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Masahiro Sugawara*

*Okayama University

H-SPACES AND SPACES OF LOOPS

MASAHIRO SUGAWARA

1. Recently, H. Samelson has shown that there exists an H-homomorphism of G into the space $\Lambda(B)$ of loops in B for an universal bundle (E, B, G, p) with a group G [3, Theorem I]; and, as an application of this result, the map $(x, y) \rightarrow xyx^{-1}y^{-1}$ is not homotopic to the constant map where x and y run through all the quaternions of norm 1 [3, Theorem II]. Our purpose of this note is to show that it holds the analogous theorem which assure the existence of an H-homomorphism of an H-space F into $\Lambda(Y)$ for some kind of F and Y; and, applying to the case $F = S_7$ the 7-sphere in Euclidean 8-space which has an H-structure by the multiplications of Cayley numbers, there is an H-homomorphism of S_7 into $\Lambda(S_8 \cup_p e^{16})$ ($S_8 \cup_p e^{16}$ is the space obtained from 8-sphere S_8 by attaching 16-cell e^{16} under the Hopf map $p: S_{15} \rightarrow S_8$ as attaching map). Therefore the same theorem of [3, Theorem II] for x, y running through Cayley numbers of norm 1 is obtained by similar processes, as an answer to the question raised in [3].

2. We first notice the following fact for a fibre space (E, B, F, p) (E is a fibre space, with base space B, fibre F and projection $p: E \rightarrow B$ (cf. [1], Ch. V for the definition)), analogous to Proposition I of [3]. Let $\Lambda(X)$ be the space of loops in the space X with a base point $x_0 \in X$, then it is known that it is an H-space (cf. § 3 below) by the natural multiplication (composition of loops) and, on the other hand, there is a natural isomorphism T between homotopy groups $\pi_i(X)$ and $\pi_{i-1}(\Lambda(X))$ (for examples, T is given by $(T\varphi)(x_1, \cdots, x_{i-1})(t) = \varphi(x_1, \cdots, x_{i-1}, t)$ for $\varphi: (I^i, \dot{I}^i) \rightarrow (X, x_0)$, I^i = the product of i-copies of $I = [0, 1]$, and $\dot{I}^i$ = its boundary), (cf. [4], [2]).

Proposition 1. *In the fibre space (E, B, F, p), suppose that the fibre F, considering as the fibre over a point $b_0 \in B$, is contractible to a point $x_0 \in F$ in E (with x_0 stationary throughout the contraction). Then there exists a (continuous) map f of F into $\Lambda(B)$ (loops having b_0 as a base point), and f induces $f_*: \pi_i(F) \rightarrow \pi_i(\Lambda(B))$ such that $\partial \circ p_*^{-1} \circ T^{-1} \circ f_* =$ the identity map, where ∂ is the boundary homomorphism*[1].

1) As well known, $\pi_{i+1}(B) \approx \pi_i(F) + \pi_{i+1}(E)$ by the hypothesis of this proposition (cf. e.g. [5], 17. 10. Theorem), and the last equation implies that the map $T \circ f_*$ gives an induces embedding of $\pi_i(F)$ onto direct summand of $\pi_i(B)$.

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Proof. Let $k: F \times I \to E$ be the contraction giving by hypothesis, then the map f is defined by

$$f(x)(t) = p \circ k(x, t). \qquad \text{for } x \in F,\ t \in I.$$

If we take $\varphi: (I^t, \dot{I}^t) \to (F, x_0)$ representing an element $\alpha \in \pi_t(F)$, the map $\bar{\varphi}: (I^{t+1}, I^t, J^t) \to (E, F, x_0)$ defined by $\bar{\varphi}(x, t) = k(\varphi(x), t)$ for $(x, t) \in I^t \times I = I^{t+1}$ $(J^t = (\dot{I}^{t+1} - I^t) \cup \dot{I}^t)$ represents $\beta \in \pi_{t+1}(E, F)$ which satisfies $\partial(\beta) = \alpha$. Because $(T(p \circ \bar{\varphi})(x))(t) = p \circ k(\varphi(x), t) = (f \circ \varphi(x))(t)$, $T \circ p_*(\beta) = f_*(\alpha)$ and the above property is obtained.

3. As usually, we say that a topological space X is an H-space (have an H-structure) if the following condition is satisfied:

(1) There exists a multiplication μ in X, i. e. a (continuous) map μ of $X \times X$ into X.

(2) μ has a homotopy-unit $x_0 \in X$ which means $\mu(x_0, x_0) = x_0$ and the two maps $x \to \mu(x, x_0)$ and $x \to \mu(x_0, x)$ of X into itself are both homotopic to the identity map (with x_0 stationary).

(3) There is an inversion, i. e. a map σ of X into itself such that $\sigma(x_0) = x_0$ and the two maps $x \to \mu(x, \sigma(x))$ and $x \to \mu(\sigma(x), x)$ are both homotopic to constant (with x_0 stationary).

(We often write xy or $x \cdot y$ instead of $\mu(x, y)$ and x^{-1} instead of $\sigma(x)$.)

We notice here that the homotopy-associative condition is not presupposed, and so, in general, the two maps $(x, y, z) \to \mu(\mu(x, y), z)$ and $(x, y, z) \to \mu(x, \mu(y, z))$ are not homotopic to each other necessarily.

A map f of an H-space into second H-space Y is said to be an H-homomorphism, if f and both the multiplications and inversions are homotopy-commutative respectively, i. e. two maps $(x, y) \to \mu'(f(x), f(y))$ and $(x, y) \to f \circ \mu(x, y)$ $(x \to f \circ \sigma(x)$ and $x \to \sigma' \circ f(x))$ of $X \times X$ into Y (X into Y) are homotopic where μ and μ' (σ and σ') are multiplications (inversions) of X and Y respectively.

4. In this note we consider the following conditions for fibre spaces (E, B, F, p):

(I) F is an H-space with multiplication μ.

(II) There exists a subset E' of E and a map $\bar{\mu}$ of $E' \times F$ into E such that E' contains F (considered as a fibre over a point $b_0 \in B$) and $\bar{\mu}$ is equal to μ on F i. e. $\bar{\mu} \mid F \times F = \mu$, and finally $\bar{\mu}$ maps a point to that contained in the same fibre i. e. $p \circ \bar{\mu}(u, x) = p(u)$ for any $u \in E'$ and $x \in F$. (We often write ux or $u \cdot x$ instead of $\bar{\mu}(u, x)$ if no confusion.)

In the case that the condition (I) is satisfied, (II) holds if we take

$E'=F$ or, more generally, $E'=$ any subset of $p^{-1}(V)$, containing F, where V is a subset of B containing b_0 such that $p^{-1}(V)$ is homeomorphic to the product $V\times F$. We say that these cases are inessential; and it is a question, for any given fibre spaces, whether a subset E' could be taken essentially or not.

Simple examples of essential cases are principal fibre bundles. In these cases, E' can be taken the whole of E as is well known.

5. Another essential example is, as using afterwards, the fiber bundle (S_{15}, S_8, S_7, p). where S_i is the i-sphere in Euclidean $(i+1)$-space and $p: S_{15}\to S_8$ is the Hopf map (cf., for examples, [5], 20.6). Let C be the (non-associative) division algebra of all Cayler numbers, and set $S_{15}=\{(c,d)\mid c,d\in C \text{ and } |c|^2+|d|^2=1\}$, $S_8=\{c\mid c\in C \text{ or } c=\infty$ the point at infinity$\}$ and $S_7=\{c\mid c\in C \text{ and } |c|=1\}$, then the Hopf map p is given by $p(c,d)=cd^{-1}$ if $d\neq 0$, $=\infty$ if $d=0$. Setting $E'=\{(c,d)\mid (c,d)\in S_{15}$ and $c\neq 0$ or $(c,d)=(0,1)\}$, it contains the fibre over ∞: $\{(c,0)\mid |c|=1\}$ whom we identify with S_7. If we consider the mutiplication μ of Cayley numbers in S_7, it becomes an H-space, and hence (I) is satisfied. The map $\bar{\mu}: E'\times S_7\to S_{15}$ can be defined as follows:

$$\bar{\mu}((c,d),c')=(c\cdot c',(dc^{-1})\cdot(cc')), \qquad \text{if } c\neq 0,$$
$$\bar{\mu}((0,1),c')=(0,c').$$

The continuity of $\bar{\mu}$ at a point (c,d) with $c\neq 0$ is clear. As the norm of $e=(dc^{-1})\cdot(cc')-c'=(dc^{-1})\cdot(cc')-c^{-1}\cdot(cc')$[1] $=((d-1)\cdot c^{-1})\cdot(cc')$ is equall to $|d-1||c'|$, if (c,d) converges to $(0,1)$, $|e|$ converges to 0 and hence e converges to 0. This shows that $\bar{\mu}$ is continuous at $(0,1)$. $\bar{\mu}\mid S_7\times S_7=\mu$ is immediate from definitions. If $c\neq 0$ and $d\neq 0$, $p\circ\bar{\mu}((c,d),c')=(cc')\cdot((dc^{-1})(cc'))^{-1}=(cc')\cdot((cc')^{-1}\cdot(dc^{-1})^{-1})=(dc^{-1})^{-1}=cd^{-1}=p(c,d)$; if $c\neq 0$, $p\circ\bar{\mu}((c,0),c')=(cc',0)=\infty=p(c,0)$; and finally $p\circ\bar{\mu}((0,1),c')=0=p(0,1)$. Therfore (II) is satisfied by the above E' and $\bar{\mu}$; and so the bundle (S_{15}, S_8, S_7, p) is an essential example, as it is clear $p(E')=S_8$.

By an essentially similar reason, the bundles (S_7, S_4, S_3, p) and (S_3, S_2, S_1, p)[2] are also essential.

6. Now we state our results.

Theorem 1. *Let X, E, E', F, Y, B be given spaces such that* $X\supset$

1) The associativity of the subalgebra of C generated by two elements is known, cf. Dickson, Linear Algebra, Cambridge Univ. Press, 1914.

2) For these bundles, cf. [5], 20.1—20.5.

$E \supset E' \supset F$, $Y \supset B$, and E is contractible to a point in X, and there exists a map $\bar{p}: (X, E, F) \to (Y, B, b_0)$ $(b_0 \in B)$ satisfying the following conditions:

(1) *$(E, B; F, \bar{p} \mid E)$ is a fibre space and it stisfies the conditions* (I) *and* (II) *and the given E' is taken as the subset of* (III).

(2) *F is contractible in E' to a homotopy-unit $x_0 \in F$ (with x_0 stationary throughout the contraction).*

Then there exists a map f of F into the space $\Lambda(B)$ of loops in B with base point b_0 such that the composition $i \circ f: F \to \Lambda(Y)$ of f and the inclusion map $i: \Lambda(B) \to \Lambda(Y)$ is an H-homomorphism.

Proof. We proof this result by an essentially same manner to proofs of Theorem I of [3]. The resulting map f is the map giving in proposition 1 by making use of the contraction k in E', i.e. $k(F \times 1) \subset E'$. Let $\mu_t: F \to F$, $0 \leqslant t \leqslant 1$, be the homotopy between $\mu_0(x) = \mu(x_0, x) = x_0 \cdot x$ and $\mu_1 =$ the identity map giving by (2) of § 3, and we define a map Φ of $F \times F \times \dot{I}^2$ into E by:

$$\Phi(x, y, t, u) = \begin{cases} x_0, & \text{for } t = 1, \ 0 \leqslant u \leqslant 1, \\ x \cdot y, & \text{for } t = 0, \ 0 \leqslant u \leqslant 1, \\ k(x \cdot y, t), & \text{for } u = 0, \ 0 \leqslant t \leqslant 1, \\ k(x, 3t) \cdot y, & \text{for } u = 1, \ 0 \leqslant t \leqslant \frac{1}{3}, \\ \mu_{3t-1}(y), & \text{for } u = 1, \ \frac{1}{3} \leqslant t \leqslant \frac{2}{3}, \\ k(y, 3t-2), & \text{for } u = 1, \ \frac{2}{3} \leqslant t \leqslant 1, \end{cases}$$

for any $x, y \in F$, where $\cdot$ of the fourth row is $\bar{\mu}$ of (II) and the others are μ the multiplication in F. It is clearly continuous. We extend this map Φ to a map $\bar{\Phi}$ of $F \times F \times I^2$ into X: for each $(x, y) \in F \times F$, we map the center $\left(\frac{1}{2}, \frac{1}{2}\right)$ of I^2 into $* \in X$ to which E is contractible in X by the hypothesis, and the segment from any $(t, u) \in \dot{I}^2$ to $\left(\frac{1}{2}, \frac{1}{2}\right)$ on the path, described by the point $\Phi(x, y, t, u)$ under the contraction of E to $*$. From this map, we obtain a map Ψ of $F \times F \times I$ into $\Lambda(Y)$ (loops with the base point b_0) as usually: $\Psi(x, y, u)(t) = \bar{p} \circ \bar{\Phi}(x, y, t, u)$, as $\bar{\Phi}(x, y, t, u) \in F$ for $t = 0, 1$. Thus we have a homotopy between Ψ_0 and $\Psi_1: F \times F \to \Lambda(Y)$, defined by $\Psi_u(x, y) = \Psi(x, y, u)$ $(0 \leqslant u \leqslant 1)$. By the definitions, $\Psi_0(x, y)$ is identical with $i \circ f(x \cdot y)$. On the other hand, by the properties $\Phi(x, y, t, 1) \in F$ for $\frac{1}{3} \leqslant t \leqslant \frac{2}{3}$ and $p(\bar{\mu}(z, y)) = p(z)$ for $z \in E'$ and $y \in F$ $(p = \bar{p} \mid E)$, $\Psi_1(x, y)$ is identical with

$$i \circ ((f(x) \cdot e_{b_0}) \cdot f(y)) = ((i \circ f(x)) \cdot e_{b_0}) \cdot (i \circ f(y)),$$

where $\cdot$ are the multipications in $\Lambda(B)$ and $\Lambda(Y)$ (i is clearly H-homomorphic), and e_{b_0} is the constant loop (i. e. $e_{b_0}(t) = b_0$ for all $0 \leqslant t \leqslant 1$) of $\Lambda(B)$ and $\Lambda(Y)$ respectively. It it clear that the last map is homotopic to the map $\Psi_2: F \times F \rightarrow \Lambda(Y)$ such that $\Psi_2(x, y) = (i \circ f(x)) \cdot (i \circ f(y))$ for $x, y \in F$; and so Ψ_0 and Ψ_2 are homotopic. Hence the homotopy-commutativity of the diagram:

$$\begin{array}{ccc} F \times F & \longrightarrow & F \\ \downarrow (i \circ f) \times (i \circ f) & & \uparrow i \circ f \\ \Lambda(Y) \times \Lambda(Y) & \longrightarrow & \Lambda(Y) \end{array}$$

is proved.

The homotopy-commutativity of $i \circ f$ and the inversion can be treated similarly, and can be proved strictly same to §5 of [3]. Therefore we shall omit its proofs, and finish the proofs of Theorem 1.

Remark. If E' is connected, the contractible condition to x_0 in (2) is an easy consequence of a contractibility to any point of E', and, if F is a CW-complex and E' is simply connected in addition, a contraction being x_0 stationary can be constructed from any contraction, by making use of Homotopy Extension Theorem (cf. e. g. [1], Ch. VII, Theorem 1.4).

7. We apply above results to the fibre space (S_{15}, S_8, S_7, p) treated in §5. Let e^{16} be an (open) 16-cell bounded by the 15-sphere S_{15} in Euclidean 16-space, and $S_8 \smile_p e^{16}$ the space constructed from S_8 attaching e^{16} by the Hopf map $p: S_{15} \rightarrow S_8$ as the attaching map (cf. e. g. [1], Ch. VI, §2). We denote by $\bar{p}$ the characteristic map $(\bar{e}^{16}, S_{15}) \rightarrow (S_8 \smile_p e^{16}, S_8)$ which is a homeomorphism on e^{16} and $\bar{p} \mid S_{15} = p$ ($\bar{e}^{16} =$ closure of e^{16}). Then, for $X = \bar{e}^{16}$, $E = S^{15}$, $Y = S_8 \smile_p e^{16}$, $B = S_8$ and $\bar{p}$, the hypotheses of Theorem 1 is fulfilled. $\bar{e}^{16}$ is clearly contractible and the condition (1) is seen in §5. The condition (2) is also immediate from Remark of §6 and the fact that E' (giving in §5) itself is contractible to the point (0, 1) by arcs of great circles. Hence, by Theorem 1, we obtain the following result:

Proposition 2. *There exists a (continuous) map f of S_7 into $\Lambda(S_8)$*

such that the composed map $g = i \circ f : S_7 \to \Lambda(S_8 \cup_p e^{16})$ *is an H-homomorphism with respect to the multiplication of Cayley numbers and loops respectively, where* $i : \Lambda(S_8) \to \Lambda(S_8 \cup_p e^{16})$ *is the inclusion map.*

8. It folds the following properties for this map $i \circ f$:

Proposition 3. *The map* $g = i \circ f$ *of Proposition* 2 *induces an isomorphism of* $\pi_i(S_7)$ *onto* $\pi_i(\Lambda(S_8 \cup_p e^{16}))$ *for* $i \leqslant 20$.

Proof. By Proposition 1, $f_* : \pi_i(S_7) \to \pi_i(\Lambda(S_8))$ is an isomorphism into. In the following diagram

$$\begin{array}{ccccc} p_*(\pi_{i+1}(S_{15})) + E\pi_i(S_7) = \pi_{i+1}(S_8) & & \overset{i_*}{\rightarrow} & & \pi_{i+1}(S_8 \cup_p e^{16}) \\ \nearrow E \quad \downarrow T & & & & \downarrow T \\ \pi_i(S_7) \quad \overset{f_*}{\rightarrow} \quad \pi_i(\Lambda(S_8)) & & \overset{i_*}{\rightarrow} & & \pi_i(\Lambda(S_8 \cup_p e^{16})) \end{array}$$

where E is the suspension homomorphism, commutative relations hold[1)] and T are isomorphic onto for all i. Therefore it is sufficient to show that kernel i_* (in the upper line) $= p_*(\pi_{i+1}(S_{15}))$ and i_* is onto. We consider now the following diagram:

$$\begin{array}{ccccccc} \cdots \to \pi_{i+2}(S_8 \cup_p e^{16}, S_8) & \overset{\partial}{\rightarrow} & \pi_{i+1}(S_8) & \overset{i_*}{\rightarrow} & \pi_{i+1}(S_8 \cup_p e^{16}) \to \cdots \\ \uparrow \bar{p}_* & & \uparrow p_* & & \\ \pi_{i+2}(\bar{e}^{16}, S_{15}) & \overset{\partial}{\rightarrow} & \pi_{i+1}(S_{15}) & & \end{array}$$

where the upper line is a homotopy sequence of the pair $(S_8 \cup_p e^{16}, S_8)$; the commutative relation holds clearly. The map $\bar{p}_*$ is isomorphic onto for $2 \leqslant i+2 \leqslant 22$ by Theorem 2.14 of Chapter VI of [1], and ∂ (in the lower line) is evidently an isomorphism for all i. Hence, for $i \leqslant 20$, kernel $i_* =$ image $\partial =$ image p_*, and ∂ (in the upper line) is isomorphic into because p_* is. By the exactness, i_* is onto, and therefore Proposition 3 is obtained.

9. We now come to the situation to solve the problem: Are the Cayley numbers of norm 1 homotopy-abelian?

Theorem 2. *The map* $\kappa : S_7 \times S_7 \to S_7$, *defined by* $\kappa(x, y) = x\, y\, x^{-1} y^{-1}$ *is not homotopic to a constant*; S_7 *is not homotopy-commutative.*

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1) The commutativity in the triangle follows from the fact that $T \circ f_*$ is the induced embedding of $\pi_i(S_7)$ into $\pi_{i+1}(S_8) \approx \pi_i(S_7) + \pi_{i+1}(S_{15})$, cf. footnote 1) of p. 5, and so is the suspension by precisely analogous arguments of Theorem 3.1 of [1], Ch. VI.

Proof of this theorem is now essentially same to it of Theorem II of [3], and we follow proofs briefly. By Proposition 2, to obtain the above result, it is sufficient to prove the fact that $d : S_7 \times S_7 \to A(S_8 \cup_p e^{16})$ giving by $d(x, y) = (g(x) \cdot g(y)) \cdot (g(x)^{-1} \cdot g(y)^{-1})$ is not homotopic to a constant map. Let $\alpha \in \pi_8(S_8 \cup_p e^{16})$ be represented by the inclusion map i, and s denote the standard map of $(I^{14}, \dot{I}^{14})$ onto $(S_7 \times S_7, S_7 \vee S_7)$, then $T[\alpha, \alpha] \in \pi_{14}(A(S_8 \cup_p e^{16})$ is represented, up to sign, by $d \circ s$, by the processes of [2] and Lemma 2 of [3] ($[\alpha, \alpha]$ is so-called Whitehead product of α). This reduces the problem to the question whether $[\alpha, \alpha] \in \pi_{15}(S_8 \cup_p e^{16})$ is not zero. This element is clearly the image of $[\iota_8, \iota_8]$ under i_* (ι_8 is the homotopy class of identity $S_8 \to S_8$), and it is known that $[\iota_8, \iota_8] - 2\nu_8$ is a suspension of a non-zero element of $\pi_{14}(S_7)$ ($\nu_8 =$ the element represented by the Hopf map) (cf. [6], Theorem (4.1)). Therefore we have $[\alpha, \alpha] = i_*[\iota_8, \iota_8] = i_*([\iota_8, \iota_8] - 2\nu_8])$ is not zero, by the properties of i_* studied in proofs of Proposition 3. Theorem III is thus obtained.

Remark. Similarly, we can apply Theorem 1 to the bundle (S_7, S_4, S_3, p) and the properties analogous to Propositions 2 and 3 consist.

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