U_pS,\ 0 \le t < r_0(v)\}$, which is an open subset of T_pS. Then*

1. $\mathrm{Exp}_p : V \to \mathrm{Exp}_p(V)$ *is a diffeomorphism.*
2. $\mathrm{Exp}_p(\bar V) = S$.
3. $\mathrm{Exp}_p(\partial V)$ *is the cut locus of p, and* $\mathrm{Exp}_p(\partial V) \bigcap \mathrm{Exp}_p(V) = \emptyset$.

The set $\partial V (\subset T_pS)$ is called the *tangent cut locus*, which is a smooth curve at $r_0(v)v$ with $r_0(v) < r_1(v)$ and $U_p \to \partial V$ ($v \mapsto r_0(v)v$) is a homeomorphism. Also, as a consequence of Proposition 2.1, we have

Proposition 2.4. *$r_1(v)$ is a smooth function of $v \in U_p$. In particular, singularities of the conjugate locus only appear at points $\mathrm{Exp}_p(r_1(v)v)$, where $v \in U_p$ is a critical point of $r_1(v)$.*

3. Liouville surfaces

A Liouville surface is, roughly speaking, a surface endowed with a riemannian metric of the form

$$(f_1(x_1) - f_2(x_2))(dx_1^2 + dx_2^2).$$

Correctly speaking, it is a two-dimensional complete riemannian manifold whose geodesic flow has a first integral which is a quadratic form on each cotangent space. For the details, see [6], [3], and [7].

Liouville surfaces needed here are those of type (A) and type (C) explained in [7, §3]. Since we only need some restricted version, we shall explain them in that form.

3.1. **Type (A) — compact case.** Let b_1 and b_2 be real constants such that $b_2 < 0 < b_1$, and let $A(\lambda)$ be a positive C^∞ function on the interval $b_2 \le \lambda \le b_1$. We shall construct a compact Liouville surface from those data.

Define the positive constants α_1 and α_2 by

$$\frac{\alpha_1}{2} = \int_0^{b_1} \frac{A(\lambda)\, d\lambda}{\sqrt{\lambda(b_1-\lambda)(\lambda-b_2)}}\,, \qquad \frac{\alpha_2}{2} = \int_{b_2}^{0} \frac{A(\lambda)\, d\lambda}{\sqrt{-\lambda(b_1-\lambda)(\lambda-b_2)}}.$$

Next, define the function $f_i(x_i)$ on the circle $\mathbb{R}/\alpha_i\mathbb{Z} = \{x_i\}$ $(i = 1, 2)$ by the conditions

$$f_i'(x_i)^2 = \frac{(-1)^{i-1} 4 f_i (b_1 - f_i)(f_i - b_2)}{A(f_i)^2},$$
$$f_i(0) = 0, \quad f_i(\alpha_i/4) = b_i, \quad f_i(-x_i) = f_i(\alpha_i/2 - x_i) = f_i(x_i)\ .$$

Define the equivalence relation $\sim$ on the torus $R = \mathbb{R}/\alpha_1\mathbb{Z} \times \mathbb{R}/\alpha_2\mathbb{Z} = \{(x_1, x_2)\}$ by

$$(-x_1, -x_2) \sim (x_1, x_2)$$

and let S be the corresponding quotient space: $S = R/\sim$. Clearly S is homeomorphic to the two-sphere S^2. There are four ramification points of the quotient map $R \to S$, and by taking $x_1^2 - x_2^2$ and $2x_1x_2$ as coordinate functions around the image of $(0, 0)$ for example, a differentiable structure is defined on S and the map $R \to S$ becomes of class C^∞.

It is not hard to verify that through the quotient map,

$$g = (f_1(x_1) - f_2(x_2))(dx_1^2 + dx_2^2)$$

and

$$F = \frac{f_2(x_2)\xi_1^2 + f_1(x_1)\xi_2^2}{f_1(x_1) - f_2(x_2)}$$

express a well-defined riemannian metric on S and a C^∞ function on the cotangent bundle T^*S respectively, where (ξ_1, ξ_2) are the fiber coordinates associated with the base coordinates (x_1, x_2). F is a first integral of the geodesic flow determined by g.

For example, if $A(\lambda) = 1$, then S is the sphere of constant curvature 1, and if $A(\lambda) = \sqrt{\lambda + a}$ $(a > -b_2)$, then S is isometric to the ellipsoid whose three principal axes have the lengths $2\sqrt{b_1 + a}$, $2\sqrt{a}$, and $2\sqrt{b_2 + a}$ respectively. In this case, the functions $(f_1(x_1), f_2(x_2))$ are equal with the elliptic coordinates of the ellipsoid. One can verify those by using the explicit form of the metric described with the elliptic coordinates.

3.2. **Type (C) — noncompact case.** Let $b < 0$ be a constant and let $B(\lambda)$ be a positive function on the interval $[b, \infty)$. As in the compact case, we shall construct a noncompact Liouville surface from those data.

Define constants α_1 and α_2 by (α_1 may equal ∞)

$$2\alpha_1 = \int_0^\infty \frac{B(\lambda)\, ds}{\sqrt{\lambda(\lambda - b)}}\ , \qquad \frac{\alpha_2}{2} = \int_b^0 \frac{B(\lambda)\, ds}{\sqrt{-\lambda(\lambda - b)}}.$$

Next, define the function $f_1(x_1)$ on the interval $(-\alpha_1, \alpha_1)$ and the function $f_2(x_2)$ on the circle $\mathbb{R}/\alpha_2\mathbb{Z} = \{x_2\}$ by the conditions

$$f_i'(x_i)^2 = \frac{(-1)^{i-1} 4 f_i(f_i - b)}{B(f_i)^2},$$
$$f_i(0) = 0, \quad f_2(\alpha_2/4) = b, \quad \lim_{x_1 \to \alpha_1} f_1(x_1) = \infty,$$
$$f_2(-x_2) = f_2(\alpha_2/2 - x_2) = f_2(x_2), \quad f_1(-x_1) = f_1(x_1)\ .$$

Define the equivalence relation $\sim$ on the cylinder $R = (-\alpha_1, \alpha_1) \times \mathbb{R}/\alpha_2\mathbb{Z} = \{(x_1, x_2)\}$ by

$$(-x_1, -x_2) \sim (x_1, x_2)$$

and put $S = R/\sim$. The quotient space S is homeomorphic to $\mathbb{R}^2$. As in the compact case, a natural differentiable structure is defined on S so that the quotient map $R \to S$ is of C^∞. Also,

$$g = (f_1(x_1) - f_2(x_2)(dx_1^2 + dx_2^2)$$

and

$$F = \frac{f_2(x_2)\xi_1^2 + f_1(x_1)\xi_2^2}{f_1(x_1) - f_2(x_2)}$$

express a riemannian metric on S and a function on the cotangent bundle T^*S respectively, and F is a first integral of the geodesic flow determined by g.

For example, the two-sheeted hyperboloid

$$\frac{u_1^2}{a_1}+\frac{u_2^2}{a_2}+\frac{u_3^2}{a_3}=1 \qquad (a_1>0>a_2>a_3)$$

is obtained by the data $b=-(a_2-a_3)$, $B(\lambda)=\sqrt{\dfrac{\lambda-a_3}{\lambda+a_1-a_3}}$, and the elliptic paraboloid

$$\frac{u_1^2}{a_1}+\frac{u_2^2}{a_2}-2u_3=0 \qquad (a_1>a_2>0)$$

corresponds to the case $b=-(a_1-a_2)$, $B(\lambda)=\sqrt{\lambda+a_1}$. One can verify those by using the explicit forms of the metrics described with the coordinates similar to the elliptic coordinate system, i.e., those obtained form the family of confocal quadrics; see [7, §A.3]. Also, if $B(\lambda)=1$, then S is the flat $\mathbb{R}^2$, and if $B(\lambda)=(\lambda-c)^{-1/2}$, $c<b$, then S has constant negative curvature -1 (see [7, §A.1]).

Remark 3.1. The simplest way to define a Liouville surface is to give functions $f_1(x_1)$ and $f_2(x_2)$ directly. The reason why we use the functions $A(\lambda)$ and $B(\lambda)$ for describing Liouville surfaces is that the conditions we need are expressed in the simplest way with those functions (see §§5,6, and 8).

Remark 3.2. We note that two systems of constants and functions $\{b_1,b_2,A(\lambda)\}$ and $\{\bar b_1,\bar b_2,\bar A(\lambda)\}$ define mutually isomorphic Liouville surfaces if and only if $\bar b_1=cb_1$, $\bar b_2=cb_2$, $\bar A(\lambda)=A(\lambda/c)$ for some $c>0$, or $\bar b_1=cb_2$, $\bar b_2=cb_1$, $\bar A(\lambda)=A(\lambda/c)$ for some $c<0$. Moreover, if $A(\lambda)$ is not constant, then "isomorphic" can be replaced by "isometric". Similarly, two systems of constants and functions $\{b,B(\lambda)\}$ and $\{\bar b,\bar B(\lambda)\}$ defines mutually isomorphic Liouville surfaces if and only if $\bar b=cb$ and $\bar B(\lambda)=B(\lambda/c)/\sqrt{c}$ for some $c>0$. (See [7, p. 4, §§3.4, 3.5].)

4. Geodesic equations

We shall use (x_1,x_2) given in the previous section as a coordinate system of S. The energy function (the hamiltonian of the geodesic flow) E is expressed as

$$2E=\frac{\xi_1^2+\xi_2^2}{f_1(x_1)-f_2(x_2)}\ .$$

Therefore, in both types (A) and (C), the geodesic equations with $2E = 1$ and $F = c$ are given by

$$\begin{aligned} \epsilon_1 \frac{dx_1}{\sqrt{f_1(x_1) - c}} &= \epsilon_2 \frac{dx_2}{\sqrt{c - f_2(x_2)}} \\ dt = \epsilon_1 \sqrt{f_1(x_1) - c}\, dx_1 &+ \epsilon_2 \sqrt{c - f_2(x_2)}\, dx_2 , \end{aligned} \tag{4.1}$$

where $\epsilon_i (= \pm 1)$ denotes the sign of dx_i/dt $(i = 1, 2)$. Geodesic orbits are determined by the first equation and the length parameter t is obtained by integrating the second formula on each orbit (for the detail, see [4]).

In this section, we shall give a rough sketch of the behavior of geodesics in both types (A) and (C). Let $\gamma(t) = (x_1(t), x_2(t))$ be a geodesic with $2E = 1$ and $F = c$. The range of c (= the image of F on the unit cotangent bundle) is $[b_2, b_1]$ for the type (A) and $[b, \infty)$ for the type (C).

4.1. **Type (A).** First, let us observe the case where $b_2 \le c < 0$. Let $\nu_2(c)$ be the unique value of x_2 in the interval $0 < x_2 \le \alpha_2/4$ such that $f_2(x_2) = c$. Then, $x_2(t)$ oscillates either between $\nu_2(c)$ and $\alpha_2/2 - \nu_2(c)$, or between $\alpha_2/2 + \nu_2(c)$ and $\alpha_2 - \nu_2(c)$, depending on the the initial position. Also, $x_1(t)$ increases or decreases monotonously. In the case where $0 < c \le b_1$, $x_1(t)$ and $x_2(t)$ exchange their roles and behave in similar ways as above. In this case we define $\nu_1(c)$ for $c > 0$ in the same way as $\nu_2(c)$.

In the case where $c = 0$, geodesics behave as follows. Let us call the four points $(x_1, x_2) = (0, 0)$, $(0, \alpha_2/2)$, $(\alpha_1/2, \alpha_2/2)$, $(\alpha_1/2, 0)$ as p_0, p_1, p_2, p_3 respectively. They are the branch points of the quotient mapping $R \to S$, and lie on a single closed geodesic L, represented by $x_1 = 0, \alpha_1/2$, or $x_2 = 0, \alpha_2/2$, in this order. Let l denote the length of L. Any geodesic $\gamma(t)$ with $c = 0$, which do not coincide with L, necessarily pass one of the four point at some time. If $\gamma(t_0) = p_i$, then $\gamma(t_0 + l/2) = p_{i+2}$ ($i + 2$ should be considered $\mod 2$). The segment $\gamma(t)$ $(t_0 < t < t_0 + l/2)$ does not intersect L, and $x_1(t)$ and $x_2(t)$ vary monotonously during this time interval. In particular, we have $\gamma(t_0 + l) = p_i$. Conversely, for any geodesic which pass one of the four points, the value c of F must be 0.

4.2. **Type (C).** The case where $b \le c < 0$ is similar to that of type (A): $\nu_2(c)$ is defined as before; $x_2(t)$ oscillates either between $\nu_2(c)$ and $\alpha_2/2 - \nu_2(c)$, or between $\alpha_2/2 + \nu_2(c)$ and $\alpha_2 - \nu_2(c)$; $x_1(t)$ increases or decreases monotonously. If $0 < c < \alpha_1$, then $x_2(t)$ varies on the circle $\mathbb{R}/\alpha_2\mathbb{Z}$ monotonously, but $x_1(t)$ varies in the interval $[\nu_1(c), \alpha_1)$

or $(-\alpha_1, -\nu_1(c)]$. Here, $\nu_1(c)$ is the unique value of x_1 such that $0 < x_1$ and $f_1(x_1) = c$. In this case, if $x_1(0) > 0$ and $x_1'(0) < 0$ for example, then $x_1(t)$ first decreases up to $\nu_1(c)$, and next increases up to α_1 monotonously.

In the case where $c = 0$, any geodesic must pass one of the two points p_1, p_2 represented by $(x_1, x_2) = (0, 0)$ and $(0, \alpha_2/2)$ respectively. The geodesic L passing through both p_1 and p_2 is represented by $x_1 = 0$, or $x_2 = 0$, $\alpha_2/2$. A geodesic which passes one of those points at $t = 0$, and which dose not coincide with L, does not intersect L at any time $t \neq 0$, and both $x_1(t)$ and $x_2(t)$ vary monotonously. Since at any point $p \in S$ not lying on L the quadratic form F on T_p^*S has just two directions of zeros, and since those directions correspond to the tangent vectors of the geodesics from the points p_1 and p_2 respectively, one can easily see that the geodesic connecting p and p_i is unique $(i = 1, 2)$. As a consequence, it turns out that the points p_1 and p_2 are poles, i.e., the exponential maps $T_{p_i}S \to S$ at those points are diffeomorphisms.

5. Cut loci on surfaces of type (A)

Let us define the functions $I_1(c)$ and $I_2(c)$ as follows. In view of the geodesic equation (4.1), these quantities are intimately related to the global behavior of the geodesics: For $b_2 < c < 0$,

$$I_1(c) = \int_0^{\alpha_1/2} \frac{dx_1}{\sqrt{f_1(x_1) - c}}, \quad I_2(c) = \int_{\nu_2(c)}^{\alpha_2/2-\nu_2(c)} \frac{dx_2}{\sqrt{c - f_2(x_2)}};$$

and for $0 < c < b_1$,

$$I_1(c) = \int_{\nu_1(c)}^{\alpha_1/2-\nu_1(c)} \frac{dx_1}{\sqrt{f_1(x_1) - c}}, \quad I_2(c) = \int_0^{\alpha_2/2} \frac{dx_2}{\sqrt{c - f_2(x_2)}}.$$

They are also expressed as

$$I_1(c) = \int_0^{b_1} \frac{A(\lambda)\, d\lambda}{\sqrt{(b_1 - \lambda)(\lambda - b_2)\lambda(\lambda - c)}},$$
$$I_2(c) = \int_{b_2}^{c} \frac{A(\lambda)\, d\lambda}{\sqrt{(b_1 - \lambda)(\lambda - b_2)(-\lambda)(-\lambda + c)}}$$

for $c < 0$, and

$$I_1(c) = \int_c^{b_1} \frac{A(\lambda)\, d\lambda}{\sqrt{(b_1 - \lambda)(\lambda - b_2)\lambda(\lambda - c)}},$$
$$I_2(c) = \int_{b_2}^{0} \frac{A(\lambda)\, d\lambda}{\sqrt{(b_1 - \lambda)(\lambda - b_2)(-\lambda)(-\lambda + c)}}$$

for $c > 0$.

We now put the following monotonicity condition on the function $A(\lambda)$:

$$\frac{d^2}{d\lambda^2}\left((b_1-\lambda)A(\lambda)\right) < 0 \qquad \text{on } [b_2, b_1] \ . \tag{5.1}$$

Let p be a point on S which is not equal to one of the four points p_1, p_2, p_3, p_4, and let $C(p)$ be the cut locus of it. Suppose that p is represented by $(x_1, x_2) = (s_1, s_2)$. Taking the symmetries of S into account, we may assume that $0 \le s_i \le \alpha_i/4$. We denote by $\tilde{p}$ the antipodal point p, which is represented by $(\alpha_1/2 - s_1, \alpha_2/2 + s_2)$.

Theorem 5.1. *Assume that the condition (5.1) is satisfied. Then, $C(p)$ is equal to a segment $\mathcal{I}$ of the coordinate line $x_2 = \alpha_2/2 - s_2$ containing the point $\tilde{p}$. The end points $x_1 = s_+$ and $x_1 = s_- + \alpha_1$ $(s_1 - \alpha_1/2 < s_- < s_1 < s_+ < s_1 + \alpha_1/2)$ of the segment $\mathcal{I}$ are determined by*

$$I_2(c) = \int_{s_1}^{s_+} \frac{dx_1}{\sqrt{f_1(x_1) - c}} = \int_{s_-}^{s_1} \frac{dx_1}{\sqrt{f_1(x_1) - c}} \ ,$$

where $c = f_2(s_2)$. Namely,

$$\mathcal{I} = \left\{(x_1, x_2) \in S \,|\, x_2 = \frac{\alpha_2}{2} - s_2,\ s_+ \le x_1 \le s_- + \alpha_1\right\} .$$

We shall first prove this theorem under the following stronger condition:

$$A'(\lambda) > 0, \quad A''(\lambda) < 0 \qquad \text{on } [b_2, b_1] \ . \tag{5.2}$$

In this case the proof is almost similar to that of the case of ellipsoids [4], so we shall only give an outline. The key point of the proof is the following inequalities.

Proposition 5.1. *If $A(\lambda)$ satisfies the condition (5.2), then:*

1. *$I_1(c) - I_2(c) > 0$ for any $c \in [b_2, b_1]$.*
2. *$(\partial/\partial c)\,(I_1(c) - I_2(c)) < 0$ for any $c \in [b_2, b_1]$.*

Proof. First, observe that $I_2(c) = I_1(c)$ if $A(\lambda)$ is constant, which is a periodic integral of the holomorphic 1-form $\mu^{-1} d\lambda$ on the elliptic curve $\mu^2 = (b_1 - \lambda)(\lambda - b_2)\lambda(\lambda - c)$. Therefore, putting

$$A_1(\lambda) = \frac{A(\lambda) - A(c)}{\lambda - c} = \int_0^1 A'(t\lambda + (1-t)c)\, dt \ ,$$

we have

$$
\begin{aligned}
&I_1(c) - I_2(c) \\
&= \int_0^{b_1} \frac{\sqrt{\lambda - c}\, A_1(\lambda)\, d\lambda}{\sqrt{(b_1-\lambda)(\lambda-b_2)\lambda}} + \int_{b_2}^{c} \frac{\sqrt{c-\lambda}\, A_1(\lambda)\, d\lambda}{\sqrt{(b_1-\lambda)(\lambda-b_2)(-\lambda)}}\,.
\end{aligned}
$$

for $c < 0$ and a similar formula for $c > 0$. Since $A_1(\lambda) > 0$, it follows that $I_1(c) - I_2(c) > 0$. Moreover, differentiating the above formula by c, we have

$$
\begin{aligned}
&\frac{\partial}{\partial c}\,(I_1(c) - I_2(c)) \\
&= \int_0^{b_1} \frac{\sqrt{\lambda - c}\, A_2(\lambda)\, d\lambda}{2\sqrt{(b_1-\lambda)(\lambda-b_2)\lambda}} + \int_{b_2}^{c} \frac{\sqrt{c-\lambda}\, A_2(\lambda)\, d\lambda}{2\sqrt{(b_1-\lambda)(\lambda-b_2)(-\lambda)}}\,,
\end{aligned}
$$

where

$$
A_2(\lambda) = \frac{A_1(\lambda) - A_1(c)}{\lambda - c} = \frac{\partial}{\partial c} A_1(\lambda) = \int_0^1 (1-t)\, A''(t\lambda + (1-t)c)\, dt\,.
$$

Since $A_2(\lambda) < 0$, the assertion (2) follows. Note that those formulas are also valid at $c = 0$. □

Let us parametrize the unit tangent vectors at p by $\eta \in \mathbb{R}/2\pi\mathbb{Z}$;

$$
v(\eta) = \cos\eta\, \frac{\partial/\partial x_1}{|\partial/\partial x_1|} + \sin\eta\, \frac{\partial/\partial x_2}{|\partial/\partial x_2|} \in T_pS.
$$

Let $\gamma_\eta(t) = (x_1(\eta,t), x_2(\eta,t))$ be the geodesic such that $\dot\gamma_\eta(0) = v(\eta)$ and let $c(\eta) = f_2(s_2)(\cos\eta)^2 + f_1(s_1)(\sin\eta)^2$ be the corresponding value of the first integral F. We put

$$
\sigma_i(\eta,t) = \int_0^t \left| \frac{df_i(x_i(\eta,s))}{ds} \right| ds \qquad (i = 1,2) \tag{5.3}
$$

and define positive times $t_1(\eta)$ and $t_2(\eta)$ by

$$
\begin{aligned}
\sigma_1(\eta, t_1(\eta)) &= 2(b_1 - \max\{c(\eta), 0\}), \\
\sigma_2(\eta, t_2(\eta)) &= 2(\min\{c(\eta), 0\} - b_2)\,.
\end{aligned} \tag{5.4}
$$

Then $t_i(\eta)$ is the positive time such that the total variation of $t \mapsto x_i(t,\eta)$ $(0 \le t \le t_i(\eta))$ is equal to the size of the range of $x_i(t,\eta)$ if the range is an interval, and is equal to $\alpha_i/2$ if the range is the whole circle. In particular, we have

$$
x_2(t_2(\eta), \eta) = \begin{cases} \frac{\alpha_2}{2} - s_2 & (c(\eta) < 0) \\ s_2 \pm \frac{\alpha_2}{2} & (c(\eta) > 0) \end{cases}
$$

and

$$x_1(t_1(\eta),\eta) = \begin{cases} s_1 \pm \frac{\alpha_1}{2} & (c(\eta) < 0) \\ \frac{\alpha_1}{2} - s_1 & (c(\eta) > 0) \end{cases} .$$

Also, let $t = r(\eta)$ be the first positive time such that $\gamma_\eta(r(\eta))$ is a conjugate point of p along the geodesic $\gamma_\eta(t)$.

The following lemma holds without the conditions (5.2) nor (5.1), which indicates that the two geodesics $\gamma_\eta(t)$ and $\gamma_{-\eta}(t)$ meets again at $t = t_2(\eta)$.

Lemma 5.1. *(1) $\gamma_\eta(t_2(\eta)) = \gamma_{-\eta}(t_2(-\eta))$ and $t_2(\eta) = t_2(-\eta)$.*
(2) $r(\eta) = t_2(\eta)$ if $\eta = 0, \pi$.

Proof. (1) When $c(\eta) < 0$, we have

$$\int_0^{t_2(\eta)} \frac{|\partial x_2(t,\eta)/\partial t|\; dt}{\sqrt{c(\eta) - f_2(x_2(t,\eta))}} = \int_0^{t_2(\eta)} \frac{|\partial x_1(t,\eta)/\partial t|\; dt}{\sqrt{f_1(x_1(t,\eta)) - c(\eta)}}$$

by the geodesic equation (4.1). This implies

$$\int_{s_2}^{\alpha_2/2 - s_2} \frac{dx_2}{\sqrt{c(\eta) - f_2(x_2)}} = \epsilon_1 \int_{s_1}^{x_1(t_2(\eta),\eta)} \frac{dx_1}{\sqrt{f_1(x_1) - c_(\eta)}} ,$$

where ϵ_1 is the sign of dx_1/dt. Since $c(\eta) = c(-\eta)$, and since ϵ_1 is common for $x_1(t,\eta)$ and $x_1(t,-\eta)$, it follows that

$$x_1(t_2(\eta),\eta) = x_1(t_2(-\eta),-\eta)\,, \text{and } \gamma_\eta(t_2(\eta)) = \gamma_{-\eta}(t_2(-\eta))\,.$$

Then, by the formula

$$t_2(\eta) = \epsilon_1 \int_{s_1}^{x_1(t_2(\eta),\eta)} \sqrt{f_1(x_1) - c(\eta)}\, dx_1 + \int_{s_2}^{\alpha_2/2 - s_2} \sqrt{c(\eta) - f_2(x_2)}\, dx_2\,,$$

which follows from (4.1), we have $t_2(\eta) = t_2(-\eta)$. The case where $c(\eta) \geq 0$ is similar. The proof of (2) is completely the same as [4, Proposition 3]; so we shall omit.

□

As in [4], one can show the following items in each case $c(\eta) < 0$, $c(\eta) >$ and $c(\eta) = 0$. (Note that the sign of c employed here is opposite to that in [4].)

(1) $\gamma_\eta(t_2(\eta)) \in \mathcal{I}$ and $\gamma_\eta(t) \notin \mathcal{I}$ for any $0 < t < t_2(\eta)$.
(2) The mapping $\eta \mapsto \gamma_\eta(t_2(\eta))$ is injective on $[0,\pi]$.
(3) $r(\eta) > t_2(\eta)$ if $0 < \eta < \pi$, $\pi < \eta < 2\pi$.

These items were proved in [4] by means of the properties of $I_3(c) = I_1(c) - I_2(c)$ described in the proposition above. Thus these are also proved in this situation in the same way. While those properties were obtained in [4] via an explicit integral expression of $I_3(c)$, we obtained them here directly from the condition (5.2).

Using the above facts, one can show that the exponential map

$$\mathrm{Exp}_p : \{tv(\eta) \in T_pM \mid 0 \le t < t_2(\eta),\, \eta \in \mathbb{R}/2\pi\mathbb{Z}\} \to M - \mathcal{I}$$

is a diffeomorphism (see [4, p.258]). Since the cut point of p along the geodesic $\gamma_\eta(t)$ appear at $t \le t_2(\eta)$ by Lemma 5.1 and Proposition 2.2, it therefore follows that

$$\mathcal{I} = \{\mathrm{Exp}_p\,(t_2(\eta)v(\eta)) \mid v \in U_pS\}$$

is the cut locus of p. Thus we have Theorem 5.1 under the condition (5.2).

Next, we shall consider the general case; $A(\lambda)$ satisfies the condition (5.1), but does not necessarily satisfy (5.2). Here, we use "projectively equivalent metrics".

Given a Liouville metric $g = (f_1(x_1) - f_2(x_2))(dx_1^2 + dx_2^2)$, if one defines a new Liouville metric $\bar{g}$ by

$$\begin{aligned}\bar{g} =& \frac{f_1(x_1) - f_2(x_2)}{(\alpha f_1 + \beta)(\alpha f_2 + \beta)}\left(\frac{dx_1^2}{\alpha f_1 + \beta} + \frac{dx_2^2}{\alpha f_2 + \beta}\right)\\ =& \left(\frac{f_1}{\beta(\alpha f_1 + \beta)} - \frac{f_2}{\beta(\alpha f_2 + \beta)}\right)\left(\frac{dx_1^2}{\alpha f_1 + \beta} + \frac{dx_2^2}{\alpha f_2 + \beta}\right),\end{aligned} \tag{5.5}$$

α, β being constants, then it is known that the geodesic orbits corresponding to the two metrics g and $\bar{g}$ coincide, i.e., the associated Levi-Civita connections are mutually projectively equivalent. (This is a classical result. See, e.g., [17].)

We now explain how they are constructed. Let S be a Liouville surface constructed from constants $b_2 < 0 < b_1$ and a positive function $A(\lambda)$ on $[b_2, b_1]$. Let α and β be any constants such that $\alpha < 0$, $\beta > 0$, and $\alpha b_1 + \beta > 0$. Put

(5.6)

$$\bar{A}(\mu) = \frac{A\left(\frac{\beta^2\mu}{1-\alpha\beta\mu}\right)}{\sqrt{\beta(\alpha b_1 + \beta)(\alpha b_2 + \beta)}}\,, \qquad \bar{b}_i = \frac{b_i}{\beta(\alpha b_i + \beta)} \quad (i = 1, 2)\,.$$

From the data $\bar{b}_1$, $\bar{b}_2$, and $\bar{A}(\mu)$ we construct positive numbers $\bar{\alpha}_1$ and $\bar{\alpha}_2$, the torus

$$\bar{R} = \mathbb{R}/\bar{\alpha}_1\mathbb{Z} \times \mathbb{R}/\bar{\alpha}_2\mathbb{Z} = \{(\bar{x}_1, \bar{x}_2)\}\,,$$

the functions $\bar{f}_1(\bar{x}_1)$ and $\bar{f}_2(\bar{x}_2)$, and the Liouville surface $\bar{S}$ with the metric $\bar{g} = (\bar{f}_1(\bar{x}_1) - \bar{f}_2(\bar{x}_2))(d\bar{x}_1^2 + d\bar{x}_2^2)$ as before. Then the diffeomorphism $\phi : S \to \bar{S}$ $((x_1, x_2) \mapsto (\bar{x}_1, \bar{x}_2))$ defined by

$$d\bar{x}_i = \frac{dx_i}{\sqrt{\alpha f_i(x_i) + \beta}}\,, \quad x_i = 0 \leftrightarrow \bar{x}_i = 0\,, \quad (i = 1, 2)$$

is a projective isomorphism, i.e., the pullback $\phi^*\bar{g}$ is of the form (5.5). Thus ϕ maps geodesic orbits of S to geodesic orbits of $\bar{S}$. The following lemma is easy.

Lemma 5.2. *If $\phi(x_1, x_2) = (\bar{x}_1, \bar{x}_2)$, then*

$$\phi(-x_1, x_2) = (-\bar{x}_1, \bar{x}_2), \quad \phi(\alpha_1/2 - x_1, x_2) = (\bar{\alpha}_1/2 - \bar{x}_1, \bar{x}_2)\,,$$
$$\phi(x_1, -x_2) = (\bar{x}_1, -\bar{x}_2), \quad \phi(x_1, \alpha_2/2 - x_2) = (\bar{x}_1, \bar{\alpha}_2/2 - \bar{x}_2)\,.$$

Now, suppose that $A(\lambda)$ satisfies the condition (5.1), i.e.,

$$\frac{d^2}{d\lambda^2}((b_1 - \lambda)A(\lambda)) < 0 \qquad \text{on } [b_2, b_1]\,.$$

We choose $\alpha < 0$ and $\beta > 0$ in the following way: $-(b_1 + \beta/\alpha) > 0$ is small enough so that

$$\max_{\lambda \in [b_2, b_1]} \frac{d^2}{d\lambda^2}((b_1 - \lambda)A(\lambda)) < -\left| b_1 + \frac{\beta}{\alpha} \right| \max_{\lambda \in [b_2, b_1]} |A''(\lambda)|\,.$$

Then the corresponding $\bar{A}(\mu)$ satisfies the condition (5.2):

Lemma 5.3. *$\bar{A}'(\mu) > 0$ and $\bar{A}''(\mu) < 0$ on $[\bar{b}_2, \bar{b}_1]$.*

Proof. From (5.6) we have

$$A(\lambda) = c\,\bar{A}\left(\frac{\lambda}{\beta(\alpha\lambda + \beta)}\right)\,, \qquad c = \sqrt{\beta(\alpha b_1 + \beta)(\alpha b_2 + \beta)}\,.$$

Differentiating this formula, we have

$$\frac{d^2}{d\lambda^2}((\alpha\lambda + \beta)A(\lambda)) = c\,\bar{A}''\left(\frac{\lambda}{\beta(\alpha\lambda + \beta)}\right)\frac{1}{(\alpha\lambda + \beta)^3}\,.$$

Since

$$(\alpha\lambda + \beta)A(\lambda) = -\alpha\left((b_1 - \lambda)A(\lambda) - \left(b_1 + \frac{\beta}{\alpha}\right)A(\lambda)\right)\,,$$

it therefore follows that $\bar{A}''(\mu) < 0$.

Now, putting $D(\lambda) = (b_1 - \lambda)A(\lambda)$. Then,

$$A(\lambda) = \frac{D(\lambda) - D(b_1)}{b_1 - \lambda} = -\int_0^1 D'(t\lambda + (1 - t)b_1)\,dt\,,$$

and hence

$$A'(\lambda) = -\int_0^1 t\, D''(t\lambda + (1-t)b_1)\, dt > 0\,.$$

Since

$$A'(\lambda) = c\,\bar{A}'\left(\frac{\lambda}{\beta(\alpha\lambda+\beta)}\right)\frac{1}{(\alpha\lambda+\beta)^2}\,,$$

we have $\bar{A}'(\mu) > 0$. □

By the proved part of the theorem, the cut locus of a point $\bar{p} = (\bar{s}_1, \bar{s}_2) \neq (0,0)$ $(0 \leq \bar{s}_i \leq \bar{\alpha}_i/4)$ is equal to the segment $\bar{\mathcal{I}}$ of the coordinate line $\bar{x}_2 = \bar{\alpha}_2/2 - \bar{s}_2$ whose end points $\bar{x}_1 = \bar{s}_+$ and $\bar{x}_1 = \bar{s}_- + \bar{\alpha}_1$ $(\bar{s}_1 - \bar{\alpha}_1/2 < \bar{s}_- < \bar{s}_1 < \bar{s}_+ < \bar{s}_1 + \bar{\alpha}_1/2)$ are given by

$$\int_{\bar{s}_2}^{\alpha_2/2-\bar{s}_2} \frac{d\bar{x}_2}{\sqrt{\bar{c} - \bar{f}_2(\bar{x}_2)}} = \int_{\bar{s}_1}^{\bar{s}_+} \frac{d\bar{x}_1}{\sqrt{\bar{f}_1(\bar{x}_1) - \bar{c}}} = \int_{\bar{s}_-}^{\bar{s}_1} \frac{d\bar{x}_1}{\sqrt{\bar{f}_1(\bar{x}_1) - \bar{c}}}\,,$$

where $\bar{c} = \bar{f}_2(\bar{s}_2)$.

Now, put $\phi^{-1}(\bar{p}) = p = (s_1, s_2)$ $(0 \leq s_i \leq \alpha_i/4)$ and $\phi^{-1}(\bar{\mathcal{I}}) = \mathcal{I}$. Then, by Lemma 5.2, $\mathcal{I}$ is a segment of the coordinate line $x_2 = \alpha_2/2 - s_2$. Since ϕ maps geodesics to geodesics and coordinate lines to coordinate lines, it follows that each point on a geodesic where the geodesic is tangent to a coordinate line is mapped to a point on the mapped geodesic having the same property. Therefore the end points $x_1 = s_+, s_- + \alpha_1$ of $\mathcal{I}$ is just equal to the ones given in the theorem.

Since $t_2(\eta)$ is well defined as before, and since

$$t_2(\eta) = t_2(-\eta), \quad \gamma_\eta\,(t_2(\eta)) = \gamma_{-\eta}\,(t_2(-\eta))$$

without the assumption (5.1), we see that $\gamma_\eta\,(t_2(\eta)) \in \mathcal{I}$ and $\gamma_\eta(t) \notin \mathcal{I}$ for any $0 < t < t_2(\eta)$. Therefore the image of Exp_p on

$$\{tv(\eta) \in T_pS \,|\, 0 \leq t < t_2(\eta),\, \eta \in \mathbb{R}/2\pi\mathbb{Z}\}$$

is equal to $S - \mathcal{I}$, and since $\phi^{-1}(\bar{S} - \bar{\mathcal{I}}) = S - \mathcal{I}$, Exp_p is injective there. Thus the cut locus of p is equal to $\mathcal{I}$. This completes the proof of the theorem.

It is easy to see that $A(\lambda) = \sqrt{\lambda + a}$ (the case of ellipsoid) satisfies the condition (5.2).

$$A(\lambda) = \frac{1}{\sqrt{b_0 - \lambda}} \qquad (b_0 > b_1)$$

is a simple example which satisfies (5.1), but does not satisfy (5.2). Perturbations of this type are treated in [13]. Another such example is

$$A(\lambda) = \frac{c}{1 - \epsilon\lambda} \qquad (c > 0,\, \epsilon > 0, \epsilon \text{ is sufficiently small})\,,$$

which corresponds to a metric on $SO(3)/SO(2)$ induced from a right-invariant symmetric 2-form on $SO(3)$ (it is sometimes called a Poisson sphere).

6. Cut loci on surfaces of Type (C) — 1

In this section we assume that the function $B(\lambda)$ ($b \le \lambda < \infty$, $b < 0$), defining the Liouville surface S of type (C) under consideration, satisfies the following condition:

$$B'(\lambda) > 0, \quad B''(\lambda) < 0 \qquad \text{on } [b, \infty)\ . \tag{6.1}$$

Note that α_1 must be equal to ∞ in this case. The main idea in this section is to embed an arbitrary large bounded region of the surface S into a Liouville surface of type (A) isometrically, with which the problem is essentially reduced to the compact case.

Lemma 6.1. *Suppose that a positive function $B(\lambda)$ on $[b, \infty)$ ($b < 0$) satisfies the condition (6.1). Put $b_2 = b$, and take any constant $b_0 > 0$. Then there is a constant $b_1 > b_0$ and a function $A(\lambda)$ on $[b_2, b_1]$ possessing the following properties:*

(i) $A(\lambda) > 0, \quad A'(\lambda) > 0, \quad A''(\lambda) < 0$ *on* $[b_2, b_1]$.
(ii) $A(\lambda) = B(\lambda)\sqrt{b_1 - \lambda}$ *if* $b_2 \le \lambda \le b_0$.

Proof. Take $\tilde{b}_0 > b_0$ and fix it. Since

$$\begin{aligned}\frac{d}{d\lambda} B(\lambda)\sqrt{b_1 - \lambda} &= B'(\lambda)\sqrt{b_1 - \lambda} - \frac{1}{2}\frac{B(\lambda)}{\sqrt{b_1 - \lambda}} \\ &\ge B'(\tilde{b}_0)\sqrt{b_1 - \tilde{b}_0} - \frac{1}{2}\frac{B(\tilde{b}_0)}{\sqrt{b_1 - \tilde{b}_0}}\end{aligned}$$

for $\lambda \in [b_2, \tilde{b}_0]$, it follows that the left hand side of the above formula is positive on $[b_2, \tilde{b}_0]$, provided $b_1 > \tilde{b}_0$ being large enough. Also,

$$\frac{d^2}{d\lambda^2} B(\lambda)\sqrt{b_1 - \lambda} = B''(\lambda)\sqrt{b_1 - \lambda} - \frac{B'(\lambda)}{\sqrt{b_1 - \lambda}} - \frac{1}{4}\frac{B(\lambda)}{(b_1 - \lambda)^{3/2}}$$

is negative on $[b_2, \tilde{b}_0]$.

Now, take a diffeomorphism $\phi : [b_2, b_1] \to [b_2, \tilde{b}_0]$ such that

$$\phi(\lambda) = \lambda \quad \text{on } [b_2, b_0]\,; \qquad \phi'(\lambda) > 0, \quad \phi''(\lambda) \le 0 \quad \text{on } [b_2, b_1]\,,$$

and put

$$A(\lambda) = B(\phi(\lambda))\sqrt{b_1 - \phi(\lambda)}\,, \qquad b_2 \le \lambda \le b_1\,.$$

Then $A(\lambda)$ possesses the desired properties. □

The following proposition is an immediate consequence of the above lemma.

Proposition 6.1. *Let $0 < b_0 < b_1$ and $A(\lambda)$ be as in Lemma 6.1. Let S_1 be the Liouville surface of type (A) constructed by b_1, b_2, and $A(\lambda)$. Then the open subset of S defined by $|x_1| < \nu_1(b_0)$ is isometrically identified to the open subset of S_1 defined by $|x_1| < \nu_1(b_0)$ in a natural way.*

Observing that the region of S_1 described in the above proposition is contained in the half of S_1 expressed as $|x_1| < \alpha_1/4$, we have the following theorem.

Theorem 6.1. *Let S be a Liouville surface of type (C) satisfying the condition (6.1). Then the cut locus of any point $p = (s_1, s_2) \in S$ is of one of the following three types: (i) the empty set; (ii) an unbounded segment of the coordinate line $x_2 = \alpha_2/2 - s_2$; (iii) the disjoint union of two unbounded segments of the coordinate line $x_2 = \alpha_2/2 - s_2$.*

Proof. Let $p \in S$ and let $q \in S$ be a cut point of p. If $b_0 > 0$ is taken large enough, then the minimal geodesics joining p and q are contained in the subset of S defined by $|x_1| < \nu_1(b_0)$. Thus the theorem follows from Proposition 6.1 and Theorem 5.1. □

For finer description of the cut loci, we define the functions $J_1(c)$ and $J_2(c)$ as follows:

$$J_1(c) = \int_{-\infty}^{\infty} \frac{dx_1}{\sqrt{f_1(x_1) - c}}, \quad J_2(c) = \int_{\nu_2(c)}^{\alpha_2/2-\nu_2(c)} \frac{dx_2}{\sqrt{c - f_2(x_2)}}$$

when $b < c < 0$, and

$$J_1(c) = 2\int_{\nu_1(c)}^{\infty} \frac{dx_1}{\sqrt{f_1(x_1) - c}}, \quad J_2(c) = \int_{0}^{\alpha_1/2} \frac{dx_2}{\sqrt{c - f_2(x_2)}}$$

when $0 < c < \infty$.

Note that the condition whether $J_1(c) < \infty$ or $J_1(c) = \infty$ do not depend on c, i.e., it is the property of each surface S. For example, $J_1(c) < \infty$ for two-sheeted hyperboloids, and $J_1(c) = \infty$ for elliptic paraboloids.

Let us first consider the case where $J_1(c) < \infty$. The following lemma is proved in the same way as Proposition 2.

Lemma 6.2. *If $c < 0$, then*

$$J_1(c) - J_2(c) > 0, \qquad \frac{d}{dc}(J_1(c) - J_2(c)) < 0\ .$$

In particular, if c is negative and increasing, then $J_1(c)$ is increasing, $J_1(c) - J_2(c)$ is decreasing, and hence $J_2(c)$ is increasing.

For $b < c < 0$, we define $\tau(c)$ by the formula:

$$J_2(c) = \int_{-\infty}^{\tau(c)} \frac{dx_1}{\sqrt{f_1(x_1) - c}} .$$

Since

$$J_1(c) - J_2(c) = \int_{\tau(c)}^{\infty} \frac{dx_1}{\sqrt{f_1(x_1) - c}} ,$$

we see that

(1) $\tau(c)$ increases when c increases;
(2) $\tau(c) < 0$ if and only if $J_2(c) < \frac{1}{2} J_1(c)$.

Also, since

$$\lim_{c \to b} J_2(c) = \pi \frac{B(b)}{\sqrt{-b}}, \quad \lim_{c \to 0} (J_1(c) - J_2(c)) = \int_b^{\infty} \frac{B(\lambda) - B(0)}{\lambda \sqrt{\lambda - b}} \, d\lambda ,$$

it follows that

$$\tau(b) := \lim_{c \to b} \tau(c), \quad \tau(0) := \lim_{c \to 0} \tau(c)$$

are well-defined, $\tau(0) > 0$, and

$$\tau(b) < 0 \quad \text{if and only if} \quad 2\pi \frac{B(b)}{\sqrt{-b}} < \int_0^{\infty} \frac{B(\lambda)\, d\lambda}{(\lambda - b)\sqrt{\lambda}} \tag{6.2}$$

Let $p \in S$ be a point represented by $(x_1, x_2) = (s_1, s_2)$, where $0 \le s_1 \le \alpha_1/4$, $0 \le s_2$, and $(s_1, s_2) \neq (0,0)$. We have thus the following theorem.

Theorem 6.2. *The cut locus $C(p)$ of the point p is as follows.*

(1) *If $s_1 \le \tau(f_2(s_2))$, then $C(p)$ is the empty set.*
(2) *If $-s_1 \le \tau(f_2(s_2)) < s_1$, then $C(p)$ is a curve segment represented by $x_2 = \alpha_2/2 - s_2$, $-\infty < x_1 \le \tilde{s}_-$, where $\tilde{s}_-$ is defined by the formula*

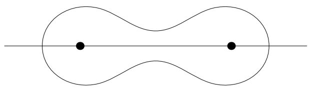
$$J_2(f_2(s_2)) = \int_{\tilde{s}_-}^{s_1} \frac{dx_1}{\sqrt{f_1(x_1) - f_2(s_2)}} .$$

(3) *If $\tau(f_2(s_2)) < -s_1$, then $C(p)$ is a disjoint union of two curve segments represented by*

$$x_2 = \alpha_2/2 - s_2, \quad x_1 \in (-\infty, \tilde{s}_-] \cup [\tilde{s}_+, \infty) ,$$

where $\tilde{s}_-$ is as above and $\tilde{s}_+$ is defined by the formula

$$J_2(f_2(s_2)) = \int_{s_1}^{\tilde{s}_+} \frac{dx_1}{\sqrt{f_1(x_1) - f_2(s_2)}} .$$



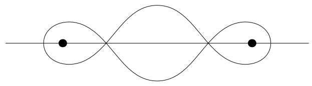


FIGURE 1. Division of surface by the type of cut loci

Remark 6.1. If $s_2 = 0$ (therefore $f_2(s_2) = 0$) in the above theorem, $\tilde{s}_-$ is defined by the formula

$$(J_1(c) - J_2(c))\,|_{c=0} = \int_{-\infty}^{\tilde{s}_-} \frac{dx_1}{\sqrt{f_1(x_1)}} + \int_{s_1}^{\infty} \frac{dx_1}{\sqrt{f_1(x_1)}} .$$

As a consequence of the above theorem, it turns out that the surface S is divided by the curve $x_1 = \tau(f_2(x_2))$ (and its natural extension) into several regions each of which contains only points having the same type of cut loci. More precisely, there are two (major) cases of divisions can arise, which depend on whether $\tau(f_2(s_2))$ may take a negative value or not (see Fig. 6.1). Namely, if $\tau(f_2(s_2)) \geq 0$ for any s_2, then S is divided into the two region (the left picture of Fig. 6.1); the bounded closed region consists of points whose cut loci are empty (i.e., the set of poles), while the cut locus of each point on the unbounded open region is a curve segment. In the picture, the two points express the points $(x_1, x_2) = (0, 0)$ and $(0, \alpha_2/2)$, and the line expresses the geodesic passing through those two points.

On the other hand, if there is s_2 such that $\tau(f_2(s_2)) < 0$, then the surface S is divided into three bounded regions and one unbounded region (the right picture of Fig. 6.1). The set of poles splits into two (closed) bounded components, and there appears a new bounded open component between them; the cut locus of each point of this region is a disjoint union of two unbounded curve segments. The cut loci of points on the unbounded region (the complement of the bounded regions) are the same as above.

For example, both two cases can occur for two-sheeted hyperboloids

$$S: \quad \frac{u_1^2}{a_1} + \frac{u_2^2}{a_2} + \frac{u_3^2}{a_3} = 1 \qquad (a_1 > 0 > a_2 > a_3) .$$

When a_1 is close to 0, then S is nearly flat and the division pattern is like the left picture, If a_2 is close to 0, then S is near "the double of the inside of hyperbola" and the division pattern is like the right picture.

Next, we shall consider the case where $J_1(c) = \infty$. This case is simpler than the above case. Elliptic paraboloids are contained in this case. Let S be a Liouville surface of type (C) satisfying $J_1(c) = \infty$

and let p be a point on S represented by $(x_1, x_2) = (s_1, s_2)$, $s_1 \geq 0$, $0 \leq s_2 \leq \alpha_2/4$, . The following theorem is easily obtained.

Theorem 6.3. *The cut locus $C(p)$ of p is as follows.*

(1) *If $s_2 \neq 0$, then $C(p)$ is the disjoint union of the two curve segments represented by*
$$x_2 = \alpha_2/2 - s_2, \quad x_1 \in (-\infty, \tilde{s}_-] \cup [\tilde{s}_+, \infty)$$
Here $\tilde{s}_\pm$ are the same as in the previous theorem.

(2) *If $s_2 = 0$ and $s_1 > 0$, then $C(p)$ is the curve segment represented by $x_2 = \alpha_2/2$, $-\infty < x_1 \leq \tilde{s}_-$.*

(3) *If $s_1 = s_2 = 0$, then $C(p)$ is empty.*

In particular, the set of poles consists of two points.

Remark 6.2. In the above theorem, the definition of $\tilde{s}_-$ when $s_1 = 0$ is similar to that in the previous theorem; but since $J_1(c) = \infty$ in this case, we use a finite part of the integral instead.

7. Cut loci on surfaces of Type (C) — 2

In this section we show another way of getting results in the previous section for a part of Liouville surfaces of type (C). The main idea used here is projectively equivalent metrics discussed in §4. Let g and $\bar{g}$ be projectively equivalent metrics as in (5.5)

We take here a type(A) metric as g, and take α and β so that $\alpha < 0$, $\beta > 0$, $\alpha b_1 + \beta \leq 0$, then $\bar{g}$ represents a complete metric on the open region
$$U = \{(x_1, x_2) \in S \mid |x_1| < \nu_1(-\beta/\alpha)\}$$
so that U becomes a Liouville surface of type(C). For example, suppose that S is the sphere of constant curvature 1: if $-\beta/\alpha = b_1$, then U is the flat $\mathbb{R}^2$; and if $0 < -\beta/\alpha < b_1$ then U is the surface of constant negative surface. The latter one is essentially the same as the Klein model for two-dimensional hyperbolic space.

In the rest of this section, we only consider the case where $-\beta/\alpha = b_1$. In this case, the corresponding data $b < 0$ and $B(\lambda)$ for $(U, \bar{g})$ are given by
$$b = \frac{k\, b_2}{\beta(\alpha b_2 + \beta)}, \qquad B(\lambda) = \frac{\sqrt{k}\, A\left(\frac{\beta^2 \lambda}{k - \alpha\beta\lambda}\right)}{\sqrt{b_1(\alpha b_2 + \beta)}}, \tag{7.1}$$
where k is an arbitrary positive constant. Since the geodesic orbits of S and U coincide, similar arguments to those in the latter half of §4 are valid. As a consequence, we have the following theorem.

Theorem 7.1. *Let S be a Liouville surface of type (A) satisfying the condition (5.1). Then the cut locus of a point $p \in U$ with respect to U is equal to the intersection of U and the cut locus of p with respect to S. More precisely, we have $J_1(c) < \infty$ and Theorem 6.2 holds in this case.*

Finally we shall show that two-sheeted hyperboloids are typical examples of such surfaces.

Proposition 7.1. *For the ellipsoid*

$$S: \quad \frac{u_1^2}{a_1} + \frac{u_2^2}{a_2} + \frac{u_3^2}{a_3} = 1 \qquad (a_1 > a_2 > a_3 > 0)\ , \tag{7.2}$$

the Liouville surface $(U, \bar{g})$ described above is isometric to a connected component of the two-sheeted hyperboloid

$$\frac{u_1^2}{c_1} + \frac{u_2^2}{c_2} + \frac{u_3^2}{c_3} = 1 \qquad (c_1 > 0 > c_2 > c_3)\ , \tag{7.3}$$

where

(7.4)

$$a_1 = (c_1 - c_2)(c_1 - c_3)k', \quad a_2 = -c_3(c_1 - c_2)k', \quad a_3 = -c_2(c_1 - c_3)k'\ ,$$

$$k' = \frac{(c_1 - c_2)(c_1 - c_3)}{c_1^3}\, k\ ,$$

and

$$\alpha = \frac{-c_1}{(c_1 - c_2)(c_1 - c_3)}\ , \qquad \beta = \frac{c_1 - c_3}{c_1}\, k\ . \tag{7.5}$$

Namely, two-sheeted hyperboloids are embedded into ellipsoids in a projectively equivalent way.

Proof. The data of the ellipsoid (7.2) are given by

$$b_2 = a_3 - a_2, \quad b_1 = a_1 - a_2, \quad A(\lambda) = \sqrt{\lambda + a_2}$$

Therefore, by (7.1) the data of $(U, \bar{g})$ are

$$b = \frac{k\, b_2}{\beta(\alpha b_2 + \beta)}\ , \qquad B(\lambda) = \sqrt{\frac{k\beta(\beta - \alpha a_2)}{-\alpha\beta b_1(\alpha b_2 + \beta)}}\sqrt{\frac{\lambda + \frac{a_2 k}{\beta^2 - \alpha\beta a_2}}{\lambda - \frac{k}{\alpha\beta}}}\ .$$

Thus, if a_1, a_2, a_3 and α, β are given as in (7.4) and (7.5) with c_1, c_2, c_3, then we have

$$b = c_3 - c_2, \qquad B(\lambda) = \sqrt{\frac{\lambda - c_3}{\lambda + c_1 - c_3}}\ ,$$

which are the data for the hyperboloid (7.3). ∎

8. Conjugate loci

In this section we investigate conjugate loci of general points for Liouville surfaces of type (A) and (C).

8.1. **Type (A).** In this subsection we assume the following condition on the positive function $A(\lambda)$:

$$\tilde{A}''(\lambda) > 0, \qquad \tilde{A}'''(\lambda) < 0 \qquad \text{on } [b_2, b_1]\,, \tag{8.1}$$

where $\tilde{A}(\lambda) = (\lambda - b_2)A(\lambda)$. Note that $\tilde{A}'(\lambda) > 0$, since $\tilde{A}'(b_2) = A(b_2) > 0$ and $\tilde{A}''(\lambda) > 0$. Note also that this condition is stronger than the condition (5.2), since

$$A(\lambda) = \frac{\tilde{A}(\lambda) - \tilde{A}(b_2)}{\lambda - b_2} = \int_0^1 \tilde{A}'(t\lambda + (1-t)b_2)\, dt\,. \tag{8.2}$$

Let $p \in S$ be a point as in §4 represented by $(x_1, x_2) = (s_1, s_2)$. Also, let $v(\eta) \in T_pS$, $t_1(\eta)$, $t_2(\eta)$, $r(\eta)$ $(\eta \in \mathbb{R}/2\pi\mathbb{Z})$ be as before. Then we have the similar result for conjugate loci as in the case of ellipsoids.

Theorem 8.1. *$r'(\eta) = 0$ if and only if $\eta = 0, \pi, \pm\pi/2$. Namely, the conjugate locus of p contains just four singular points, which are cusps. Moreover, those singular points lie on the coordinate lines passing through the antipodal point $\tilde{p}$ of p.*

We shall prove $r'(\eta) \neq 0$ for $0 < \eta < \pi/2$ and the singularity of the conjugate locus at $\eta = 0$ is a cusp; other cases will be similarly proved. Let $\eta = \eta_0$ be the unique value in $(0, \pi/2)$ such that $c(\eta) = 0$. Put

$$\tilde{I}_1(c) = \int_0^{b_1} \frac{\sqrt{\lambda - b_2}\, A(\lambda)\ d\lambda}{\sqrt{(b_1 - \lambda)\lambda(\lambda - c)}},$$
$$\tilde{I}_2(c) = \int_{b_2}^{c} \frac{\sqrt{\lambda - b_2}\, A(\lambda)\ d\lambda}{\sqrt{(b_1 - \lambda)(-\lambda)(-\lambda + c)}}$$

for $c < 0$, and

$$\tilde{I}_1(c) = \int_c^{b_1} \frac{\sqrt{\lambda - b_2}\, A(\lambda)\ d\lambda}{\sqrt{(b_1 - \lambda)\lambda(\lambda - c)}},$$
$$\tilde{I}_2(c) = \int_{b_2}^{0} \frac{\sqrt{\lambda - b_2}\, A(\lambda)\ d\lambda}{\sqrt{(b_1 - \lambda)(-\lambda)(-\lambda + c)}}$$

for $c > 0$. Then, we have the following lemma.

Lemma 8.1. (1) *$(\partial/\partial c)\,(\tilde{I}_1(c) - \tilde{I}_2(c)) > 0$ for any $c \in [b_2, b_1]$.*
(2) *$(\partial^2/\partial c^2)\,(\tilde{I}_1(c) - \tilde{I}_2(c)) < 0$ for any $c \in [b_2, b_1]$.*

Proof. Since

$$\tilde{I}_1(c) = \int_0^{b_1} \frac{\tilde{A}(\lambda)\, d\lambda}{\sqrt{(b_1-\lambda)(\lambda-b_2)\lambda(\lambda-c)}}$$

for $c<0$, etc., the lemma is proved in the same way as Proposition 5.1. □

As before, let $t \mapsto \gamma_\eta(t) = (x_1(t,\eta), x_2(t,\eta))$ be the geodesic with $\dot{\gamma}_\eta(0) = v(\eta)$. We first assume $0 < \eta \le \eta_0$ and consider the behavior of the geodesics around the point $t = r(\eta)$. The behavior of the function $t \mapsto x_2(t,\eta)$ is as follows: 1) Starting at $x_2(0,\eta) = s_2$, it increases. 2) After reaching the maximum where $f_2(x_2) = c(\eta)$, it turns to decrease. 3) Then, after passing through $t = t_2(\eta)$ where $x_2(t,\eta) = \alpha_2/2 - s_2$, it will reach the point $t = r(\eta)$ before it reaches the next turning point where $f(x_2) = c(\eta)$. It is because the points $\gamma_\eta(t)$ where $f(x_2) = c(\eta)$ are mutually conjugate along the geodesic (Lemma 5.1 (2)).

Therefore, for t near $r(\eta)$ we have from (4.1):

$$\begin{aligned} t &= \int_{s_1}^{x_1(t,\eta)} \frac{f_1(x_1)-b_2}{\sqrt{f_1(x_1)-c(\eta)}}\, dx_1 + \int_{\nu_2}^{\alpha_2/2-\nu_2} \frac{b_2-f_2(x_2)}{\sqrt{c(\eta)-f_2(x_2)}}\, dx_2 \\ &\qquad + \int_{x_2(t,\eta)}^{\alpha_2/2-s_2} \frac{b_2-f_2(x_2)}{\sqrt{c(\eta)-f_2(x_2)}}\, dx_2 \\ &= -\int_{x_1(t,\eta)}^{s_1+\alpha_1/2} \frac{f_1(x_1)-b_2}{\sqrt{f_1(x_1)-c(\eta)}}\, dx_1 - \int_{x_2(t,\eta)}^{\alpha_2/2-s_2} \frac{f_2(x_2)-b_2}{\sqrt{c(\eta)-f_2(x_2)}}\, dx_2 \\ &\qquad + \tilde{I}_1(c(\eta)) - \tilde{I}_2(c(\eta))\,, \end{aligned}$$

where $\nu_2 = \nu_2(c(\eta))$. Differentiating both sides in η and putting $t = r(\eta)$, we have

$$\begin{aligned} 0 &= -\int_{x_1(r(\eta),\eta)}^{s_1+\alpha_1/2} \frac{f_1(x_1)-b_2}{(f_1(x_1)-c(\eta))^{3/2}}\, dx_1 \\ &+ \int_{x_2(r(\eta),\eta)}^{\alpha_2/2-s_2} \frac{f_2(x_2)-b_2}{(c(\eta)-f_2(x_2))^{3/2}}\, dx_2 + 2\frac{\partial}{\partial c}\left(\tilde{I}_1(c(\eta)) - \tilde{I}_2(c(\eta))\right). \end{aligned} \tag{8.3}$$

Since each term in the second line of the above formula is positive, it follows that

$$x_1(r(\eta),\eta) < s_1 + \alpha_1/2\,.$$

Now, assume that $r'(\eta) = 0$ at some η. Then, differentiating the formula (8.3) by η, one obtains:

(8.4)
$$0 = -\int_{x_1(r(\eta),\eta)}^{s_1+\alpha_1/2} \frac{f_1(x_1) - b_2}{(f_1(x_1) - c(\eta))^{5/2}}\, dx_1 \\ - \int_{x_2(r(\eta),\eta)}^{\alpha_2/2-s_2} \frac{f_2(x_2) - b_2}{(c(\eta) - f_2(x_2))^{5/2}}\, dx_2 + \frac{4}{3}\frac{\partial^2}{\partial c^2}\left(\tilde{I}_1(c(\eta)) - \tilde{I}_2(c(\eta))\right).$$

Since each term of the right-hand side of the above equality is negative, it is a contradiction. Thus we conclude $r'(\eta) \neq 0$ for any $\eta \in (0, \eta_0]$

Next, we shall prove $r'(\eta) \neq 0$ for $\eta \in (\eta_0, \pi/2)$. In this case, the function $t \mapsto x_1(t,\eta)$ behaves as follows: Starting at $x_1(0,\eta)$, it increases; after reaching the maximum, it turns to decrease; and it passes the point $t = r(\eta)$ before it reaches the next turning point where $f_1(x_1) = c(\eta)$. Then, for t near $r(\eta)$ we have

$$t = -\int_{x_1(t,\eta)}^{\alpha_1/2-s_1} \frac{f_1(x_1) - b_2}{\sqrt{f_1(x_1) - c(\eta)}}\, dx_1 \\ - \int_{x_2(t,\eta)}^{s_2+\alpha_2/2} \frac{f_2(x_2) - b_2}{\sqrt{c(\eta) - f_2(x_2)}}\, dx_2 + \tilde{I}_1(c(\eta)) - \tilde{I}_2(c(\eta)).$$

Differentiating this formula twice in η, we have $r'(\eta) \neq 0$ in the same way as above. Thus we have proved $r'(\eta) \neq 0$ for $0 < \eta < \pi/2$.

Next, we shall show that $r''(0) > 0$. For $0 < \eta < \eta_0$ and t near $r(\eta)$ we have from (4.1):

$$\int_{x_1(t,\eta)}^{s_1+\alpha_1/2} \frac{dx_1}{\sqrt{f_1(x_1) - c(\eta)}} \\ + \int_{x_2(t,\eta)}^{\alpha_2/2-s_2} \frac{dx_2}{\sqrt{c(\eta) - f_2(x_2)}} = I_1(c(\eta)) - I_2(c(\eta)).$$

Differentiating this formula in η and putting $t = r(\eta)$, we obtain:

(8.5)
$$\int_{x_1(r(\eta),\eta)}^{s_1+\alpha_1/2} \frac{dx_1}{(f_1(x_1) - c(\eta))^{3/2}} \\ - \int_{x_2(r(\eta),\eta)}^{\alpha_2/2-s_2} \frac{dx_2}{(c(\eta) - f_2(x_2))^{3/2}} = 2\frac{\partial}{\partial c}\left(I_1(c(\eta)) - I_2(c(\eta))\right).$$

The second integral of the left-hand side of this formula is less than

(8.6)
$$\frac{\alpha_2/2 - s_2 - x_2(r(\eta),\eta)}{(c(\eta) - f_2(\alpha_2/2 - s_2))^{3/2}} = \frac{\alpha_2/2 - s_2 - x_2(r(\eta),\eta)}{(\sin\eta)^3\,(f_1(s_1) - f_2(s_2)))^{3/2}}.$$

Lemma 8.2.

$$\alpha_2/2 - s_2 - x_2(r(\eta),\eta) = \frac{1}{3}\frac{\partial^2 x_2}{\partial t \partial \eta}(r(0),0)\, r''(0)\eta^3 + O(\eta^4)\,.$$

Proof. We have

$$\frac{d}{d\eta}x_2(r(\eta),\eta) = \frac{\partial x_2}{\partial t}(r(\eta),\eta)\, r'(\eta)\,.$$

Since $r'(0) = 0$ and $(\partial x_2/\partial t)(r(0),0) = 0$, it therefore follows that

$$\frac{d^3}{d\eta^3}x_2(r(\eta),\eta)\big|_{\eta=0} = 2\,\frac{\partial^2 x_2}{\partial t \partial \eta}(r(0),0)\, r''(0)\,,$$

which indicates the lemma. □

The above lemma and the formula (8.6) implies that, if $r''(0) = 0$, then

$$\lim_{\eta\to 0}\int_{x_2(r(\eta),\eta)}^{\alpha_2/2-s_2} \frac{dx_2}{(c(\eta) - f_2(x_2))^{3/2}} = 0\,.$$

However, the first integral of the right-hand side of the equality (8.5) being nonnegative, and the left-hand side being negative at $\eta = 0$, it is a contradiction. Thus we have $r''(0) \neq 0$. We also have

$$x_1(r(\eta),\eta) - x_1(r(0),0) = \frac{1}{2}\frac{\partial x_1}{\partial t}(r(0),0)\, r''(0)\,\eta^2 + O(\eta^3)\,. \tag{8.7}$$

This formula combined with the one in the above lemma indicates that the curve $\eta \to \gamma_\eta(r(\eta))$ is a cusp at $\eta = 0$. This completes the proof of Theorem 8.1.

8.2. **Type (C).** In this subsection we assume the following condition on the positive function $B(\lambda)$, which is analogous to (8.1):

$$\tilde{B}''(\lambda) > 0, \qquad \tilde{B}'''(\lambda) < 0 \qquad \text{on } [b,\infty)\,, \tag{8.8}$$

where $\tilde{B}(\lambda) = (\lambda - b)B(\lambda)$.

Theorem 8.2. *The conjugate locus of a point $p \in S$ is not empty if and only if its cut locus is not empty. Moreover, the singular points of the conjugate locus coincide with the end points of the cut locus (therefore they are at most two points), and they are cusps.*

The theorem is proved in the same way as §5. We first prove the following lemma, which is a counterpart of Lemma 6.1.

Lemma 8.3. *Put $b_2 = b$, and take any constant $b_0 > 0$. Then there is a constant $b_1 > b_0$ and a positive function $A(\lambda)$ on $[b_2, b_1]$ possessing the following properties:*

(i) $\tilde{A}''(\lambda) > 0, \quad \tilde{A}'''(\lambda) < 0$ *on* $[b_2, b_1], \quad \tilde{A}(\lambda) = (\lambda - b_2)A(\lambda)\,.$

(ii) $A(\lambda) = B(\lambda)\sqrt{b_1 - \lambda}$ *if* $b_2 \le \lambda \le b_0$.

Proof. Take constants b_1 and $\tilde{b}_0$ so that $b_0 < \tilde{b}_0 < b_1$. Let $\phi(\lambda)$ be a function on $[b_2, b_1]$ such that $0 \le \phi(\lambda) \le 1$, and

$$\phi(\lambda) = 1 \quad \text{on } [b_2, b_0], \qquad \phi(\lambda) = 0 \quad \text{on } [\tilde{b}_0, b_1]\,.$$

Let $C(\lambda)$ be a function on $[b_2, b_1]$ defined by

$$C(\lambda) = \phi(\lambda)\,(d^3/d\lambda^3)\left(\sqrt{b_1 - \lambda}\tilde{B}(\lambda)\right) + (1 - \phi(\lambda))(-\epsilon)\,,$$

where ϵ is a small positive constant, and then define $A(\lambda)$ by

$$\tilde{A}(\lambda) = (\lambda - b_2)A(\lambda)\,, \quad \frac{d^3}{d\lambda^3}\tilde{A}(\lambda) = C(\lambda)\,,$$

$$\frac{d^k}{d\lambda^k}\tilde{A}(\lambda)\big|_{\lambda=b_2} = \frac{d^k}{d\lambda^k}\left(\sqrt{b_1 - \lambda}\,\tilde{B}(\lambda)\right)\big|_{\lambda=b_2} \quad (k = 0, 1, 2)$$

Since

$$\frac{d^2}{d\lambda^2}\left(\sqrt{b_1 - \lambda}\,\tilde{B}(\lambda)\right) = \sqrt{b_1 - \lambda}\,\tilde{B}''(\lambda) - \frac{\tilde{B}'(\lambda)}{\sqrt{b_1 - \lambda}} - \frac{1}{4}\frac{\tilde{B}(\lambda)}{(b_1 - \lambda)^{3/2}}\,,$$

it is positive on $[b_2, b_0]$, if b_1 is taken to be sufficiently large. Also, we have

$$\frac{d^3}{d\lambda^3}\left(\sqrt{b_1 - \lambda}\,\tilde{B}(\lambda)\right) < 0$$

on $[b_2, b_0]$. Therefore, taking $\tilde{b}_0$ sufficiently near b_0 and taking $\epsilon > 0$ sufficiently small, we have

$$\tilde{A}''(\lambda) > 0, \quad \tilde{A}'''(\lambda) < 0 \qquad \text{on } [b_2, b_1]\,.$$

□

Thus, Theorem 8.2 follows from the above lemma, Proposition 6.1, and Theorem 8.1. It is directly verified that $B(\lambda) = \sqrt{\lambda + a_1}$, the case of elliptic paraboloid, satisfies the condition (8.8). The case of two-sheeted hyperboloid also satisfies (8.8), which is a consequence of the following general result.

Proposition 8.1. *If $A(\lambda)$ satisfies the condition (8.1), then the function $B(\lambda)$ defined by the formula (7.1) satisfies (8.8).*

Proof. From (7.1) we have

$$\tilde{B}(\lambda) = c\tilde{A}(\mu)(\lambda + e), \tag{8.9}$$

where c is a positive constant and

$$\mu = \frac{\beta^2\lambda}{k - \alpha\beta\lambda}, \qquad e = \frac{k}{-\alpha\beta}\,.$$

Then we have

$$\tilde{B}''(\lambda) = c\tilde{A}''(\mu)\left(\frac{d\mu}{d\lambda}\right)^2 (\lambda + e)\,,$$

$$\tilde{B}'''(\lambda) = c\tilde{A}'''(\mu)\left(\frac{d\mu}{d\lambda}\right)^3 (\lambda + e) + c'\tilde{A}''(\lambda)(\lambda + e)^{-4}\,,$$

where $c' < 0$. Therefore the proposition follows. □

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