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## Evaluation of the Convolution Sums $\sum_{1+24m=n} \sigma(1) \sigma(m)$ and $\sum_{3l+8m=n} \sigma(1) \sigma(m)$

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Math. J. Okayama Univ. **49** (2007), 93–111**EVALUATION OF THE CONVOLUTION SUMS**

$$\sum_{l+24m=n} \sigma(l)\sigma(m) \quad \text{AND} \quad \sum_{3l+8m=n} \sigma(l)\sigma(m)$$

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ABSTRACT. The convolution sums  $\sum_{l+24m=n} \sigma(l)\sigma(m)$  and  $\sum_{3l+8m=n} \sigma(l)\sigma(m)$  are evaluated for all  $n \in \mathbb{N}$ , and their evaluations used to determine the number of representations of a positive integer  $n$  by the form  $x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + 8(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2)$ .

## 1. INTRODUCTION

Let  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$  denote the sets of natural numbers, integers, real numbers, complex numbers respectively. For  $k, n \in \mathbb{N}$  we set

$$(1.1) \quad \sigma_k(n) = \sum_{d|n} d^k,$$

where  $d$  runs through the positive divisors of  $n$ . If  $n \notin \mathbb{N}$  we set  $\sigma_k(n) = 0$ . We write  $\sigma(n)$  for  $\sigma_1(n)$ . For  $a, b \in \mathbb{N}$  with  $a \leq b$  we define the convolution sum  $W_{a,b}(n)$  by

$$(1.2) \quad W_{a,b}(n) := \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ al + bm = n}} \sigma(l)\sigma(m).$$

Set  $g = \gcd(a, b)$ . Clearly

$$(1.3) \quad W_{a,b}(n) = \begin{cases} W_{a/g,b/g}(n/g), & \text{if } g \mid n, \\ 0, & \text{if } g \nmid n. \end{cases}$$

Hence we may suppose that  $\gcd(a, b) = 1$ . When  $a = 1$  and  $b = k \in \mathbb{N}$  we have

$$(1.4) \quad W_{1,k}(n) = \sum_{\substack{m \in \mathbb{N} \\ m < n/k}} \sigma(m)\sigma(n - km)$$

and we write  $W_k(n)$  for  $W_{1,k}(n)$ . The sum  $W_k(n)$  has been evaluated for  $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 12$  and  $16$ , see [1] for references. The sum  $W_{2,3}(n)$  was evaluated in [4] and the sum  $W_{3,4}(n)$  in [1]. In this paper we determine

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$W_{24}(n)$  and  $W_{3,8}(n)$ . These determinations are given in Theorem 2.1 in Section 2. The proof of Theorem 2.1 is given in Section 3. Some related convolution sums are evaluated in [5], [6].

For  $k, n \in \mathbb{N}$  we let

$$(1.5) \quad N_k(n) := \text{card}\{(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \mathbb{Z}^8 \mid n = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2 + k(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2)\}.$$

The values of  $N_1(n)$ ,  $N_2(n)$ ,  $N_3(n)$ ,  $N_4(n)$  and  $N_6(n)$  are known, see [10], [4], [13], [1] and [3], respectively. An elementary evaluation of  $N_1(n)$  is given in [9]. In Section 4 we use the evaluations of  $W_8(n)$  (see [14]),  $W_{24}(n)$  and  $W_{3,8}(n)$  to determine  $N_8(n)$ , see Theorem 2.2 in Section 2.

## 2. STATEMENTS OF THEOREMS 2.1 AND 2.2

We begin by defining the quantities  $c_{1,24}(n)$  and  $c_{3,8}(n)$  ( $n \in \mathbb{N}$ ), which will be central to everything that we do.

**Definition 2.1.** For  $n \in \mathbb{N}$  we define  $c_{1,24}(n)$  by

$$\begin{aligned} & 61 \sum_{n=1}^{\infty} c_{1,24}(n)q^n \\ &= 34q \prod_{n=1}^{\infty} (1 + q^n)(1 - q^{2n})(1 - q^{3n})^3(1 - q^{4n})^3(1 - q^{6n})(1 - q^{12n-6}) \\ (2.1) &+ 30q \prod_{n=1}^{\infty} (1 + q^n)^3(1 - q^{2n})^2(1 - q^{3n})(1 - q^{4n})^2(1 - q^{6n})^3(1 - q^{12n-6})^2 \\ &- 3q \prod_{n=1}^{\infty} (1 - q^{2n-1})^2(1 + q^{3n})^6(1 - q^{4n})^2(1 - q^{6n})^6(1 - q^{12n-6})^6 \\ &+ 4q^2 \prod_{n=1}^{\infty} (1 + q^n)(1 + q^{2n})^2(1 - q^{3n})^3(1 - q^{4n})^4(1 + q^{6n})(1 - q^{12n}) \\ &- 2q^2 \prod_{n=1}^{\infty} (1 + q^n)^2(1 - q^{2n})^3(1 + q^{3n})^2(1 - q^{4n})(1 - q^{6n})^3(1 - q^{12n}) \end{aligned}$$

and  $c_{3,8}(n)$  by

$$\sum_{n=1}^{\infty} c_{3,8}(n)q^n$$

$$\begin{aligned}
&= q \prod_{n=1}^{\infty} (1 - q^{2n-1})^2 (1 + q^{3n})^6 (1 - q^{4n})^2 (1 - q^{6n})^6 (1 - q^{12n-6})^6 \\
(2.2) &+ 2q^2 \prod_{n=1}^{\infty} (1 - q^n)^2 (1 + q^{2n})^5 (1 + q^{3n})^6 (1 - q^{6n})^6 (1 - q^{12n-6})^3 \\
&+ 42q^2 \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n})(1 + q^{3n})^3 (1 - q^{6n})^6 \\
&- 30q^2 \prod_{n=1}^{\infty} (1 - q^n)(1 - q^{2n})^3 (1 + q^{3n})^3 (1 - q^{4n-2})^2 (1 - q^{6n})^2 (1 - q^{12n})^2 \\
&+ 4q^3 \prod_{n=1}^{\infty} (1 + q^n)(1 - q^{4n})^2 (1 - q^{6n-3})^3 (1 - q^{12n})^6 \\
&- 52q^3 \prod_{n=1}^{\infty} (1 - q^{2n})^2 (1 - q^{4n-2})^2 (1 - q^{12n})^6.
\end{aligned}$$

Clearly the coefficients of  $q^n$  ( $n \in \mathbb{N}$ ) on the right hand sides in Definition 2.1 are integers. Hence

$$(2.3) \quad 61c_{1,24}(n) \in \mathbb{Z}, \quad c_{3,8}(n) \in \mathbb{Z} \quad (n \in \mathbb{N}).$$

The first thirty values of  $61c_{1,24}(n)$  and  $c_{3,8}(n)$  are given in the following table.

$n$	$61c_{1,24}(n)$	$c_{3,8}(n)$	$n$	$61c_{1,24}(n)$	$c_{3,8}(n)$
1	61	1	16	448	448
2	132	12	17	234	-2766
3	117	-63	18	1188	108
4	112	112	19	860	1100
5	6	126	20	672	672
6	-36	-396	21	1848	3288
7	-136	344	22	3024	-1296
8	-224	-224	23	3048	648
9	-291	-831	24	672	672
10	-648	1512	25	811	-7289
11	-348	-588	26	2136	3336
12	-336	-336	27	1173	-447
13	-322	2198	28	-1792	-1792
14	-672	-1632	29	-2130	9030
15	-618	-258	30	-4536	1944

We note that  $c_{1,24}(1) = c_{3,8}(1) = 1$ . The table suggests that  $61c_{1,24}(4n) = c_{3,8}(4n)$  for all  $n \in \mathbb{N}$ , and we prove this at the end of Section 4, see (4.11).

In Section 3 we prove the following theorem.

**Theorem 2.1.** *Let  $n \in \mathbb{N}$ . Then*

$$\begin{aligned} & \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+24m=n}} \sigma(l)\sigma(m) \\ &= \frac{1}{1920}\sigma_3(n) + \frac{1}{640}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{640}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{160}\sigma_3\left(\frac{n}{4}\right) \\ & \quad + \frac{9}{640}\sigma_3\left(\frac{n}{6}\right) + \frac{1}{30}\sigma_3\left(\frac{n}{8}\right) + \frac{9}{160}\sigma_3\left(\frac{n}{12}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{24}\right) \\ & \quad + \left(\frac{1}{24} - \frac{n}{96}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{24}\right) - \frac{61}{1920}c_{1,24}(n) \end{aligned}$$

and

$$\begin{aligned} & \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ 3l+8m=n}} \sigma(l)\sigma(m) \\ &= \frac{1}{1920}\sigma_3(n) + \frac{1}{640}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{640}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{160}\sigma_3\left(\frac{n}{4}\right) \\ & \quad + \frac{9}{640}\sigma_3\left(\frac{n}{6}\right) + \frac{1}{30}\sigma_3\left(\frac{n}{8}\right) + \frac{9}{160}\sigma_3\left(\frac{n}{12}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{24}\right) \\ & \quad + \left(\frac{1}{24} - \frac{n}{32}\right)\sigma\left(\frac{n}{3}\right) + \left(\frac{1}{24} - \frac{n}{12}\right)\sigma\left(\frac{n}{8}\right) - \frac{1}{1920}c_{3,8}(n), \end{aligned}$$

where  $c_{1,24}(n)$  and  $c_{3,8}(n)$  are defined in (2.1) and (2.2) respectively.

Making use of Theorem 2.1, we prove the following result in Section 4.

**Theorem 2.2.** *Let  $n \in \mathbb{N}$ . Then*

$$\begin{aligned} N_8(n) &= \frac{3}{10}\sigma_3(n) + \frac{9}{10}\sigma_3\left(\frac{n}{2}\right) + \frac{27}{10}\sigma_3\left(\frac{n}{3}\right) + \frac{18}{5}\sigma_3\left(\frac{n}{4}\right) \\ & \quad + \frac{81}{10}\sigma_3\left(\frac{n}{6}\right) + \frac{96}{5}\sigma_3\left(\frac{n}{8}\right) + \frac{162}{5}\sigma_3\left(\frac{n}{12}\right) + \frac{864}{5}\sigma_3\left(\frac{n}{24}\right) \\ & \quad - \frac{9}{4}c_{1,8}(n) - \frac{81}{4}c_{1,8}\left(\frac{n}{3}\right) + \frac{549}{40}c_{1,24}(n) + \frac{9}{40}c_{3,8}(n), \end{aligned}$$

where  $c_{1,8}(n)$  is given by

$$(2.4) \quad \sum_{n=1}^{\infty} c_{1,8}(n)q^n = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4.$$

### 3. PROOF OF THEOREM 2.1

The Eisenstein series  $L(q)$ ,  $M(q)$  and  $N(q)$  are defined by

$$(3.1) \quad L(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n, \quad q \in \mathbb{C}, \quad |q| < 1,$$

$$(3.2) \quad M(q) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \quad q \in \mathbb{C}, \quad |q| < 1,$$

$$(3.3) \quad N(q) := 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n, \quad q \in \mathbb{C}, \quad |q| < 1,$$

see for example [11, eqn. (25)], [12, p. 140]. Ramanujan's discriminant function  $\Delta(q)$  is given by

$$(3.4) \quad \Delta(q) := q \prod_{n=1}^{\infty} (1 - q^n)^{24} = \frac{1}{1728} (M(q)^3 - N(q)^2),$$

see for example [11, eqn. (44)], [12, p. 144]. The Jacobi theta function  $\varphi(q)$  is defined by

$$(3.5) \quad \varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}, \quad q \in \mathbb{C}, \quad |q| < 1.$$

Set

$$(3.6) \quad p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}$$

and

$$(3.7) \quad k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}.$$

It is shown in [4, eqn. (3.84), p. 501] that

$$(3.8) \quad L(q) - 2L(q^2) = -(1 + 14p + 24p^2 + 14p^3 + p^4)k^2$$

and in [1, eqn. (3.12), p. 34] that

$$(3.9) \quad L(q) - 12L(q^{12}) = -(11 + 34p + 36p^2 + 16p^3 + 2p^4)k^2.$$

The following result is proved in [2, Theorem 9, p. 179].

**Duplication principle.**

$$p(q^2) = \frac{1 + p - p^2 - ((1 - p)(1 + p)(1 + 2p))^{1/2}}{p^2},$$

$$k(q^2) = \frac{(1 + p - p^2 + ((1 - p)(1 + p)(1 + 2p))^{1/2})k}{2}.$$

Applying the duplication principle to (3.9), we obtain

$$(3.10) \quad \begin{aligned} L(q^2) - 12L(q^{24}) &= -\frac{1}{2} \left( 13 + 26p + 12p^2 - p^3 - \frac{1}{2}p^4 \right) k^2 \\ &\quad - \frac{9}{2} (1 + p) ((1 - p)(1 + p)(1 + 2p))^{1/2} k^2. \end{aligned}$$

Appealing to (3.8), (3.10) and the trivial identity

$$L(q) - 24L(q^{24}) = (L(q) - 2L(q^2)) + 2(L(q^2) - 12L(q^{24})),$$

we obtain

$$(3.11) \quad \begin{aligned} L(q) - 24L(q^{24}) &= - \left( 14 + 40p + 36p^2 + 13p^3 + \frac{1}{2}p^4 \right) k^2 \\ &\quad + 9(1 + p) ((1 - p)(1 + p)(1 + 2p))^{1/2} k^2. \end{aligned}$$

Next applying the duplication principle to (3.8) we obtain

$$(3.12) \quad L(q^2) - 2L(q^4) = - \left( 1 + 2p + 6p^2 + 5p^3 - \frac{1}{2}p^4 \right) k^2,$$

see [1, eqn. (3.40), p. 39]. Then, from (3.8) and (3.12), we obtain

$$(3.13) \quad L(q) - 4L(q^4) = -(3 + 18p + 36p^2 + 24p^3)k^2.$$

Applying duplication to (3.13), we have

$$(3.14) \quad \begin{aligned} L(q^2) - 4L(q^8) &= - \left( \frac{3}{2} + 3p + 9p^2 + \frac{15}{2}p^3 - \frac{3}{4}p^4 \right) k^2 \\ &\quad - \left( \frac{3}{2} + \frac{3}{2}p - 3p^2 \right) ((1 - p)(1 + p)(1 + 2p))^{1/2} k^2. \end{aligned}$$

Recall from [4, eqn. (3.87), p. 502], [1, eqn. (3.9), p. 33] that

$$(3.15) \quad L(q) - 3L(q^3) = -(2 + 16p + 36p^2 + 16p^3 + 2p^4)k^2.$$

Then, using the simple identity

$$3L(q^3) - 8L(q^8) = (L(q) - 2L(q^2)) + 2(L(q^2) - 4L(q^8)) - (L(q) - 3L(q^3)),$$

we obtain

$$(3.16) \quad \begin{aligned} 3L(q^3) - 8L(q^8) &= -\left(2 + 4p + 6p^2 + 13p^3 - \frac{5}{2}p^4\right)k^2 \\ &\quad - (3 + 3p - 6p^2)((1-p)(1+p)(1+2p))^{1/2}k^2. \end{aligned}$$

Squaring (3.11) and (3.16), we deduce the following result.

**Lemma 3.1.**

$$(a) \quad \begin{aligned} (L(q) - 24L(q^{24}))^2 &= \left(277 + 1444p + 2932p^2 + 3082p^3 + 1945p^4 \right. \\ &\quad \left. + 814p^5 + 205p^6 + 13p^7 + \frac{1}{4}p^8\right)k^4 \\ &\quad + (252 + 972p + 1368p^2 + 882p^3 \\ &\quad \left. + 243p^4 + 9p^5)((1-p)(1+p)(1+2p))^{1/2}k^4. \end{aligned}$$

$$(b) \quad \begin{aligned} (3L(q^3) - 8L(q^8))^2 &= \left(13 + 52p + 40p^2 - 26p^3 + 85p^4 \right. \\ &\quad \left. + 298p^5 + 175p^6 - 137p^7 + \frac{25}{4}p^8\right)k^4 \\ &\quad + (12 + 36p + 36p^2 + 66p^3 - 9p^4 \\ &\quad \left. - 171p^5 + 30p^6)((1-p)(1+p)(1+2p))^{1/2}k^4. \end{aligned}$$

From [1, eqns. (3.14)-(3.19), p. 34] we have

$$(3.17) \quad \begin{aligned} M(q) &= (1 + 124p + 964p^2 + 2788p^3 + 3910p^4 \\ &\quad + 2788p^5 + 964p^6 + 124p^7 + p^8)k^4, \end{aligned}$$

$$(3.18) \quad \begin{aligned} M(q^2) &= (1 + 4p + 64p^2 + 178p^3 + 235p^4 \\ &\quad + 178p^5 + 64p^6 + 4p^7 + p^8)k^4, \end{aligned}$$

$$(3.19) \quad \begin{aligned} M(q^3) &= (1 + 4p + 4p^2 + 28p^3 + 70p^4 \\ &\quad + 28p^5 + 4p^6 + 4p^7 + p^8)k^4, \end{aligned}$$

$$(3.20) \quad \begin{aligned} M(q^4) &= \left(1 + 4p + 4p^2 - 2p^3 + 10p^4 + 28p^5 \right. \\ &\quad \left. + \frac{31}{4}p^6 - \frac{29}{4}p^7 + \frac{1}{16}p^8\right)k^4, \end{aligned}$$

$$(3.21) \quad M(q^6) = (1 + 4p + 4p^2 - 2p^3 - 5p^4$$



$$(3.22) \quad \begin{aligned} & -2p^5 + 4p^6 + 4p^7 + p^8)k^4, \\ M(q^{12}) &= \left(1 + 4p + 4p^2 - 2p^3 - 5p^4 - 2p^5 \right. \\ & \left. + \frac{1}{4}p^6 + \frac{1}{4}p^7 + \frac{1}{16}p^8\right)k^4. \end{aligned}$$

Applying the duplication principle to (3.20), we obtain

$$(3.23) \quad \begin{aligned} M(q^8) &= \left(\frac{17}{32} + \frac{17}{8}p + \frac{1}{4}p^2 - \frac{107}{16}p^3 - \frac{55}{32}p^4 \right. \\ & \left. + \frac{163}{16}p^5 + \frac{151}{64}p^6 - \frac{269}{64}p^7 + \frac{1}{256}p^8\right)k^4 \\ & + \left(\frac{15}{32} + \frac{45}{32}p + \frac{45}{16}p^2 + \frac{105}{32}p^3 - \frac{225}{64}p^4 \right. \\ & \left. - \frac{315}{64}p^5 + \frac{15}{32}p^6\right)((1-p)(1+p)(1+2p))^{1/2}k^4. \end{aligned}$$

Applying the duplication principle to (3.22), we obtain

$$(3.24) \quad \begin{aligned} M(q^{24}) &= \left(\frac{17}{32} + \frac{17}{8}p + \frac{17}{8}p^2 - \frac{17}{16}p^3 - \frac{85}{32}p^4 \right. \\ & \left. - \frac{17}{16}p^5 + \frac{1}{64}p^6 + \frac{1}{64}p^7 + \frac{1}{256}p^8\right)k^4 \\ & + \left(\frac{15}{32} + \frac{45}{32}p + \frac{15}{16}p^2 - \frac{15}{32}p^3 - \frac{45}{64}p^4 \right. \\ & \left. - \frac{15}{64}p^5\right)((1-p)(1+p)(1+2p))^{1/2}k^4. \end{aligned}$$

From (3.17)-(3.24) we deduce the following result.

**Lemma 3.2.**

$$(a) \quad \begin{aligned} & \frac{47}{50}M(q) - \frac{9}{50}M(q^2) - \frac{27}{50}M(q^3) - \frac{18}{25}M(q^4) \\ & - \frac{81}{50}M(q^6) - \frac{96}{25}M(q^8) - \frac{162}{25}M(q^{12}) + \frac{13536}{25}M(q^{24}) \\ & = \left(277 + \frac{6104}{5}p + \frac{10034}{5}p^2 + \frac{10208}{5}p^3 + 2197p^4 \right. \\ & \left. + \frac{9776}{5}p^5 + \frac{4391}{5}p^6 + \frac{677}{5}p^7 + \frac{1}{4}p^8\right)k^4 \end{aligned}$$

$$\begin{aligned}
& + \left( 252 + 756p + \frac{2484}{5}p^2 - \frac{1332}{5}p^3 - \frac{1836}{5}p^4 \right. \\
& \quad \left. - 108p^5 - \frac{9}{5}p^6 \right) ((1-p)(1+p)(1+2p))^{1/2} k^4. \\
\text{(b)} \quad & - \frac{3}{50}M(q) - \frac{9}{50}M(q^2) + \frac{423}{50}M(q^3) - \frac{18}{25}M(q^4) \\
& - \frac{81}{50}M(q^6) + \frac{1504}{25}M(q^8) - \frac{162}{25}M(q^{12}) - \frac{864}{25}M(q^{24}) \\
& = \left( 13 + \frac{224}{5}p - \frac{646}{5}p^2 - \frac{1552}{5}p^3 + 337p^4 \right. \\
& \quad \left. + \frac{3416}{5}p^5 + \frac{461}{5}p^6 - \frac{1153}{5}p^7 + \frac{25}{4}p^8 \right) k^4 \\
& + \left( 12 + 36p + \frac{684}{5}p^2 + \frac{1068}{5}p^3 - \frac{936}{5}p^4 \right. \\
& \quad \left. - 288p^5 + \frac{141}{5}p^6 \right) ((1-p)(1+p)(1+2p))^{1/2} k^4.
\end{aligned}$$

From Lemmas 3.1 and 3.2 we obtain

**Lemma 3.3.**

$$\begin{aligned}
\text{(a)} \quad & (L(q) - 24L(q^{24}))^2 \\
& - \left( \frac{47}{50}M(q) - \frac{9}{50}M(q^2) - \frac{27}{50}M(q^3) - \frac{18}{25}M(q^4) \right. \\
& \quad \left. - \frac{81}{50}M(q^6) - \frac{96}{25}M(q^8) - \frac{162}{25}M(q^{12}) + \frac{13536}{25}M(q^{24}) \right) \\
& = \frac{18}{5}p(1-p)(1+p)(1+2p)(2+p)(31+51p+17p^2)k^4 \\
& \quad + \frac{9}{5}p(2+p)^2(30+91p+61p^2+p^3)((1-p)(1+p)(1+2p))^{1/2}k^4.
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad & (3L(q^3) - 8L(q^8))^2 \\
& - \left( -\frac{3}{50}M(q) - \frac{9}{50}M(q^2) + \frac{423}{50}M(q^3) - \frac{18}{25}M(q^4) \right. \\
& \quad \left. - \frac{81}{50}M(q^6) + \frac{1504}{25}M(q^8) - \frac{162}{25}M(q^{12}) - \frac{864}{25}M(q^{24}) \right)
\end{aligned}$$

$$\begin{aligned}
 &= \frac{18}{5}p(1-p)(1+p)(1+2p)(2+p)(1+21p-13p^2)k^4. \\
 &\quad -\frac{9}{5}p^2(2+p)(28+27p-63p^2-p^3)((1-p)(1+p)(1+2p))^{1/2}k^4.
 \end{aligned}$$

Before proving the next lemma, we recall from [1, eqns. (3.28)-(3.33), p. 36]

$$(3.25) \quad \Delta(q) = \frac{1}{16}p(1-p)^{12}(1+p)^4(1+2p)^3(2+p)^3k^{12},$$

$$(3.26) \quad \Delta(q^2) = \frac{1}{256}p^2(1-p)^6(1+p)^2(1+2p)^6(2+p)^6k^{12},$$

$$(3.27) \quad \Delta(q^3) = \frac{1}{16}p^3(1-p)^4(1+p)^{12}(1+2p)(2+p)k^{12},$$

$$(3.28) \quad \Delta(q^4) = \frac{1}{65536}p^4(1-p)^3(1+p)(1+2p)^3(2+p)^{12}k^{12},$$

$$(3.29) \quad \Delta(q^6) = \frac{1}{256}p^6(1-p)^2(1+p)^6(1+2p)^2(2+p)^2k^{12},$$

$$(3.30) \quad \Delta(q^{12}) = \frac{1}{65536}p^{12}(1-p)(1+p)^3(1+2p)(2+p)^4k^{12}.$$

**Lemma 3.4.**

$$\begin{aligned}
 \text{(a)} \quad &\sum_{n=1}^{\infty} c_{1,24}(n)q^n \\
 &= \frac{1}{244}p(1-p)(1+p)(1+2p)(2+p)(31+51p+17p^2)k^4 \\
 &\quad + \frac{1}{488}p(2+p)^2(30+91p+61p^2+p^3)((1-p)(1+p)(1+2p))^{1/2}k^4.
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad &\sum_{n=1}^{\infty} c_{3,8}(n)q^n \\
 &= \frac{1}{4}p(1-p)(1+p)(1+2p)(2+p)(1+21p-13p^2)k^4 \\
 &\quad - \frac{1}{8}p^2(2+p)(28+27p-63p^2-p^3)((1-p)(1+p)(1+2p))^{1/2}k^4.
 \end{aligned}$$

*Proof.* We just prove part (a) as part (b) can be treated similarly.

We have

$$\begin{aligned} q \prod_{n=1}^{\infty} (1+q^n)(1-q^{2n})(1-q^{3n})^3(1-q^{4n})^3(1-q^{6n})(1-q^{12n-6}) \\ = q \prod_{n=1}^{\infty} (1-q^n)^{-1}(1-q^{2n})^2(1-q^{3n})^3(1-q^{4n})^3(1-q^{6n})^2(1-q^{12n})^{-1} \\ = \Delta(q)^{-1/24} \Delta(q^2)^{1/12} \Delta(q^3)^{1/8} \Delta(q^4)^{1/8} \Delta(q^6)^{1/12} \Delta(q^{12})^{-1/24} \\ = 2^{-3} p(1-p)(1+p)^2(1+2p)(2+p)^2 k^4. \end{aligned}$$

Similarly

$$\begin{aligned} q \prod_{n=1}^{\infty} (1+q^n)^3(1-q^{2n})^2(1-q^{3n})(1-q^{4n})^2(1-q^{6n})^3(1-q^{12n-6})^2 \\ = q \prod_{n=1}^{\infty} (1-q^n)^{-3}(1-q^{2n})^5(1-q^{3n})(1-q^{4n})^2(1-q^{6n})^5(1-q^{12n})^{-2} \\ = \Delta(q)^{-1/8} \Delta(q^2)^{5/24} \Delta(q^3)^{1/24} \Delta(q^4)^{1/12} \Delta(q^6)^{5/24} \Delta(q^{12})^{-1/12} \\ = 2^{-3} p(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{3/2}(2+p)^2 k^4, \end{aligned}$$

$$\begin{aligned} q \prod_{n=1}^{\infty} (1-q^{2n-1})^2(1+q^{3n})^6(1-q^{4n})^2(1-q^{6n})^6(1-q^{12n-6})^6 \\ = q \prod_{n=1}^{\infty} (1-q^n)^2(1-q^{2n})^{-2}(1-q^{3n})^{-6}(1-q^{4n})^2(1-q^{6n})^{18}(1-q^{12n})^{-6} \\ = \Delta(q)^{1/12} \Delta(q^2)^{-1/12} \Delta(q^3)^{-1/4} \Delta(q^4)^{1/12} \Delta(q^6)^{3/4} \Delta(q^{12})^{-1/4} \\ = 2^{-2} p(1-p)(1+p)(1+2p)(2+p) k^4, \end{aligned}$$

$$\begin{aligned} q^2 \prod_{n=1}^{\infty} (1+q^n)(1+q^{2n})^2(1-q^{3n})^3(1-q^{4n})^4(1+q^{6n})(1-q^{12n}) \\ = q^2 \prod_{n=1}^{\infty} (1-q^n)^{-1}(1-q^{2n})^{-1}(1-q^{3n})^3(1-q^{4n})^6(1-q^{6n})^{-1}(1-q^{12n})^2 \\ = \Delta(q)^{-1/24} \Delta(q^2)^{-1/24} \Delta(q^3)^{1/8} \Delta(q^4)^{1/4} \Delta(q^6)^{-1/24} \Delta(q^{12})^{1/12} \\ = 2^{-5} p^2(1-p)^{1/2}(1+p)^{3/2}(1+2p)^{1/2}(2+p)^3 k^4, \end{aligned}$$

and

$$\begin{aligned} & q^2 \prod_{n=1}^{\infty} (1 + q^n)^2 (1 - q^{2n})^3 (1 + q^{3n})^2 (1 - q^{4n}) (1 - q^{6n})^3 (1 - q^{12n}) \\ &= q^2 \prod_{n=1}^{\infty} (1 - q^n)^{-2} (1 - q^{2n})^5 (1 - q^{3n})^{-2} (1 - q^{4n}) (1 - q^{6n})^5 (1 - q^{12n}) \\ &= \Delta(q)^{-1/12} \Delta(q^2)^{5/24} \Delta(q^3)^{-1/12} \Delta(q^4)^{1/24} \Delta(q^6)^{5/24} \Delta(q^{12})^{1/24} \\ &= 2^{-4} p^2 (1 - p)^{1/2} (1 + p)^{1/2} (1 + 2p)^{3/2} (2 + p)^2 k^4. \end{aligned}$$

Thus, appealing to Definition 2.1, we obtain

$$\begin{aligned} & 61 \sum_{n=1}^{\infty} c_{1,24}(n) q^n \\ &= 34 \cdot 2^{-3} p (1 - p) (1 + p)^2 (1 + 2p) (2 + p)^2 k^4 \\ &\quad + 30 \cdot 2^{-3} p (1 - p)^{1/2} (1 + p)^{3/2} (1 + 2p)^{3/2} (2 + p)^2 k^4 \\ &\quad - 3 \cdot 2^{-2} p (1 - p) (1 + p) (1 + 2p) (2 + p) k^4 \\ &\quad + 4 \cdot 2^{-5} p^2 (1 - p)^{1/2} (1 + p)^{3/2} (1 + 2p)^{1/2} (2 + p)^3 k^4 \\ &\quad - 2 \cdot 2^{-4} p^2 (1 - p)^{1/2} (1 + p)^{1/2} (1 + 2p)^{3/2} (2 + p)^2 k^4 \\ &= \frac{1}{4} p (1 - p) (1 + p) (1 + 2p) (2 + p) (31 + 51p + 17p^2) k^4 \\ &\quad + \frac{1}{8} p (2 + p)^2 (30 + 91p + 61p^2 + p^3) ((1 - p) (1 + p) (1 + 2p))^{1/2} k^4. \end{aligned}$$

Dividing both sides by 61, we obtain the assertion of part (a). □

From Lemmas 3.3 and 3.4 we obtain

**Lemma 3.5.**

$$\begin{aligned} \text{(a)} \quad & (L(q) - 24L(q^{24}))^2 \\ &= 529 + \frac{1}{5} \sum_{n=1}^{\infty} (1128\sigma_3(n) - 216\sigma_3\left(\frac{n}{2}\right) - 648\sigma_3\left(\frac{n}{3}\right) \\ &\quad - 864\sigma_3\left(\frac{n}{4}\right) - 1944\sigma_3\left(\frac{n}{6}\right) - 4608\sigma_3\left(\frac{n}{8}\right) \\ &\quad - 7776\sigma_3\left(\frac{n}{12}\right) + 649728\sigma_3\left(\frac{n}{24}\right) + 4392c_{1,24}(n)) q^n. \end{aligned}$$

$$\text{(b)} \quad (3L(q^3) - 8L(q^8))^2$$

$$\begin{aligned}
&= 25 + \frac{1}{5} \sum_{n=1}^{\infty} (-72\sigma_3(n) - 216\sigma_3\left(\frac{n}{2}\right) + 10152\sigma_3\left(\frac{n}{3}\right) \\
&\quad - 864\sigma_3\left(\frac{n}{4}\right) - 1944\sigma_3\left(\frac{n}{6}\right) + 72192\sigma_3\left(\frac{n}{8}\right) \\
&\quad - 7776\sigma_3\left(\frac{n}{12}\right) - 41472\sigma_3\left(\frac{n}{24}\right) + 72c_{3,8}(n))q^n.
\end{aligned}$$

*Proof.* We just prove part (b) as part (a) can be proved similarly. By Lemma 3.3(b) and Lemma 3.4(b) we have

$$\begin{aligned}
&(3L(q^3) - 8L(q^8))^2 \\
&= -\frac{3}{50}M(q) - \frac{9}{50}M(q^2) + \frac{423}{50}M(q^3) - \frac{18}{25}M(q^4) \\
&\quad - \frac{81}{50}M(q^6) + \frac{1504}{25}M(q^8) - \frac{162}{25}M(q^{12}) - \frac{864}{25}M(q^{24}) \\
&\quad + \frac{72}{5} \sum_{n=1}^{\infty} c_{3,8}(n)q^n.
\end{aligned}$$

Then, appealing to (3.2), we obtain the asserted formula of part (b).  $\square$

We are now ready to prove Theorem 2.1.

*Proof of Theorem 2.1.* We just prove the first identity as the second identity can be treated similarly.

We begin by recalling the classical identity

$$(3.31) \quad L(q)^2 = 1 + \sum_{n=1}^{\infty} (240\sigma_3(n) - 288n\sigma(n))q^n,$$

see for example [7], [8]. Mapping  $q \rightarrow q^{24}$  in (3.31), we obtain

$$(3.32) \quad L(q^{24})^2 = 1 + \sum_{n=1}^{\infty} \left( 240\sigma_3\left(\frac{n}{24}\right) - 12n\sigma\left(\frac{n}{24}\right) \right) q^n.$$

Also

$$\begin{aligned}
(3.33) \quad L(q)L(q^{24}) &= \left( 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n \right) \left( 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^{24n} \right) \\
&= 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n - 24 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{24}\right)q^n \\
&\quad + 576 \sum_{n=1}^{\infty} W_{24}(n)q^n.
\end{aligned}$$

Thus, from (3.31)–(3.33), we have

$$\begin{aligned}
 & (L(q) - 24L(q^{24}))^2 \\
 (3.34) \quad & = 529 + \sum_{n=1}^{\infty} (240\sigma_3(n) + 138240\sigma_3\left(\frac{n}{24}\right) \\
 & \quad + 1152\left(1 - \frac{n}{4}\right)\sigma(n) + 1152(1 - 6n)\sigma\left(\frac{n}{24}\right) \\
 & \quad - 27648W_{24}(n))q^n.
 \end{aligned}$$

Equating the coefficients of  $q^n$  ( $n \in \mathbb{N}$ ) on the right hand sides of Lemma 3.5(a) and (3.34), we obtain

$$\begin{aligned}
 & \frac{1}{5} \left( 1128\sigma_3(n) - 216\sigma_3\left(\frac{n}{2}\right) - 648\sigma_3\left(\frac{n}{3}\right) - 864\sigma_3\left(\frac{n}{4}\right) \right. \\
 & \quad \left. - 1944\sigma_3\left(\frac{n}{6}\right) - 4608\sigma_3\left(\frac{n}{8}\right) - 7776\sigma_3\left(\frac{n}{12}\right) \right. \\
 (3.35) \quad & \quad \left. + 649728\sigma_3\left(\frac{n}{24}\right) + 4392c_{1,24}(n) \right) \\
 & = 240\sigma_3(n) + 138240\sigma_3\left(\frac{n}{24}\right) \\
 & \quad + 1152\left(1 - \frac{n}{4}\right)\sigma(n) + 1152(1 - 6n)\sigma\left(\frac{n}{24}\right) \\
 & \quad - 27648W_{24}(n).
 \end{aligned}$$

Solving (3.35) for  $W_{24}(n)$  we obtain the first identity.

#### 4. PROOF OF THEOREM 2.2

Let  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . For  $l \in \mathbb{N}_0$  we set

$$(4.1) \quad r(l) = \text{card}\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid l = x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_3x_4 + x_4^2\}$$

so that  $r(0) = 1$ . It is known that [9, Theorem 13, p. 266], [10, p. 12]

$$(4.2) \quad r(l) = 12\sigma(l) - 36\sigma\left(\frac{l}{3}\right), \quad l \in \mathbb{N}.$$

By (1.5) and (4.1) we have

$$\begin{aligned}
 N_8(n) & = \sum_{\substack{(l,m) \in \mathbb{N}_0^2 \\ l+8m=n}} r(l)r(m) \\
 & = r(n)r(0) + r(0)r\left(\frac{n}{8}\right) + \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+8m=n}} r(l)r(m).
 \end{aligned}$$

Thus

$$\begin{aligned}
 N_8(n) &= \left(12\sigma(n) - 36\sigma\left(\frac{n}{3}\right) + 12\sigma\left(\frac{n}{8}\right) - 36\sigma\left(\frac{n}{24}\right)\right) \\
 (4.3) \quad &+ 144 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+8m=n}} \sigma(l)\sigma(m) - 432 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+8m=n}} \sigma\left(\frac{l}{3}\right)\sigma(m) \\
 &- 432 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+8m=n}} \sigma(l)\sigma\left(\frac{m}{3}\right) + 1296 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+8m=n}} \sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right).
 \end{aligned}$$

By a result of Williams [14, Theorem 1] the first sum is

$$\begin{aligned}
 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+8m=n}} \sigma(l)\sigma(m) &= \sum_{m < n/8} \sigma(m)\sigma(n-8m) \\
 (4.4) \quad &= \frac{1}{192}\sigma_3(n) + \frac{1}{64}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{16}\sigma_3\left(\frac{n}{4}\right) + \frac{1}{3}\sigma_3\left(\frac{n}{8}\right) \\
 &+ \left(\frac{1}{24} - \frac{n}{32}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{8}\right) - \frac{1}{64}c_{1,8}(n),
 \end{aligned}$$

where  $c_{1,8}(n)$  is defined in (2.4).

By Theorem 2.1 the second sum is

$$\begin{aligned}
 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+8m=n}} \sigma\left(\frac{l}{3}\right)\sigma(m) &= \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ 3l+8m=n}} \sigma(l)\sigma(m) \\
 (4.5) \quad &= \frac{1}{1920}\sigma_3(n) + \frac{1}{640}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{640}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{160}\sigma_3\left(\frac{n}{4}\right) \\
 &+ \frac{9}{640}\sigma_3\left(\frac{n}{6}\right) + \frac{1}{30}\sigma_3\left(\frac{n}{8}\right) + \frac{9}{160}\sigma_3\left(\frac{n}{12}\right) + \frac{3}{10}\sigma_3\left(\frac{n}{24}\right) \\
 &+ \left(\frac{1}{24} - \frac{n}{32}\right)\sigma\left(\frac{n}{3}\right) + \left(\frac{1}{24} - \frac{n}{12}\right)\sigma\left(\frac{n}{8}\right) - \frac{1}{1920}c_{3,8}(n).
 \end{aligned}$$

By Theorem 2.1 the third sum is

$$\begin{aligned}
 \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+8m=n}} \sigma(l)\sigma\left(\frac{m}{3}\right) &= \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+24m=n}} \sigma(l)\sigma(m) \\
 (4.6) \quad &= \frac{1}{1920}\sigma_3(n) + \frac{1}{640}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{640}\sigma_3\left(\frac{n}{3}\right) + \frac{1}{160}\sigma_3\left(\frac{n}{4}\right)
 \end{aligned}$$



$$\begin{aligned}
 & + \frac{9}{640} \sigma_3 \left( \frac{n}{6} \right) + \frac{1}{30} \sigma_3 \left( \frac{n}{8} \right) + \frac{9}{160} \sigma_3 \left( \frac{n}{12} \right) + \frac{3}{10} \sigma_3 \left( \frac{n}{24} \right) \\
 & + \left( \frac{1}{24} - \frac{n}{96} \right) \sigma(n) + \left( \frac{1}{24} - \frac{n}{4} \right) \sigma \left( \frac{n}{24} \right) - \frac{61}{1920} c_{1,24}(n).
 \end{aligned}$$

From (4.4) with  $n$  replaced by  $n/3$ , the fourth sum is

$$\begin{aligned}
 (4.7) \quad & \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+8m=n}} \sigma \left( \frac{l}{3} \right) \sigma \left( \frac{m}{3} \right) = \sum_{\substack{(l,m) \in \mathbb{N}^2 \\ l+8m=n/3}} \sigma(l) \sigma(m) \\
 & = \frac{1}{192} \sigma_3 \left( \frac{n}{3} \right) + \frac{1}{64} \sigma_3 \left( \frac{n}{6} \right) + \frac{1}{16} \sigma_3 \left( \frac{n}{12} \right) + \frac{1}{3} \sigma_3 \left( \frac{n}{24} \right) \\
 & + \left( \frac{1}{24} - \frac{n}{96} \right) \sigma \left( \frac{n}{3} \right) + \left( \frac{1}{24} - \frac{n}{12} \right) \sigma \left( \frac{n}{24} \right) - \frac{1}{64} c_{1,8} \left( \frac{n}{3} \right).
 \end{aligned}$$

Using the evaluations (4.4), (4.5), (4.6) and (4.7) in the formula (4.3) for  $N_8(n)$ , we obtain after some simplification the assertion of Theorem 2.2.  $\square$

Using the simple identity

$$\sigma(3n) = 4\sigma(n) - 3\sigma \left( \frac{n}{3} \right), \quad n \in \mathbb{N},$$

it is easy to show that

$$W_{24}(3n) + 3W_{3,8}(n) = 4W_8(n)$$

and

$$3W_{24}(n) + W_{3,8}(3n) = 4W_8(n).$$

Appealing to Theorem 2.1 and (4.4), we obtain after some calculation

$$\begin{aligned}
 (4.8) \quad c_{1,8}(n) & = \frac{61}{120} c_{1,24}(3n) + \frac{1}{40} c_{3,8}(n) \\
 & = \frac{61}{40} c_{1,24}(n) + \frac{1}{120} c_{3,8}(3n), \quad n \in \mathbb{N}.
 \end{aligned}$$

Next using the easily proved identity

$$\sigma(4n) = 7\sigma(n) - 6\sigma \left( \frac{n}{2} \right), \quad n \in \mathbb{N},$$

we can show that

$$W_{24}(4n) = 7W_6(n) - 6W_3 \left( \frac{n}{2} \right)$$

and

$$W_{3,8}(4n) = 7W_{2,3}(n) - 6W_3 \left( \frac{n}{2} \right).$$

Appealing to [4, Theorem 1, p. 493] for the evaluations of  $W_6(n)$  and  $W_{2,3}(n)$ , and to [9, Theorem 3, p. 248] for the evaluation of  $W_3(n)$ , we obtain after some calculation

$$(4.9) \quad c_{1,6}(n) = \frac{61}{112} c_{1,24}(4n) = \frac{1}{112} c_{3,8}(4n),$$

where

$$(4.10) \quad \sum_{n=1}^{\infty} c_{1,6}(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^2 (1 - q^{2n})^2 (1 - q^{3n})^2 (1 - q^{6n})^2.$$

This establishes that

$$(4.11) \quad 61c_{1,24}(n) = c_{3,8}(n), \text{ if } n \equiv 0 \pmod{4},$$

which was asserted in Section 2.

It was proved in [4, p. 509] that

$$(4.12) \quad c_{1,6}(2^\alpha \cdot 3^\beta) = (-1)^{\alpha+\beta} \cdot 2^\alpha \cdot 3^\beta, \quad \alpha, \beta \in \mathbb{N}_0.$$

From (4.9) and (4.12) we deduce

$$(4.13) \quad c_{1,24}(2^\alpha \cdot 3^\beta) = (-1)^{\alpha+\beta} \cdot 2^{\alpha+2} \cdot 3^\beta \cdot 7/61, \quad \alpha \geq 2, \beta \geq 0$$

and

$$(4.14) \quad c_{3,8}(2^\alpha \cdot 3^\beta) = (-1)^{\alpha+\beta} \cdot 2^{\alpha+2} \cdot 3^\beta \cdot 7, \quad \alpha \geq 2, \beta \geq 0.$$

From (2.4) we deduce that

$$(4.15) \quad c_{1,8}(n) = 0, \quad n \equiv 0 \pmod{2}.$$

Appealing to (4.8) and (4.15) with  $n = 2 \cdot 3^\beta$  ( $\beta \in \mathbb{N}_0$ ) we obtain

$$(4.16) \quad c_{1,24}(2 \cdot 3^{\beta+1}) = -\frac{3}{61} c_{3,8}(2 \cdot 3^\beta)$$

and

$$(4.17) \quad c_{1,24}(2 \cdot 3^\beta) = -\frac{1}{183} c_{3,8}(2 \cdot 3^{\beta+1}).$$

As  $c_{1,24}(2) = \frac{132}{61}$ ,  $c_{3,8}(2) = 12$ ,  $c_{1,24}(6) = -\frac{36}{61}$  and  $c_{3,8}(6) = -396$ , we deduce from (4.16) and (4.17) that for  $\beta \in \mathbb{N}_0$

$$(4.18) \quad c_{1,24}(2 \cdot 3^{2\beta}) = 2^2 \cdot 3^{2\beta+1} \cdot 11/61,$$

$$(4.19) \quad c_{1,24}(2 \cdot 3^{2\beta+1}) = -2^2 \cdot 3^{2\beta+2}/61,$$

$$(4.20) \quad c_{3,8}(2 \cdot 3^{2\beta}) = 2^2 \cdot 3^{2\beta+1},$$

$$(4.21) \quad c_{3,8}(2 \cdot 3^{2\beta+1}) = -2^2 \cdot 3^{2\beta+2} \cdot 11.$$

There does not appear to be a simple formula for  $c_{1,24}(3^\beta)$  or for  $c_{3,8}(3^\beta)$  ( $\beta \in \mathbb{N}_0$ ).

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EVALUATION OF TWO CONVOLUTION SUMS

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