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## The Galois Action on the Torsor of Homotopy Classes of Paths on a Projective Line minus a Finite Number of Points

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**THE GALOIS ACTION ON THE TORSOR OF HOMOTOPY CLASSES OF PATHS ON A PROJECTIVE LINE MINUS A FINITE NUMBER OF POINTS**

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0. INTRODUCTION.

**0.1.** Deligne on a conference in Schloss Ringberg considered the mixed Hodge structure on the fundamental group of  $P^1 \setminus \{0, 1, -1, \infty\}$ . He showed that the motivic Galois Lie algebra associated to this mixed Hodge structure contains a free Lie subalgebra on generators in degree  $1, 3, 5, \dots, 2n + 1, \dots$  corresponding to  $\log 2, \zeta(3), \zeta(5), \dots, \zeta(2n + 1), \dots$ .

In [W1] and [DW] we were studying actions of Galois groups on fundamental groups. In this note we are studying the action of the Galois group  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  on the torsor of ( $\ell$ -adic) paths from  $\overrightarrow{01}$  to  $-1$  on  $P_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$ . We show that the associated graded Lie algebra of the image of  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_{\ell} \infty))$  contains a free Lie subalgebra over  $\mathbb{Q}_{\ell}$  on generators in degree  $1, 3, 5, \dots, 2n + 1, \dots$ . We use the idea working modulo 2 from Deligne’s talk in Schloss Ringberg.

In [W1] section 5 we were studying some general aspects of actions of Galois groups on torsors of paths. To make this paper self contained we recall some definitions and results from [W1] in sections 1 and 2.

1. TORSORS OF PATHS.

**1.1.** Let  $K$  be a number field and let  $a_1, \dots, a_{n+1}$  be  $K$ -points of a projective line  $P_K^1$ . Let  $V = P_K^1 \setminus \{a_1, \dots, a_n, a_{n+1}\}$ . For simplicity we assume that  $a_{n+1} = \infty$ .

We denote by  $\widehat{V}(K)$  the set of  $K$ -points of  $V$  and of tangential base points defined over  $K$ . Let  $z, v \in \widehat{V}(K)$ . Let  $\pi_1(V_{\bar{K}}, v)$  be the  $\ell$ -completion of the etale fundamental group of  $V_{\bar{K}}$  and let  $\pi(V_{\bar{K}}, z, v)$  be the set of  $\ell$ -adic

paths from  $v$  to  $z$  on  $V_{\bar{K}}$ . The set  $\pi(V_{\bar{K}}, z, v)$  is a  $\pi_1(V_{\bar{K}}, v)$ -torsor. The Galois group  $G_K := \text{Gal}(\bar{K}/K)$  acts on  $\pi_1(V_{\bar{K}}, v)$  and on  $\pi(V_{\bar{K}}, z, v)$  in a compatible way, i.e.,  $\sigma(p \cdot S) = \sigma(p) \cdot \sigma(S)$ , where  $\sigma \in G_K$ ,  $p \in \pi(V_{\bar{K}}, z, v)$  and  $S \in \pi_1(V_{\bar{K}}, v)$ .

Let us fix a path  $p \in \pi(V_{\bar{K}}, z, v)$ . We define a bijection of sets

$$t_p : \pi(V_{\bar{K}}, z, v) \rightarrow \pi_1(V_{\bar{K}}, v)$$

setting  $t_p(q) := p^{-1} \cdot q$  (the composition of paths is from right to left). The bijection  $t_p$  is not  $G_K$ -equivariant. Using the bijection  $t_p$  we transport the action of  $G_K$  on  $\pi(V_{\bar{K}}, z, v)$  into the action of  $G_K$  on  $\pi_1(V_{\bar{K}}, v)$ .

Let  $\sigma \in G_K$ . We set

$$f_p(\sigma) := p^{-1} \cdot \sigma(p).$$

The element  $f_p(\sigma) \in \pi_1(V_{\bar{K}}, v)$ . Let us define a new action of  $G_K$  on  $\pi_1(V_{\bar{K}}, v)$  setting

$$\sigma_p(S) := f_p(\sigma) \cdot \sigma(S).$$

Observe that

$$(\tau \cdot \sigma)_p = \tau_p \cdot \sigma_p,$$

i.e., we have an action of  $G_K$  on  $\pi_1(V_{\bar{K}}, v)$ . We have

$$t_p(\sigma(q)) = \sigma_p(t_p(q)),$$

i.e., the bijection  $t_p$  is  $G_K$ -equivariant if we equip  $\pi_1(V_{\bar{K}}, v)$  with the new action of  $G_K$ .

**1.2.** We fix generators of  $\pi_1(V_{\bar{K}}, v)$  in the following way. At each missing point  $a_i$  we choose a tangential base point  $v_i$  defined over  $K$ . Let  $\gamma_i$  be a path from  $v$  to  $v_i$ . Then  $x_i$  is the composition of the path  $\gamma_i$  + a small loop around  $a_i$  in the opposit clockwise direction + the path  $\gamma_i^{-1}$ . We can assume that  $x_{n+1} \cdot x_n \cdot \dots \cdot x_1 = 1$ .

To study the action of  $G_K$  on the torsor  $\pi(V_{\bar{K}}, z, v)$ , i.e., the action

$$(\ )_p : G_K \rightarrow \text{Aut}_{\text{set}}(\pi_1(V_{\bar{K}}, v))$$

it is very convenient to embed  $\pi_1(V_{\bar{K}}, v)$  into the ring of formal power series in non-commuting variables.

Let  $\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}$  (resp.  $\mathbb{Q}_\ell\{X_1, \dots, X_n\}$ ) be a  $\mathbb{Q}_\ell$ -algebra of formal power series (resp. of polynomials) in non-commuting variables  $X_1, \dots, X_n$ . Let

$$k : \pi_1(V_{\bar{K}}, v) \rightarrow \mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}$$

be a continuous multiplicative embedding given by  $k(x_i) = e^{X_i}$  for  $i = 1, \dots, n$ .

Let us set

$$\Lambda_p(\sigma) := k(f_p(\sigma)).$$

The action of  $G_K$  on  $\pi_1(V_{\bar{K}}, v)$  induces a homomorphism

$$G_K \rightarrow \text{Aut}_{\mathbb{Q}_\ell\text{-algebra}}(\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}).$$

The action of  $G_K$  on  $\pi(V_{\bar{K}}, z, v)$ , i.e., the action  $( )_p$  induces a homomorphism

$$\varphi_p : G_K \rightarrow \text{Aut}_{\mathbb{Q}_\ell\text{-linear}}(\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}).$$

Let  $\omega \in \mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}$  and let  $\sigma \in G_K$ . Then

$$\varphi_p(\sigma)(\omega) = \Lambda_p(\sigma) \cdot \sigma(\omega).$$

**1.3.** We shall study the Lie algebras of derivations of free Lie algebras.

Let  $\text{Lie}(V)$  be a free Lie algebra over  $\mathbb{Q}_\ell$  on free generators  $X_1, \dots, X_n$ . Let  $L(V) := \varprojlim^n \text{Lie}(V)/\Gamma^n \text{Lie}(V)$ . We identify  $\text{Lie}(V)$  (resp.  $L(V)$ ) with the Lie algebra of Lie elements of  $\mathbb{Q}_\ell\{X_1, \dots, X_n\}$  (resp. of  $\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}$ ).

If  $L$  is a Lie algebra then we denote by  $\text{Der } L$  the Lie algebra of derivations of  $L$ .

Let  $\underline{n} := \{1, \dots, n\}$ . We set

$$\text{Der}^* \text{Lie}(V) := \{D \in \text{Der } \text{Lie}(V) \mid \forall i \in \underline{n} \exists A_i \in \text{Lie}(V), D(X_i) = [X_i, A_i]\}$$

and

$$\text{Der}^* L(V) := \{D \in \text{Der } L(V) \mid \forall i \in \underline{n} \exists A_i \in L(V), D(X_i) = [X_i, A_i]\}.$$

The derivation  $D \in \text{Der}^* \text{Lie}(V)$  such that  $D(X_i) = [X_i, A_i]$  for  $i \in \underline{n}$  we denote by  $D_{(A_1, \dots, A_n)}$  or  $D_{(A_i)_{i \in \underline{n}}}$ . Let  $\langle X_i \rangle$  be a vector subspace of  $\text{Lie}(V)$  generated by  $X_i$ . Observe that we have an isomorphism of vector spaces

$$\text{Der}^* \text{Lie}(V) \approx \bigoplus_{i=1}^n (\text{Lie}(V) / \langle X_i \rangle)$$

which maps  $D_{(A_1, \dots, A_n)}$  onto  $(A_1, \dots, A_n)$ . We introduce on  $\bigoplus_{i=1}^n (\text{Lie}(V) / \langle X_i \rangle)$  a new bracket  $\{ \}$  defined in the following way

$$\{(A_i)_{i \in \underline{n}}, (B_i)_{i \in \underline{n}}\} := ([A_i, B_i] + D_{(A_j)_{j \in \underline{n}}}(B_i) - D_{(B_j)_{j \in \underline{n}}}(A_i))_{i \in \underline{n}}.$$

**Lemma 1.3.1.** The vector space  $\bigoplus_{i=1}^n (\text{Lie}(V) / \langle X_i \rangle)$  equip with the bracket  $\{ \}$  is a Lie algebra isomorphic to the Lie algebra  $\text{Der}^* \text{Lie}(V)$ . The isomorphism of Lie algebras maps  $(A_i)_{i \in \underline{n}}$  onto  $D_{(A_j)_{j \in \underline{n}}}$ .  $\square$

The vector space  $\bigoplus_{i=1}^n (\text{Lie}(V) / \langle X_i \rangle)$  equip with the Lie bracket  $\{ \}$  we shall denote by  $(\bigoplus_{i=1}^n (\text{Lie}(V) / \langle X_i \rangle), \{ \})$ .

We define a semi-direct product of Lie algebras

$$\text{Lie}(V) \tilde{\times} \text{Der}^* \text{Lie}(V)$$

defining a Lie bracket  $\{ \}$  on the product of vector spaces  $\text{Lie}(V) \times \text{Der}^* \text{Lie}(V)$  in the following way

$$\{(\lambda, D_\beta), (\lambda_1, D_{\beta_1})\} := ([\lambda, \lambda_1] + D_\beta(\lambda_1) - D_{\beta_1}(\lambda), [D_\beta, D_{\beta_1}]).$$

Hence the Lie bracket in a semi-direct product of Lie algebras

$$\text{Lie}(V) \tilde{\times} \left( \bigoplus_{i=1}^n (\text{Lie}(V) / \langle X_i \rangle), \{ \} \right)$$

is given by

$$\{(\lambda, \beta), (\lambda_1, \beta_1)\} := ([\lambda, \lambda_1] + D_\beta(\lambda_1) - D_{\beta_1}(\lambda), \{\beta, \beta_1\}).$$

We recall that  $\mathbb{Q}_\ell\{X_1, \dots, X_n\}$  is a  $\mathbb{Q}_\ell$ -algebra of polynomials in non-commuting variables  $X_1, \dots, X_n$ . Observe that any derivation of the Lie algebra  $\text{Lie}(V)$  (resp.  $L(V)$ ) induces a derivation of the  $\mathbb{Q}_\ell$ -algebra  $\mathbb{Q}_\ell\{X_1, \dots, X_n\}$  (resp.  $\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}$ ). Let  $\omega \in \mathbb{Q}_\ell\{X_1, \dots, X_n\}$  (resp.  $\omega \in \mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}$ ). We denote by  $L_\omega$  the left multiplication by  $\omega$  in the corresponding  $\mathbb{Q}_\ell$ -algebra. We denote by  $L_{\text{Lie}(V)}$  (resp.  $L_{L(V)}$ ) the set of left multiplications by elements of  $\text{Lie}(V)$  (resp.  $L(V)$ ). Observe that the semi-direct product

$$L_{\text{Lie}(V)} \tilde{\times} \text{Der}^* \text{Lie}(V) \subset \text{End}_{\mathbb{Q}_\ell\text{-linear}}(\mathbb{Q}_\ell\{X_1, \dots, X_n\}).$$

Notice that the Lie algebras  $\text{Lie}(V) \tilde{\times} \text{Der}^* \text{Lie}(V)$  and  $L_{\text{Lie}(V)} \tilde{\times} \text{Der}^* \text{Lie}(V)$  are obviously isomorphic. The same is true if we replace  $\text{Lie}(V)$  by  $L(V)$  and  $\mathbb{Q}_\ell\{X_1, \dots, X_n\}$  by  $\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}$ .

#### 1.4. Using the representations

$$(1.4.1) \quad G_K \rightarrow \text{Aut}_{\mathbb{Q}_\ell\text{-algebra}}(\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\})$$

and

$$\varphi_p : G_K \rightarrow \text{Aut}_{\mathbb{Q}_\ell\text{-linear}}(\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\})$$

we shall define filtrations of the Galois group  $G_K$ . We set

$$\begin{aligned} G_m &= G_m(V, v) \\ &:= \ker(\psi_m : G_K \rightarrow \text{Aut}_{\mathbb{Q}_\ell\text{-algebra}}(\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\} / I^{m+1})), \end{aligned}$$

where  $I$  is the augmentation ideal of the  $\mathbb{Q}_\ell$ -algebra  $\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}$  and  $\psi_m$  is induced by the action (1.4.1) of  $G_K$  on the  $\mathbb{Q}_\ell$ -algebra  $\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}$ .

We set

$$\begin{aligned} H_m &= H_m(V, z, v) \\ &:= \ker(\varphi_{p,m} : G_m \rightarrow \text{Aut}_{\mathbb{Q}_\ell\text{-linear}}(\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\} / I^m)), \end{aligned}$$

where  $\varphi_{p,m}$  is induced by  $\varphi_p$ .

We set

$$G_\infty := \bigcap_{m=1}^\infty G_m \text{ and } H_\infty := \bigcap_{m=1}^\infty H_m.$$

2. LIE ALGEBRAS OF ACTIONS OF GALOIS GROUPS ON TORSORS.

**2.1.** We have seen in section 1 that the action of  $G_K$  on the torsor  $\pi(V_{\bar{K}}, z, v)$  leads to the Galois representation

$$\varphi_p : G_K \rightarrow \text{Aut}(\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}),$$

where  $\varphi_p(\sigma)(\omega) = \Lambda_p(\sigma) \cdot \sigma(\omega)$ . It is shown in [W1] Lemma 5.1.7 that for  $\sigma \in \text{Gal}(\bar{K}/K(\mu_\ell^\infty))$ .

$$(2.1.1) \quad \log \varphi_p(\sigma) = L_{\log \varphi_p(\sigma)(1)} + \log \sigma.$$

Moreover we have

$$(2.1.2) \quad (\log \sigma)(X_i) = [X_i, \log \varphi_{\gamma_i}(\sigma)(1)]$$

for  $i = 1, \dots, n$  (see [W1] Proposition 5.1.8). Passing with the representation  $\varphi_p$  to Lie algebras we get a homomorphism of Lie algebras

$$\text{Lie}\varphi_p : \text{Lie}(H_1/H_\infty \otimes \mathbb{Q}) \rightarrow \text{End}_{\mathbb{Q}_\ell\text{-linear}}(\mathbb{Q}_\ell\{\{X_1, \dots, X_n\}\}).$$

It follows from (2.1.1) and (2.1.2) that  $\text{Lie}\varphi_p$  factors through

$$\text{Lie}\varphi_p : \text{Lie}(H_1/H_\infty \otimes \mathbb{Q}) \rightarrow L_{L(V)} \tilde{\times} \text{Der}^* L(V).$$

We recall that we have a canonical isomorphism

$$L_{L(V)} \tilde{\times} \text{Der}^* L(V) \approx L(V) \tilde{\times} \left( \bigoplus_{i=1}^n (L(V)/\langle X_i \rangle), \{ \} \right).$$

Let  $\sigma \in \text{Gal}(\bar{K}/K(\mu_\ell^\infty))$ . We shall calculate coordinates of  $(\text{Lie} \varphi_p)(\sigma)$  in  $L(V) \tilde{\times} \left( \bigoplus_{i=1}^n (L(V)/\langle X_i \rangle), \{ \} \right)$ .

**Lemma 2.1.3.** Let  $\sigma \in \text{Gal}(\bar{K}/K(\mu_\ell^\infty))$ . Then

$$(\text{Lie}\varphi_p)(\sigma) = (\log \varphi_p(\sigma)(1), (\log \varphi_{\gamma_i}(\sigma)(1))_{i \in \underline{n}}).$$

*Proof.* The lemma follows from (2.1.1) and (2.1.2). □

We pass with the morphism  $\text{Lie}\varphi_p$  to associated graded Lie algebras. Then we get a morphism

$$\text{grLie}\varphi_p : \text{grLie}(H_1/H_\infty \otimes \mathbb{Q}) \rightarrow L_{\text{Lie}(V)} \tilde{\times} \text{Der}^* \text{Lie}(V).$$

Let us set

$$\phi_p := \text{grLie}\varphi_p.$$

**Lemma 2.1.4.** Let  $\sigma \in H_n$ . Then

- i)  $\log \varphi_p(\sigma)(1) \equiv \log \Lambda_p(\sigma) \pmod{\Gamma^{n+1}\text{Lie}(V)}$ ,
- ii) the class of  $\log \Lambda_p(\sigma) \pmod{\Gamma^{n+1}\text{Lie}(V)}$  does not depend on a choice of a path  $p$  from  $v$  to  $z$ .

*Proof.* The lemma is already proved in [W1]. □

Let  $\sigma \in H_n$ . We denote by  $\mathcal{L}(z, v)(\sigma)$  the class of  $\log \Lambda_p(\sigma) \pmod{\Gamma^{n+1}\text{Lie}(V)}$ .

Now we can calculate coordinates of  $\phi_p(\sigma)$  in  $L_{\text{Lie}(V)} \tilde{\times} \text{Der}^* \text{Lie}(V) \approx \text{Lie}(V) \tilde{\times} \left( \bigoplus_{i=1}^n (\text{Lie}(V) / \langle X_i \rangle), \{ \} \right)$ .

**Lemma 2.1.5.** Let  $\sigma \in H_n$ . Then

$$\begin{aligned} \phi_p(\sigma) &= (\mathcal{L}(z, v)(\sigma), (\mathcal{L}(v_i, v)(\sigma))_{i \in \mathbb{N}}) \\ &\text{in } \text{Lie}(V) \tilde{\times} \left( \bigoplus_{i=1}^n (\text{Lie}(V) / \langle X_i \rangle), \{ \} \right). \end{aligned}$$

*Proof.* The lemma follows from Lemmas 2.1.3 and 2.1.4. □

It follows from Lemma 2.1.5 that the morphism of Lie algebras

$$\phi_p : \text{grLie}(H_1/H_\infty \otimes \mathbb{Q}) \rightarrow L_{\text{Lie}(V)} \tilde{\times} \text{Der}^* \text{Lie}(V).$$

does not depend on a choice of a path  $p$  from  $v$  to  $z$ , hence we shall denote it by  $\phi_{z,v}$ .

We set

$$t_V(z, v) := \text{image}(\phi_{z,v}).$$

Observe that the Lie algebra  $t_V(v, v)$  is the associated graded Lie algebra of the image of  $\text{Gal}(\bar{K}/K(\mu_{\ell^\infty}))$  in  $\text{Aut}(\pi_1(V_{\bar{K}}, v))$ . This Lie algebra was studied in [W1] section 15. To indicate the importance of the Lie algebra  $t_V(v, v)$  we set

$$\delta_V(v) := t_V(v, v).$$

### 3. EXAMPLES.

Let  $V = P_{\mathbb{Q}}^1 \setminus \{0, 1, \infty\}$ . In the fundamental group  $\pi_1(V_{\mathbb{Q}}, \overrightarrow{01})$  we have two generators  $x$  - loop around 0 and  $y$  - loop around 1. We embed  $\pi_1(V_{\mathbb{Q}}, \overrightarrow{01})$  into  $\mathbb{Q}_\ell \{ \{X, Y\} \}$  mapping  $x$  onto  $e^X$  and  $y$  onto  $e^Y$ .

**Proposition 3.1.** The Lie algebras  $\delta_V(\overrightarrow{01})$  and  $t_V(\overrightarrow{10}, \overrightarrow{01})$  are isomorphic.

*Proof.* It follows from Lemma 2.1.5 that

$$\phi_{\overrightarrow{01}, \overrightarrow{01}}(\sigma) = (0, (0, \mathcal{L}(\overrightarrow{10}, \overrightarrow{01})(\sigma)))$$

and

$$\phi_{\overrightarrow{10}, \overrightarrow{01}}(\sigma) = (\mathcal{L}(\overrightarrow{10}, \overrightarrow{01})(\sigma), (0, \mathcal{L}(\overrightarrow{10}, \overrightarrow{01})(\sigma)))$$

in  $\text{Lie}(V) \tilde{\times} ((\text{Lie}(V)/ \langle X \rangle) \oplus (\text{Lie}(V)/ \langle Y \rangle), \{ \})$ . It is clear that the map  $\delta_V(\overrightarrow{01}) \rightarrow t_V(\overrightarrow{10}, \overrightarrow{01})$  sending  $(0, (0, \mathcal{L}))$  to  $(\mathcal{L}, (0, \mathcal{L}))$  is an isomorphism of the corresponding Lie algebras.  $\square$

**Proposition 3.2.** The Lie algebra  $t_V(-1, \overrightarrow{01})$  contains a free Lie subalgebra on free generators in degree  $1, 3, 5, \dots, 2n + 1, \dots$ .

*Proof.* The proof is based on Deligne’s ideas indicated in [D]. It follows from Lemma 2.1.5 that

$$(3.2.1) \quad \phi_{-1, \overrightarrow{01}}(\sigma) = (\mathcal{L}(-1, \overrightarrow{01})(\sigma), (0, \mathcal{L}(\overrightarrow{10}, \overrightarrow{01})(\sigma)))$$

in  $\text{Lie}(V) \tilde{\times} ((\text{Lie}(V)/ \langle X \rangle) \oplus (\text{Lie}(V)/ \langle Y \rangle), \{ \})$ . Let  $I_n$  be a vector subspace of  $\text{Lie}(V)$  generated by Lie brackets of the Lie algebra  $\text{Lie}(V)$  which contain at least  $n$   $Y$ ’s. Let us set

$$\mathcal{I}_n := I_n \oplus (I_n \oplus I_n).$$

Observe that  $\mathcal{I}_n$  is a Lie ideal of the Lie algebra  $\text{Lie}(V) \tilde{\times} ((\text{Lie}(V)/ \langle X \rangle) \oplus (\text{Lie}(V)/ \langle Y \rangle), \{ \})$ .

Let  $n > 1$  and let  $\sigma \in H_n$ . It follows from the definition of  $\ell$ -adic polylogarithms in [W1] section 11 and from the definition of the filtration  $\{H_k\}_{k \in \mathbb{N}}$  of  $G_{\mathbb{Q}}$  that

$$(3.2.2) \quad \mathcal{L}(\overrightarrow{10}, \overrightarrow{01})(\sigma) \equiv \ell_n(\overrightarrow{10})(\sigma)[\dots[Y, X], X^{n-2}] \pmod{I_2 + \Gamma^{n+1}L(V)}$$

and

$$(3.2.3) \quad \mathcal{L}(-1, \overrightarrow{01})(\sigma) \equiv \ell_n(-1)(\sigma)[\dots[Y, X], X^{n-2}] \pmod{I_2 + \Gamma^{n+1}L(V)}.$$

It follows from the work of Soulé (see [S1] and [S2]) and the relation between  $\ell$ -adic polylogarithms and classes of Soulé (see [W1] Corollary 14.3.3 and also [NW] Remark 2 and [W2] Proposition 3.4) that  $\ell_{2n+1}(\overrightarrow{10}) \neq 0$  and  $\ell_{2n}(\overrightarrow{10}) = 0$ . In [W1] Corollary 11.2.3 and also in [W2] Theorem 2.1 we have proved the identity

$$2^{n-1}(\ell_n(-1) + \ell_n(1)) = \ell_n(1)$$



after the restriction to  $H_n$ . ( $\ell_n(1)$  denotes  $\ell_n(\overrightarrow{10})$ .) Hence we get that

$$(3.2.4) \quad \ell_n(1) = \frac{2^{n-1}}{1 - 2^{n-1}} \ell_n(-1)$$

for  $n > 1$ . This implies that  $\ell_{2n}(-1) = 0$ .

Let  $n > 1$  and let  $\sigma \in H_n$ . It follows from (3.2.1) - (3.2.4) that in the Lie algebra  $t_V(-1, \overrightarrow{01})$  there is an element of the form

$$(\ell_n(-1)(\sigma)[\cdot, [Y, X], X^{n-2}] + u_n, (0, \frac{2^{n-1}}{1 - 2^{n-1}} \ell_n(-1)(\sigma)[\cdot, [Y, X], X^{n-2}] + \omega_n))$$

where  $u_n, \omega_n \in I_2$ . Let us take  $\sigma \in H_{2n+1}$  such that  $\ell_{2n+1}(-1)(\sigma) \neq 0$ . Multiplying by  $(1 - 2^{2n})$  and dividing by  $\ell_{2n+1}(-1)(\sigma)$  we get an element of the form

$$z_{2n+1} := ((1 - 2^{2n})[\cdot, [Y, X], X^{2n-1}] + u_{2n+1}, (0, 2^{2n}[\cdot, [Y, X], X^{2n-1}] + w_{2n+1}))$$

$(u_{2n+1}, w_{2n+1} \in I_2)$  in the Lie algebra  $t_V(-1, \overrightarrow{01})$ .

Let  $n = 1$ . It follows from [W1] Proposition 11.0.8 that  $\ell_1(-1) = \ell(2)$ . The  $\ell$ -adic logarithm  $\ell(2)$  is the Kummer character associated to 2 (see [W1] Proposition 14.1.0.). Hence there is an element  $\sigma \in H_1$  such that  $\ell(2)(\sigma) \neq 0$ . Therefore we get that  $\mathcal{L}(\overrightarrow{10}, \overrightarrow{01})(\sigma) = 0$  and  $\mathcal{L}(-1, \overrightarrow{01})(\sigma) = \ell(2)(\sigma)Y$ . Hence the element

$$z_1 := (Y, (0, 0))$$

belongs to  $t_V(-1, \overrightarrow{01})$ .

Let us set  $t_{2n+1} = ((1 - 2^{2n})[\cdot, [Y, X], X^{2n-1}], (0, 2^{2n}[\cdot, [Y, X], X^{2n-1}]))$  for  $n > 1$  and  $t_1 = (Y, (0, 0))$ . Observe that for any Lie bracket of length  $r$  in the Lie algebra  $\text{Lie}(V) \tilde{\times} ((\text{Lie}(V) / \langle X \rangle) \oplus (\text{Lie}(V) / \langle Y \rangle), \{ \})$  we have

$$\{ \dots \{ z_{i_1}, z_{i_2} \} \dots z_{i_r} \} \equiv \{ \dots \{ t_{i_1}, t_{i_2} \} \dots, t_{i_r} \} \pmod{\mathcal{I}_{r+1}}$$

Let us set  $s_{2n+1} = [\cdot, [Y, X], X^{2n-1}]$  for  $n > 0$  and  $s_1 = Y$ . Notice that the elements  $t_1, t_3, \dots$  and  $s_1, s_2, \dots$  have integer coefficients. Observe that

$$\{ \dots \{ t_{i_1}, t_{i_2} \} \dots, t_{i_r} \} \equiv ([\dots [s_{i_1}, s_{i_2}] \dots, s_{i_r}], (0, 0)) \pmod{2},$$

where  $[ \ , \ ]$  is the standard Lie bracket in the free Lie algebra  $\text{Lie}(V)$ .

The Hall basic Lie elements in  $s_1, s_3, \dots, s_{2n+1}, \dots$  in the free Lie algebra  $\text{Lie}(V)$  are linearly independent. Hence the Hall basic Lie elements  $z_1, z_3, \dots, z_{2n+1}, \dots$  in the Lie algebra  $\text{Lie}(V) \tilde{\times} ((\text{Lie}(V) / \langle X \rangle) \oplus (\text{Lie}(V) / \langle Y \rangle), \{ \})$  are linearly independent. Hence the elements  $\overrightarrow{z_1}, \overrightarrow{z_3}, \dots, \overrightarrow{z_{2n+1}}, \dots$  are free generators of a free Lie subalgebra of  $t_V(-1, \overrightarrow{01})$ . □

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