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SOME REMARKS ON WEAKLY REGULAR MODULES

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Let A be a ring with 1, and M a right A-module. In [7], M is defined to be *locally projective*, if for any $m \in M$ there exist $m_i \in M$ and $f_i \in \mathrm{Hom}_A(M,A)$ such that $m = \sum_{i=1}^{k} m_i f_i(m)$. Following the first author [4], we call M a *weakly regular module* if M satisfies one of the following equivalent conditions: 1) For any $m \in M$ there exist $s_i \in \mathrm{End}_A(M)$ and $f_i \in \mathrm{Hom}_A(M,A)$ such that $m = \sum_{i=1}^{n} s_i(m) f_i(m)$; 2) M_A is locally projective and every $\mathrm{End}_A(M)$-A-submodule of M is ideal pure; 3) M_A is locally projective and $TI = TI^2$ for each left ideal I of A, where T denotes the trace ideal of M_A.

In this paper we shall first consider the weak regularity of certain scalar extensions of weakly regular modules. We shall show that if M_A is weakly regular and B is a finite normalizing, separable extension of A such that B_A is flat, then $M \otimes_A B$ is weakly regular. We shall also prove that if G is a locally finite group and M_A is a weakly regular module without $|g|$-torsion for all $g \in G$, then the right $A[G]$-module $M \otimes_A A[G]$ is weakly regular. In the latter part of this paper, we deal with submodules, factor modules, extensions and direct products of weakly regular modules.

Throughout the paper, A will denote an associative ring with 1, and all the modules considered will be unital. Unadorned $\otimes$ means $\otimes_A$, unless otherwise stated.

Noting that every locally projective module is flat (see [7]), as a combination of [4, Theorem 7] and [1, Lemmas 19.1 and 19.18], we readily obtain the following

Lemma 1. *Let M be a right A-module and let $S = \mathrm{End}_A(M)$. Then the following are equivalent:*

1) *M_A is weakly regular.*

2) *M_A is locally projective and M/N_A is flat for each S-A-submodule N of M.*

3) *M_A is locally projective and for every S-A-submodule N of M the functor* $\mathrm{Hom}_S(M/N, \ )$: S-Mod $\to$ A-Mod *preserves injectives.*

Let B be a ring extension of A. If the mapping $\sum_j x_j \otimes y_j \to \sum_j x_j y_j$ from $B \otimes B$ to B splits as an B-B-homomorphism, we say B/A is a *sepa-*

rable extension. A ring extension B/A is a *finite normalizing extension* if and only if there exist finitely many elements $b_1, \cdots, b_n$ in B such that $B = \sum_{i=1}^{n} Ab_i$ and $Ab_i = b_iA$ for all i.

Lemma 2. *Let B/A be a separable extension, and M a right B-module. If M_A is flat, then so is M_B.*

Proof. It is easy to see that $M \otimes B_B$ is flat. By hypothesis, B is isomorphic to a B-B-direct summand of $B \otimes B$. Hence $M(= M \otimes_B B)$ is isomorphic to a direct summand of $M \otimes B(= M \otimes_B B \otimes B)$, which implies that M_B is flat.

We are now in a position to prove the first main theorem.

Theorem 1. *Let B/A be a finite normalizing, separable extension such that B_A is flat. If M_A is weakly regular, then so is $M \otimes B_B$.*

Proof. It is clear that $M \otimes B_B$ is locally projective. Let $B = \sum_{i=1}^{n} Ab_i$ and $Ab_i = b_iA$ for all i. We set $S = \mathrm{End}_A(M)$ and $T = \mathrm{End}_B(M \otimes B)$. We shall first show that $(M \otimes B)/N_A$ is flat for each T-B-submodule N of $M \otimes B$. Since $M \otimes B_A$ is flat, it suffices to show that $(M \otimes B)a \cap N \subseteq Na$ for all $a \in A$ (see [1, Lemma 19.18]). By induction on k, we shall show that if $(m_1 \otimes b_1 + \cdots + m_k \otimes b_k)a \in N$ then $(m_1 \otimes b_1 + \cdots + m_k \otimes b_k)a \in Na$. Suppose that $(m_1 \otimes b_1)a \in N$. Then $b_1 a = a' b_1$ with some $a' \in A$. Since M_A is weakly regular, we can write $m_1 a' = \sum_i s_i(m_1 a') f_i(m_1 a')$ with some $s_i \in S$ and $f_i \in \mathrm{Hom}_A(M, A)$. Then we see that $m_1 \otimes b_1 a = \sum_i s_i(m_1 a') f_i(m_1 a') \otimes b_1 = \sum_i (s_i \otimes 1)(m_1 \otimes b_1) a c_i a \in Na$, where $f_i(m_1) b_1 = b_1 c_i$, $c_i \in A$. Now, assume that $k > 1$ and our assertion is true for $k-1$. Choose $s'_j \in S$ and $c'_j \in A$ such that $m_k \otimes b_k a = \sum_j (s'_j \otimes 1)(m_k \otimes b_k a) c'_j a$. Setting $y = \sum_j (s'_j \otimes 1)(m_1 \otimes b_1 + \cdots + m_k \otimes b_k) a c'_j \in N$, we get $v = (m_1 \otimes b_1 + \cdots + m_k \otimes b_k - y)a \in N$. Since we can write $v = m'_1 \otimes b_1 + \cdots + m'_{k-1} \otimes b_{k-1}$ with some $m'_i \in M$, by induction hypothesis, there exists $z \in N$ such that $v = za$. Hence $(m_1 \otimes b_1 + \cdots + m_k \otimes b_k)a = (y+z)a \in Na$. This completes our induction. Thus $(M \otimes B)/N_A$ is flat, and so Lemma 2 proves that it is a flat B-module. Therefore, $M \otimes B_B$ is weakly regular by Lemma 1.

Obviously, $\mathrm{Mat}_n(A)$ is a finite normalizing, separable extension of A. For any monic polynomial f in $A[X]$ with $Af = fA$, it is known that $A[X]/A[X]f$ is separable over A if and only if the derivative f' of f is invertible in $A[X]$ modulo $A[X]f$ (see, e.g. [6, Theorem 1.8]). Hence we have the following

Corollary 1. *Let M_A be a weakly regular module.*

(1) *For any positive integer n, $M \otimes \mathrm{Mat}_n(A)$ is a weakly regular $\mathrm{Mat}_n(A)$-module.*

(2) *Let f be a monic polynomial in $A[X]$ such that $Af = fA$ and the derivative f' of f is invertible in $A[X]$ modulo $A[X]f$, and let $B = A[X]/A[X]f$. Then $M \otimes B_B$ is weakly regular.*

It is well known that a group ring $B = A[G]$ of a finite group G is separable over A if and only if the order of G is invertible in A (see, e.g. [5, Corollary 1, p.128]). Hence, if M_A is weakly regular and the order of G is invertible in A then $M \otimes B_B$ is weakly regular by Theorem 1. This result will be generalized as follows:

Theorem 2. *Let M_A be a weakly regular module, and G a locally finite group. If M has no $|g|$-torsion for all $g \in G$, then the $A[G]$-module $M \otimes A[G]$ is weakly regular.*

In preparation for the proof of Theorem 2, we establish the following two lemmas which generalize [2, Propositions 2.1 and 2.2].

Lemma 3. *Let $0 \to N \to M \to L \to 0$ be an exact sequence of right A-modules, and $M^* = \mathrm{Hom}_A(M,A)$. If M_A is locally projective, then the following are equivalent:*

1) *L_A is flat.*

2) *$u \in NM^*(u)$ for all $u \in N$.*

Proof. 1) ⇒ 2). If $u \in N$, then $u \in N \cap MM^*(u) = NM^*(u)$ by [1, Lemma 19.18].

2) ⇒ 1). Let I be an arbitrary left ideal of A. If $u \in N \cap MI$, then $M^*(u) \subseteq M^*(MI) \subseteq I$, and so $u \in NM^*(u) \subseteq NI$. Hence, again by [1, Lemma 19.18], L is flat.

Lemma 4. *Let $0 \to N \to M \to L \to 0$ be an exact sequence of right A-modules. If M is locally projective, then the following are equivalent:*

1) *L_A is flat.*

2) *Given any $u \in L$, there exists $\theta \in \mathrm{Hom}_A(M,N)$ such that $\theta(u) = u$.*

3) *Given any $u_1, \cdots, u_n \in N$, there exists $\theta \in \mathrm{Hom}_A(M,N)$ such that $\theta(u_i) = u_i$ $(1 \leq i \leq n)$.*

Proof. 1) ⇒ 2). Let $u \in N$. By Lemma 3, we can represent $u = \sum_{i=1}^{p} m_i f_i(u)$ with some $m_i \in N$ and $f_i \in M^*$. Then $\sum_{i=1}^{p} m_i f_i$ is a desired map.

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2) ⇒ 3). We proceed by induction on n; assume that $n > 1$ and 3) holds for $k < n$. Choose $\theta_n \in \mathrm{Hom}_A(M,N)$ such that $\theta_n(u_n) = u_n$, and let $v_i = u_i - \theta_n(u_i)$ $(1 \leq i \leq n-1)$. Then, by induction hypothesis, there exists $\theta' \in \mathrm{Hom}_A(M,N)$ such that $\theta'(v_i) = v_i$ $(1 \leq i \leq n-1)$. It is easy to see that $\theta' + \theta_n - \theta'\theta_n$ has the desired property.

2) ⇒ 1). Let $u \in N$. Since M is locally projective, we can write $u = \sum_{i=1}^{q} c_i h_i(u)$ with some $c_i \in M$ and $h_i \in M^*$. By hypothesis, there exists $\theta \in \mathrm{Hom}_A(M,N)$ such that $\theta(u) = u$. Thus we obtain $u = \theta(u) = \sum_{i=1}^{q} \theta(c_i)h_i(u)$, and hence L is flat by Lemma 3.

Proof of Theorem 2. Let x be an arbitrary element in $M \otimes A[G]$. Then we may write $x = m_1 \otimes g_1 + \cdots + m_n \otimes g_n$ with some $m_i \in M$ and $g_i \in G$. Since G is locally finite, $g_1, \cdots, g_n$ generate a finite subgroup H of G. As was seen in the proof of Theorem 1, $(M \otimes A[H])/N_A$ is flat for each $\mathrm{End}_{A[H]}(M \otimes A[H])$-$A[H]$-submodule N of $M \otimes A[H]$. Since M_A is weakly regular, for any $m \in M$ there exist $s_i \in \mathrm{End}_A(M)$ and $f_i \in \mathrm{Hom}(M,A)$ such that

$$|H|m = \textstyle\sum_i s_i(|H|m)f_i(|H|m) = |H|^2 \sum_i s_i(m)f_i(m).$$

Since M has no $|H|$-torsion, we obtain $m = |H| \sum_i s_i(m)f_i(m)$. Therefore, the right multiplication by $|H|$ is an automorphism of M_A. Hence this map induces an automorphism of $(M \otimes A[H])$. Now, let y be an arbitrary element in N. By Lemma 4, there exists an A-homomorphism $\theta : M \otimes A[H] \to N$ such that $\theta(yh) = yh$ for all $h \in H$. Then the map $\hat{\theta} : M \otimes A[H] \to N$ defined by

$$\hat{\theta}(z) = |H|^{-1} \textstyle\sum_{h \in H} \theta(zh)h^{-1}$$

is an $A[H]$-homomorphism with $\hat{\theta}(y) = y$. Hence $(M \otimes A[H])/N_{A[H]}$ is flat again by Lemma 4, and therefore weakly regular by Lemma 1. Thus there exist $s_i \in \mathrm{End}_{A[H]}(M \otimes A[H])$ and $f_i \in \mathrm{Hom}_{A[H]}(M \otimes A[H], A[H])$ such that $x = \sum_i s_i(x)f_i(x)$. Identifying $M \otimes A[G]$ with $M \otimes A[H] \otimes_{A[H]} A[G]$, we get $x = \sum_i (s_i \otimes 1)(x)(f_i \otimes 1)(x)$ where $s_i \otimes 1 \in \mathrm{End}_{A[G]}(M \otimes A[G])$ and $f_i \otimes 1 \in \mathrm{Hom}_{A[G]}(M \otimes A[G], A[G])$. This completes the proof.

In the rest of this paper, we consider the weak regularity of submodules, factor modules, extensions and of direct products of weakly regular modules.

Theorem 3. (1) *Let M_A be a weakly regular module, and N an A-submdoule of M.*

(i) *If N is ideal pure, then it is weakly regular.*

(ii) *If M/N_A is locally projective, then it is weakly regular.*

(2) *Let $0 \to N \xrightarrow{\alpha} M \xrightarrow{\beta} L \to 0$ be an exact sequence of right A-modules. If M is locally projective and both N and L are weakly regular, then M is weakly regular.*

Proof. (1) (i) Since M/N_A is flat by [1, Lemma 19.18], for each $m \in N$ there exists $\theta \in \mathrm{Hom}_A(M,N)$ such that $\theta(m) = m$ (Lemma 4). On the other hand, since M_A is weakly regular, there exist $s_i \in \mathrm{End}_A(M)$ and $f_i \in \mathrm{Hom}_A(M,L)$ such that $m = \sum_i s_i(m)f_i(m)$. Setting $h_i = \theta s_i \mid N \in \mathrm{End}_A(N)$ and $f_i' = f_i \mid N \in \mathrm{Hom}_A(N,A)$, we get $m = \sum_i h_i(m)f_i'(m)$. This shows that N_A is weakly regular.

(ii) Let $\nu : M \to L = M/N$ be the natural homomorphism. Take an arbitrary element u in L. Then, L_A being locally projective, there exist $\psi \in \mathrm{Hom}_A(L,M)$ such that $\nu\psi(u) = u$. Since M_A is weakly regular, there exist $s_i \in \mathrm{End}_A(M)$ and $f_i \in \mathrm{Hom}_A(M,A)$ such that $\psi(u) = \sum_i s_i\psi(u)f_i\psi(u)$. Then, we have $u = \nu\psi(u) = \sum_i \nu s_i\psi(u)f_i\psi(u)$ where $s_i\psi \in \mathrm{Eod}_A(L)$ and $f_i\psi \in \mathrm{Hom}_A(L,A)$.

(2) Let m be an arbitrary element in M: Since L is weakly regular and M is locally projective, we see that $\beta(m) = \sum_i s_i\beta(m)f_i\beta(m)$ with some $s_i \in \mathrm{End}_A(L)$ and $f_i \in \mathrm{Hom}_A(L,A)$ and that $\beta s_i'(m) = s_i\beta(m)$ with some $s_i' \in \mathrm{End}_A(M)$. Then we have $m' = m - \sum_i s_i'(m)f_i\beta(m) \in \mathrm{Ker}\,\beta = \mathrm{Im}\,\alpha$. Since $N' = \mathrm{Im}\,\alpha$ is weakly regular, there exist $t_j \in \mathrm{End}_A(N')$ and $h_j \in \mathrm{Hom}_A(N',A)$ such that $m' = \sum_j t_j(m')h_j(m')$. On the other hand, by Lemma 4, there exists $\theta \in \mathrm{Hom}_A(M,N)$ such that $\theta(m') = m'$. Hence we have

$$m = \sum_i s_i'(m)f_i\beta(m) + \sum_j t_j\theta(m')h_j\theta(m') \in \mathrm{End}_A(M)(m)\mathrm{Hom}_A(M,A)(m)$$

where $t_j\theta \in \mathrm{End}_A(M)$ and $h_j\theta \in \mathrm{Hom}_A(M,A)$. This shows that M_A is weakly regular.

Corollary 2. *Let M_A be a locally projective module. If $M = \sum_{i \in I} M_i$ with weakly regular A-submodules M_i, then M itself is weakly regular.*

Proof. By [4, Proposition 3.2)], the external direct sum $E = \bigoplus_{i \in I} M_i$ is weakly regular. Hence, the homomorphic image M of E is weakly regular by Theorem 3 (1) (ii).

Following [7], we say A is *strongly left coherent* if and only if any direct product of (locally) projective right A-modules is locally projective. For example, every left Noetherian ring is strongly left coherent. We conclude this paper with the following.

Theorem 4. *Let A be a commutative strongly coherent ring. Then any direct product of weakly regular A-modules is weakly regular.*

Proof. Let $\{M_\lambda \mid \lambda \in \Lambda\}$ be a set of weakly regular A-modules, and set $M = \prod_{\lambda\in\Lambda} M_\lambda$. Let $m = (m_\lambda)$ be an arbitrary element in M. By [7, Theorem 4.2], there exist $f_1, \cdots, f_n \in M^*$ such that $M^*(m) = \sum_{i=1}^n Af_i(m)$. Since M_λ is weakly regular and $M_\lambda^*(m_\lambda) \subseteq M^*(m)$, for each $\lambda \in \Lambda$ we can find $s_i^\lambda \in \mathrm{End}_A(M_\lambda)$ such that $m = \sum_{i=1}^n s_i^\lambda(m_\lambda)f_i(m)$. Now, setting $s_i = (s_i^\lambda) \in \mathrm{End}_A(M)$, we have $m = \sum_{i=1}^n s_i(m)f_i(m)$. This shows that M_A is weakly regular.

REFERENCES

[1] F.W. ANDERSON and K.R. FULLER: Rings and Categories of Modules, Springer-Verlag, Berlin-New York, 1974.

[2] S.U. CHASE: Direct products of modules, Trans. Amer. Math. Soc. **97** (1960), 457—473.

[3] Y. HIRANO: On fully right idempotent rings and direct sums of simple rings, Math. J. Okayama Univ. **22** (1980), 43—49.

[4] T. MABUCHI: Weakly regular modules, Osaka J. Math. **17** (1980), 35—40.

[5] Y. MIYASHITA: Galois extensions and crossed products, J. Fac. Sci. Hokkaido Univ. Ser. I **20** (1968), 122—134.

[6] Y. MIYASHITA: On skew polynomial rings, J. Math. Soc. Japan **31** (1979), 317—330.

[7] B. ZIMMERMANN-HUISGEN: Pure submodules of direct products of free modules, Math. Ann. **224** (1976), 233—245.

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