

# *Mathematical Journal of Okayama University*

---

*Volume 3, Issue 2*

1953

*Article 4*

MARCH 1954

---

## Some remarks on homotopy groups of rotation groups

Masahiro Sugawara\*

\*Okayama University

Copyright ©1953 by the authors. *Mathematical Journal of Okayama University* is produced by  
The Berkeley Electronic Press (bepress). <http://escholarship.lib.okayama-u.ac.jp/mjou>

## SOME REMARKS ON HOMOTOPY GROUPS OF ROTATION GROUPS

MASAHIRO SUGAWARA

The purpose of this note is to remark slightly that the 15-sphere  $S^{15}$  has a 8 everywhere independent continuous vector fields, which can be proved by simple relations of homotopy groups  $\pi_r(R_n)$  of rotation groups  $R_n$ . Additionally, calculations of  $\pi_r(R_n)$  is continued from the previous note [5], whose notations are used in this note.

1. Concerning to the composition of two homomorphisms  $\Delta: \pi_{r+1}(S^n) \rightarrow \pi_r(R_n)$ , the boundary homomorphism of the fibre bundle  $\{R_{n+1}, \hat{p}, S^n, R_n, R_n\}$ , and  $p_*: \pi_r(R_n) \rightarrow \pi_r(S^{n-1})$ , the induced map of the natural projection  $p: R_n \rightarrow S^{n-1}$ , it holds the following relation.

**Lemma 1.**  $E^{n+3} p_* \Delta(\alpha) = 0,$  *if  $n$  is odd,*  
 $= 2E^{n+2}\alpha,$  *if  $n$  is even,*

for  $\alpha \in \pi_{r+1}(S^n)$ , where  $E^p: \pi_k(S^1) \rightarrow \pi_{k+p}(S^{1+p})$  is the  $p$ -fold iteration of suspension homomorphism  $E$ .

It is known that the homomorphism  $J: \pi_r(R_n) \rightarrow \pi_{r+n}(S^n)$  satisfies the relation  $J\Delta(\alpha) = [\alpha, \iota_n]$ , where  $[\alpha, \beta]$  is Whitehead product, [9, (3.6)]. As  $J(\beta)$  is represented by the Hopf construction of the mapping  $S^r \times S^{n-1} \rightarrow S^{n-1}$  of type  $(p_*(\beta), \iota_{n-1})^1$ ,  $H_0(J(\beta)) = (-1)^{(r+1)n} E(p_*(\beta) * \iota_{n-1}) = (-1)^{(r+1)n} E^{n+1} p_* \beta^2$ . On the other hand, if  $n$  is even,  $E^2 H_0[\alpha, \iota_n] = 2(-1)^n E(\alpha * \iota_n) = 2E^{n+2}\alpha$ , and, if  $n$  is odd,  $E^2 H_0[\alpha, \iota_n] = 0$ . Thus the lemma holds.

We consider the above relation for the case  $r = 14$ ,  $n = 8$  and  $\alpha = \nu'_8 \in \pi_{15}(S^8)$  represented by the Hopf map  $S^{15} \rightarrow S^8$ . Then  $\{\nu'_8\} = \infty \subset \pi_{15}(S^8)$  and  $E^i \nu'_8$  is a generator of  $\pi_{i+15}(S^{i+8}) = 240$  for  $i > 0^2$ , and hence  $E^{n+3} p_* \Delta(\nu'_8) = 2E^{n+2} \nu'_8$  is a element of order 120. Therefore  $p_* \Delta(\nu'_8)$  generates  $\pi_{14}(S^7)$ , as  $\pi_{14}(S^7) = 120^3$ . This shows that  $p_* \Delta: \pi_{15}(S^8) \rightarrow \pi_{14}(S^7)$  is onto.

Let  $\alpha$  be an element of  $\pi_{14}(R_8)$ . By the above property, there exists  $\beta \in \pi_{15}(S^8)$  such that  $p_* \Delta \beta = p_* \alpha$ , and hence  $p_*(\alpha - \Delta \beta) = 0$ . Therefore, by the exactness of the homotopy sequence:  $\pi_{14}(R_8)$

1) Cf. [8], proofs of Corolary 5.14.  
 2) Cf. [6], (3.6), (3.11) and (2.24), where  $H_0: \pi_r(S^n) \rightarrow \pi_{r+1}(S^{2n})$  is the generalized Hopf homomorphism.  
 3) Cf. [2], Théorème 3. The subgroup generated by  $\alpha$  is denoted by  $\{\alpha\}$ .

$\xrightarrow{i'_*} \pi_{14}(R_8) \xrightarrow{p_*} \pi_{14}(S^7)$ , there exists an element  $\alpha' \in \pi_{14}(R_7)$  such that  $i'_* \alpha' = \alpha - \Delta \beta$ . Hence  $(ii')_* \alpha' = i_* \alpha - i_* \Delta \beta = i_* \alpha$  by the exactness of the sequence:  $\pi_{15}(S^8) \xrightarrow{\Delta} \pi_{14}(R_8) \xrightarrow{i'_*} \pi_{14}(R_7)$ . This shows that any map  $f: S^{14} \rightarrow R_8$  is homotopic in  $R_9$  to some map of  $S^{14}$  into  $R_7$ .

It is known that  $S^{15}$  admits 7 everywhere independent continuous vector fields, i. e., 7-field, and so the characteristic map  $T_{16}: S^{14} \rightarrow R_{15}$  of the fibre bundle  $\{R_{16}, p, S^{15}, R_{15}, R_{15}\}$  is homotopic in  $R_{15}$  to a map of  $S^{14}$  into  $R_7$ . As shown above, last map is homotopic in  $R_9$  to a map  $f_0: S^{14} \rightarrow R_7$ , and hence  $T_{16}$  is homotopic to  $f_0$  in  $R_{15}$ . This shows that  $S^{15}$  admits a continuous 8-fields<sup>1)</sup>. Therefore, by the analogous proofs of [4, 27.12], it folds

**Proposition 1<sup>2)</sup>.** *If  $n = 16m + 15$ ,  $S^n$  admits 8 everywhere independent continuous vector fields.*

2. In [5],  $\pi_7(R_n)$  is calculated halfway. A. Borel proved that there is a factorization  $Spin(9)/Spin(7) = S^{15}$ , and hence  $\pi_i(R_9) \approx \pi_i(R_7)$  for  $i \leq 13$  as  $Spin(n)$  is a covering group of  $R_n$ , [1, Théorèmes 3, 4]. This shows that the case i) of Theorem 1 of [5] is not valid and, therefore, it follows from [5, 3.2, 3.4 and 3.6]:

**Proposition 2<sup>3)</sup>.**  $\pi_7(R_6) = \infty = \{\tau_7\}$ ,  $\pi_7(R_8) = \infty = \{\delta_7\}$ ,  $\pi_7(R_7) = \infty = \{\varepsilon_7\}$ ,  $\pi_7(R_9) = \infty + \infty = \{\varepsilon_7\} + \{\zeta_7\}$  and  $\pi_7(R_n) = \infty = \{\zeta_7\}$  for  $n \geq 9$ , where the relations  $2\delta_7 = \tau_7$  in  $\pi_7(R_6)$ ,  $2\varepsilon_7 = \delta_7$  in  $\pi_7(R_7)$ , and  $2\varepsilon_7 = \zeta_7$  in  $\pi_7(R_9)$  are hold.

Before continuing the calculation of  $\pi_r(R_n)$  for  $r \geq 9$ , we remark slightly at a left distributive law for homotopy groups. In general, if  $X$  is any space,  $(\alpha_1 + \alpha_2) \circ \beta$  is not equal to  $\alpha_1 \circ \beta + \alpha_2 \circ \beta$  for  $\alpha_1, \alpha_2 \in \pi_n(X)$  and  $\beta \in \pi_r(S^n)$ . However, if  $X$  is a topological group  $G$ , above two elements are equal. Let  $f_1, f_2: S^n \rightarrow G$  be representatives of  $\alpha_1, \alpha_2$  and  $h: S^r \rightarrow S^n$  of  $\beta$ , respectively. As  $G$  is a topological group, by [4, 17.6],  $f_1 + f_2$  is homotopic to  $f_1 \cdot f_2$  where  $(f_1 \cdot f_2)(y) = f_1(y) \cdot f_2(y)$  for  $y \in S^n$ . Hence  $(f_1 + f_2) \circ h$  is homotopic to  $(f_1 \cdot f_2) \circ h = f_1(h) \cdot f_2(h)$ , and the latter is homotopic to  $f_1(h) + f_2(h) = f_1 \circ h + f_2 \circ h$  by the same reason. Therefore we have

**Lemma 2.** *If  $G$  is a topological group,  $(\alpha_1 + \alpha_2) \circ \beta = \alpha_1 \circ \beta + \alpha_2 \circ \beta$ , for  $\alpha_1, \alpha_2 \in \pi_n(G)$  and  $\beta \in \pi_r(S^n)$ .*

1) Cf. [4], 27.12 and 27.6.

2) Yoshihiro Saito has constructed practically a 8 independent vector fields over  $S^{15}$ .

3) These results agree with those of Serre and Peachter, [2, Lemme 3].

3. For  $\pi_9(R_n)$ , it holds the following

**Proposition 3.**  $\pi_9(R_3) = 3 = \{\alpha_9\}$ , where  $\alpha_9 = \alpha_3 \circ \mu_3 \circ \mu_6$ .  $\pi_9(R_4) = 3 + 3 = \{\alpha_9\} + \{\beta_9\}$ , where  $\beta_9 = \beta_3 \circ \mu_3 \circ \mu_6$ .  $\pi_9(R_5) = 0$ .  $\pi_9(R_6) = 2 = \{\delta_9\}$ , where  $p_* \delta_9 = \nu_5 \circ \eta_8$  a generator of  $\pi_9(S^5)$ .  $\pi_9(R_7) = 2 + 2 = \{\delta_9\} + \{\varepsilon_9\}$ , where  $\varepsilon_9 = \varepsilon_7 \circ \eta_7 \circ \eta_8$  and hence  $p_* \varepsilon_9 = 12\nu_6 \in \pi_9(S^6)$ .  $\pi_9(R_8) = 2 + 2 + 2 = \{\delta_9\} + \{\varepsilon_9\} + \{\zeta_9\}$ , where  $\zeta_9 = \zeta_7 \circ \eta_7 \circ \eta_8$ .  $\pi_9(R_9) = 2 + 2 = \{\delta_9\} + \{\xi_9\}$ .  $\pi_9(R_{10}) = 2 + \infty = \{\delta_9\} + \{\xi_9\}$ , where  $p_* \xi_9 = 2\iota_9 \in \pi_9(S^9)$ .  $\pi_9(R_n) = 2 = \{\delta_9\}$  for  $n \geq 11$ .

For  $R_3$  and  $R_4$ , it follows immediately from  $\pi_9(S^3) = 3 = \{\mu_3 \circ \mu_6\}^1$ . For the case  $R_5$ , in the homotopy sequence of the factorization  $R_5/R_4 = S^4: \pi_9(R_4) \xrightarrow{i_*} \pi_9(R_5) \xrightarrow{p_*} \pi_9(S^4) \xrightarrow{\Delta} \pi_9(R_4)$ ,  $i_* \pi_9(R_4) = i_* (\{\alpha_3 \circ \mu_3 \circ \mu_6\} + \{\beta_3 \circ \mu_3 \circ \mu_6\}) = \{i_*(\alpha_3 \circ \mu_3 \circ \mu_6) + i_*(\beta_3 \circ \mu_3 \circ \mu_6)\} = 0$  by [5, 2.6] and  $\Delta$  is isomorphic onto, and hence  $\pi_9(R_5) = 0$ .  $\pi_9(R_6) = 2$  is followed immediately from  $\pi_9(R_5) = \pi_9(R_6) = 0$ , and moreover  $\delta_9 = \delta_8 \circ \eta_8$ . For  $R_7$ ,  $i_*: \pi_9(R_6) \rightarrow \pi_9(R_7)$  is isomorphic into as  $\pi_{10}(S^6) = 0$ , and the image of  $p_*: \pi_9(R_7) \rightarrow \pi_9(S^6)$  is the subgroup  $\{12\nu_6\} = 2$  of  $\pi_9(S^6)$  and, moreover, the element  $\varepsilon_9 = \varepsilon_7 \circ \eta_7 \circ \eta_8$  has the properties that it is of order two and  $p_* \varepsilon_9 = (p_* \varepsilon_7) \circ \eta_7 \circ \eta_8 = \eta_6 \circ \eta_7 \circ \eta_8 = 12\nu_6$ . Therefore  $\pi_9(R_7) = 2 + 2$ , and  $\pi_9(R_8) = 2 + 2 + 2$  as  $R_8$  is equivalent to the product  $S^7 \times R_7$ .  $\pi_9(R_9) = 2 + 2$  is followed from the fact that  $i_*: \pi_9(R_8) \rightarrow \pi_9(R_9)$  is onto and its kernel is equal to  $T_{9*} \pi_9(S^7) = \{(-\varepsilon_7 + 2\zeta_7) \circ \eta_7 \circ \eta_8\} = \{\varepsilon_9\}$ . In the homotopy sequence:  $\pi_9(R_9) \xrightarrow{i_*} \pi_9(R_{10}) \xrightarrow{p_*} \pi_9(S^9) \rightarrow \pi_9(R_9) \xrightarrow{i_*^8} \pi_9(R_{10})$ , kernel  $i_*^9 = T_{10*} \pi_9(S^8) = \{(a\delta_8 + \zeta_8) \circ \eta_8\} = \{a\delta_9 + \zeta_9\}$  where  $a = 0$  or  $1$ , and hence image  $i_*^9 = 2$ . On the other hand, image  $p_* = \infty = \{2\iota_9\} \subset \pi_9(S^9)$  as kernel  $i_*^8 = 2$ , and so  $\pi_9(R_{10}) = 2 + \infty$ . Moreover we can take as a generator  $\xi_9$  of this infinite cyclic part the element represented by the characteristic map  $T_{11}: S^9 \rightarrow R_{10}$ , because  $pT_{11}: S^9 \rightarrow S^9$  represents  $2\iota_9$ , [4, 23.4].  $\pi_9(R_{11}) = 2$  is followed immediately from the fact that the kernel of  $i_*: \pi_9(R_{10}) \rightarrow \pi_9(R_{11})$  is equal to  $T_{11*} \pi_9(S^9) = \{\xi_9\}$ .

4. Furthermore,  $\pi_r(R_n)$  can be calculated partly for  $r = 10$  and 11.

**Proposition 4.**  $\pi_{10}(R_3) = 15 = \{\alpha_{10}\}$ , and  $\pi_{10}(R_4) = 15 + 15 = \{\alpha_{10}\} + \{\beta_{10}\}$ , where  $\alpha_{10} = \alpha_3 \circ \lambda_3^{10}$  and  $\beta_{10} = \beta_3 \circ \lambda_3^{10,2}$ .  $\pi_{10}(R_5) = 15 + 8 = \{\beta_{10}\} + \{\gamma_{10}\}$ , where  $p_* \gamma_{10} = 3\nu_4 \circ \nu_7 \in \pi_{10}(S^4)^{23}$  and  $2\beta_{10} = \alpha_{10}$  in  $\pi_{10}(R_5)$ .  $\pi_{10}(R_6)$

1) Where  $\mu_8$  is a generator of  $\pi_6(S^3) = 6$  and  $\mu_{3+p} = E^p \mu_3$ , cf. [3], Théorème 1.  
 2) By [3, Théorème 1],  $\pi_{10}(S^3) = 15 = \{\lambda_3^{10}\}$ ,  $\pi_{11}(S^3) = 2 = \{\lambda_3^{11}\}$  and  $\pi_{10}(S^4) = 3 + 24 = \{\mu_4 \circ \mu_7\} + \{\nu_4 \circ \nu_7\}$ .

$= 15 + 8 + 2 = \{\beta_{10}\} + \{r_{10}\} + \{\delta_{10}\}$ , where  $\delta_{10} = \delta_8 \circ \eta_8 \circ \eta_9$  and hence  $p_* \delta_{10} = \nu_3 \circ \eta_8 \circ \eta_9 \in \pi_{10}(S^7)$ .  $\pi_{10}(R_7) = A + 8 + 2 = \{\bar{\beta}_{10}\} + \{r_{10}\} + \{\delta_{10}\}^{12}$ , where  $A = 3$  or  $0$ .  $\pi_{10}(R_8) = A + 8 + 2 + 24 = \{\bar{\beta}_{10}\} + \{r_{10}\} + \{\delta_{10}\} + \{\zeta_{10}\}$ , where  $\zeta_{10} = \zeta_7 \circ \nu_7$ .  $\pi_{10}(R_9) = A + 2 + 8 = \{\bar{\beta}_{10}\} + \{\delta_{10}\} + \{\bar{\zeta}_{10}\}$ ,  $\pi_{10}(R_{10}) = A + 2 + 4 \{\bar{\beta}_{10}\} + \{\delta_{10}\} + \{\bar{\zeta}_{10}\}$ ,  $\pi_{10}(R_{11}) = A + 2 + 2 = \{\bar{\beta}_{10}\} + \{\delta_{10}\} + \{\bar{\xi}_{10}\}$ , and  $\pi_{10}(R_n) = A + 2 = \{\bar{\beta}_{10}\} + \{\delta_{10}\}$  for  $n \geq 12$ .

**Proposition 5.**  $\pi_{11}(R_3) = 2 = \{\alpha_{11}\}$  and  $\pi_{11}(R_4) = 2 + 2 = \{\alpha_{11}\} + \{\beta_{11}\}$ , where  $\alpha_{11} = \alpha_3 \circ \lambda_3^{11}$  and  $\beta_{11} = \beta_3 \circ \lambda_3^{11}$ .  $\pi_{11}(R_5) = 2 = \{\beta_{11}\}$  and  $\pi_{11}(R_6)$  is equal to i)  $2 + 2 = \{\beta_{11}\} + \{\delta_{11}\}$  or ii)  $4 = \{\delta_{11}\}$ , where  $\delta_{11} = \delta_5 \circ \nu_8$  and hence  $p_* \delta_{11} = \nu_5 \circ \nu_8$  a generator of  $\pi_{11}(S^5)$ .  $\pi_{11}(R_7) = B + 2 + \infty = \{\bar{\beta}_{11}\} + \{\bar{\delta}_{11}\} + \{\varepsilon_{11}\}$ , where  $B$  is equal to i)  $2$  or ii)  $0$  and  $p_* \varepsilon_{11} = a_0[\epsilon_6, \epsilon_6] \in \pi_{11}(S^6)$  and  $a_0 = 5$  or  $15$  according to  $A = 3$  or  $0$  respectively.  $\pi_{11}(R_n) = \pi_{11}(R_n)$  for  $8 \leq n \leq 11$  and  $n \geq 13$ , and  $\pi_{11}(R_{12}) = B + 2 + \infty + \infty = \{\bar{\beta}_{11}\} + \{\bar{\delta}_{11}\} + \{\varepsilon_{11}\} + \{\chi_{11}\}$ , where  $p_* \chi_{11} = 2\epsilon_{11} \in \pi_{11}(S^{11})$ .

We follow proofs briefly. For  $r = 10$  and  $11$ , as  $E: \pi_r(S^3) \rightarrow \pi_{r+1}(S^4)$  is isomorphic onto, the kernel of  $i_*: \pi_r(R_4) \rightarrow \pi_r(R_5)$  is equal to  $T_{7*} \pi_r(S^3) = \{(-\alpha_3 + 2\beta_3) \circ \lambda_3^r\} = -\alpha_r + 2\beta_r$  by Lemma 2. For  $\pi_{10}(R_7)$ , it can be shown that  $\pi_{11}(W_{11}) = \infty + 2$  whose infinite cyclic part is isomorphic onto  $\pi_{11}(S^6)$  by the induced map of natural projection, where  $W_{11} = R_7/R_5$  is the vector bundle over  $S^6$ , and hence the image of  $\Delta: \pi_{11}(S^6) \rightarrow \pi_{10}(R_6)$  is equal to the image of  $i_* \Delta: \pi_{11}(W_{11}) \rightarrow \pi_{10}(R_5) \rightarrow \pi_{10}(R_6)$ . It is known that the latter subgroup contains 5-cyclic group [3, Proposition 17.3], and therefore  $\pi_{10}(R_7) = A + 8 + 2$ .  $i_*: \pi_{10}(R_8) \rightarrow \pi_{10}(R_9)$  is onto and its kernel is equal to  $T_{9*}: \pi_{10}(S^7) = \{(-\varepsilon_7 + 2\zeta_7) \circ \nu_7\} = \{-\bar{\tau}_{10} + a_1 \delta_{10} + 2\bar{\zeta}_{10}\}$ .  $i_*: \pi_{10}(R_9) \rightarrow \pi_{10}(R_{10})$  is also onto and its kernel is equal to  $a\delta_{10} + 4\bar{\zeta}_{10}$ , where  $a = 0$  or  $1$ , and hence  $\pi_{10}(R_{10}) = A + 2 + 4$  or  $A + 8$  corresponding to  $a = 0$  or  $1$  respectively. The kernel of  $i_*: \pi_{10}(R_{10}) \rightarrow \pi_{10}(R_{11})$  is equal  $2$  or  $0$ ; and the kernel of  $i_*: \pi_{10}(R_{11}) \rightarrow \pi_{10}(R_{12})$  is generated by the element represented by  $T_{12}$  which is homotopic to  $T_3'': S^{10} \rightarrow R_8$ , and  $T_3''$  satisfies the property that  $pT_3'': S^{10} \rightarrow S^7$  is the suspension of the map  $S^7 \rightarrow S^4$  with Hopf invariant  $1^3$ , and hence  $T_{12}$  represents the image of  $\zeta_{10} + a_2 \delta_{10} + a_3 \beta_{10}$ , where  $a_2 = 0$  or  $1$  and  $a_3 = 0$  or  $1$  or  $2$ . On the other

1) Cf. footnote 2) of p. 131.

2) We denote by  $\bar{\alpha}$  the element  $i_* \alpha$ , where  $i_*: \pi_r(R_n) \rightarrow \pi_r(R_{n+1})$ .

3)  $T_{2k+1}'': S^{n-1} \rightarrow R_{8k} (n = 8k + 3)$  is same to  $\beta_0 | S^{n-1}$  being homotopic to  $H(w)$  of [7, §3, 1], and  $H(w)$  represents the suspension of an element of  $\pi_7(S^4)$  whose Hopf invariant is odd. In proofs of latter fact, if  $k = 2$ , it can easily be seen that  $H(w)$  represents the suspension of an element of Hopf invariant  $1$ .

hand, as the element  $\chi_{11} \in \pi_{11}(R_{12})$  represented by  $T_{13}$  satisfies  $p_* \chi_{11} = 2\epsilon_{11} \in \pi_{11}(S^{11})$ , the kernel of  $i_* : \pi_{10}(R_{11}) \rightarrow \pi_{10}(R_{12})$  has at most order two. These properties show that  $a = a_3 = 0$  and Prop. 4.

From Lemma 1, it follows that the composition of  $\pi_{13}(S^6) \xrightarrow{A} \pi_{12}(R_6) \xrightarrow{p_*} \pi_{12}(S^5)$  is onto, and hence  $i_* : \pi_{11}(R_5) \rightarrow \pi_{11}(R_6)$  is isomorphic into. The kernel of  $i_* : \pi_{11}(R_6) \rightarrow \pi_{11}(R_7)$  is equal to  $\{\delta_5 \circ \nu_5 \circ \nu_8\} = 2\delta_{11} =$  i) 0 or ii)  $\{\beta_{11}\}$ .  $i_* : \pi_{11}(R_n) \rightarrow \pi_{11}(R_{n+1})$  is isomorphic into, as its kernel is equal to  $\zeta_8 \circ \gamma_8 = 0$  for  $n = 9$ ,  $\hat{\zeta}_9 \circ \gamma_9 \circ \gamma_{10} = 2\bar{\zeta}_{10} \circ \gamma_{10} = 0$  for  $n = 10$ , and as the element represented by  $T'_7 : S^{12} \rightarrow R_{12}$  is transformed to  $\gamma_{11} \in \pi_{12}(S^{11})$  by  $p_* : \pi_{12}(R_{12}) \rightarrow \pi_{12}(S^{11})$  for  $n = 11$ .

## BIBLIOGRAPHY

- [ 1 ] BOREL, A., Le plane projectif des octaves et les sphères comme espaces homogènes, C. R., Paris, 230 (1950), 1378 - 1380.
- [ 2 ] SERRE, J.-P., Quelques calculs de groupes d'homotopie, *ibid.*, 236 (1953), 2475 - 2477.
- [ 3 ] ———, Groupes de Lie et puissances réduites de Steenrod, *Amer. Jour. Math.*, 75 (1953), 409 - 448.
- [ 4 ] STEENROD, N. E., The Topology of Fibre Bundles, Princeton Univ. Press, 1951.
- [ 5 ] SUGAWARA, M., On the homotopy groups of rotation groups, *Math. Jour. Okayama Univ.*, 3 (1953), 11 - 21.
- [ 6 ] TODA, H., Generalized Whitehead products and homotopy groups of spheres, *Jour. Inst. Poly., Osaka City Univ.*, 3 (1952), 43 - 82.
- [ 7 ] WHITEHEAD, G. W., On families of continuous vector fields over spheres, *Ann. Math.*, 47 (1946), 779 - 785; also 48 (1947), 782 - 783.
- [ 8 ] ———, Generalization of the Hopf invariant, *ibid.*, 51 (1950), 192 - 237.
- [ 9 ] WHITEHEAD, J. H. C., On certain theorems of G. W. Whitehead, *ibid.*, 58 (1953), 418 - 428.

DEPARTMENT OF MATHEMATICS,  
OKAYAMA UNIVERSITY

(Received January 15, 1954)