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DIFFERENTIAL RINGS WITH CENTRAL DERIVATIVES OF HIGHER ORDER

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Let R be a differential ring with the derivation $d: x \to x'$, and Z the center of R. We say that R is d-prime provided if I, J are differential ideals of R and $IJ = 0$ then $I = 0$ or $J = 0$, or equivalently, if $x, y \in R$ and $x^{(h)}Ry = 0$ for all $h \geq 0$ then $x = 0$ or $y = 0$. If R is d-prime then it is easy to see that R is either of prime characteristic p or torsion free ($p = \infty$). A differential ideal P of R is said to be d-prime if the factor ring R/P is d-prime. We say that R is d-semiprime if the intersection of all d-prime ideals of R is zero, or equivalently, if R is differentially isomorphic to a subdirect sum of d-prime rings (see, e.g., [3, § 3]).

The main objective of this paper is to prove the following

Theorem 1. *Let R be a d-prime ring of characteristic p, S a differential subring of R, and n a non-negative integer such that $p > n+1$. If $S^{(n)} \neq 0$ and $S^{(n)} \subseteq Z$, then $S \subseteq Z$.*

In advance of proving Theorem 1, we state two lemmas. We can easily see the first one, whose proof may be omitted.

Lemma 1. *If a d-prime ring R contains a commutative, non-zero differential right (or left) ideal, then R is commutative.*

Lemma 2. *Let R be a d-prime ring of characteristic p, S a differential subring of R, and n a positive integer. Suppose $S^{(n)} \subseteq Z$ and $S^{(n-1)} \not\subseteq Z$. If $p > n$ then $S^{(n+1)} = 0$, and if $p > n+1$ then $S^{(n)} = 0$.*

Proof. Throughout the proof, x, y will denote arbitrary elements in S, and r an arbitrary element in R.

Since $0 = [(x^{(n-1)}y^{(n)})^{(n)}, r] = y^{(2n)}[x^{(n-1)}, r]$, we have $(y^{(2n)})^{(h)}R[x^{(n-1)}, r] = 0$ for all $h \geq 0$. Noting that R is d-prime and $S^{(n-1)} \not\subseteq Z$, we get $S^{(2n)} = 0$. Suppose now that $S^{(n+k)} = 0$ for some $k > 1$. Then

$$0 = [(x^{(k-2)}y^{(n)})^{(n)}, r] = \sum_{i=0}^{n} \binom{n}{i} y^{(n+i)}[x^{(n+k-i-2)}, r].$$

Since $y^{(n+i)} = 0$ for $i > k-1$ and $x^{(n+k-i-2)} \in Z$ for $i < k-1$, this gives

Mathematical Journal of Okayama University, Vol. 27 [1985], Iss. 1, Art. 3

$\binom{n}{k-1} y^{(n+k-1)}[x^{(n-1)}, r] = 0$, and so $y^{(n+k-1)}R[x^{(n-1)}, r] = 0$. Hence $S^{(n+k-1)} = 0$, which proves that $S^{(n+1)} = 0$. Then, we can easily see that no non-zero element in $S^{(n)}$ is a zero-divisor in R. Since

$$0 = [(x^{(n-1)}x)^{(n)}, r] = (n+1)x^{(n)}[x^{(n-1)}, r],$$

if $p > n+1$ then we get $S^{(n)} = 0$, by Brauer's trick.

Proof of Theorem 1. In case $n = 0$, there is nothing to prove. If $n > 0$, Lemma 2 shows that $S^{(n-1)} \subseteq Z$. Hence $S \subseteq Z$, by induction.

As an application of Theorem 1, we obtain the following

Theorem 2. *Let R be a d-prime ring of characteristic p, U a non-zero differential ideal of R, and n a non-negative integer such that $p > n+1$. If $R^{(n)} \neq 0$ and $U^{(n)} \subseteq Z$, then R is commutative.*

Proof. We can apply the argument employed in the proof of [2, Theorem] to see that $R^{(n)} \neq 0$ implies $U^{(n)} \neq 0$. Then $U \subseteq Z$ by Theorem 1, and hence R is commutative by Lemma 1.

Corollary 1 (cf. [3, Corollary 1]). *Let R be a d-prime ring of characteristic p, U a non-zero differential ideal of R, and n an even positive integer such that $p > n+1$. If $R^{(n-1)} \neq 0$ and $U^{(n)} \subseteq Z$, then R is commutative.*

Proof. By making use of the same method as in the proof of [1, Theorem], we can prove that $R^{(n)} \neq 0$. Hence, R is commutative by Theorem 2.

Theorem 3. *Let n be a positive integer. Let R be a $(n+1)!$-torsion free d-semiprime ring, and U a differential ideal of R with $l(U) = 0$. If $K_n = \{x \in R \mid (RxR)^{(n)} = 0\}$ is commutative and $U^{(n)} \subseteq Z$ then R is commutative.*

Proof. As is easily seen, $\bigcap_{\lambda \in \Lambda} P_\lambda = 0$ with d-prime ideals P_λ such that $U \not\subseteq P_\lambda$ and R/P_λ is of characteristic $> n+1$. Put $\Lambda_1 = \{\lambda \in \Lambda \mid R^{(n)} \not\subseteq P_\lambda\}$, and let D be the commutator ideal of R (note that D is a differential ideal of R). If $\lambda \in \Lambda_1$ then Theorem 2 proves that $D \subseteq P_\lambda$. Since $(RDR)^{(n)} \subseteq P_\lambda$ for all $\lambda \in \Lambda \backslash \Lambda_1$, we see that $(RDR)^{(n)} \subseteq \bigcap_{\lambda \in \Lambda} P_\lambda = 0$. This means $D \subseteq K_n$, and so D is a commutative ideal. Now, by Lemma 1, we can easily see that $D \subseteq \bigcap_{\lambda \in \Lambda} P_\lambda = 0$. This proves that R is commutative.

By making use of Corollary 1 instead of Theorem 2, the proof of Theorem 3 gives the following

Corollary 2 (cf. [3, Theorem 3]). *Let n be an even positive integer. Let R be a $(n+1)!$-torsion free d-semiprime ring, and U a differential ideal of R with $l(U) = 0$. If K_{n-1} is commutative and $U^{(n)} \subseteq Z$, then R is commutative.*

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REFERENCES

[1] L. O. CHUNG and J. LUH: Nilpotency of derivations, Canad. Math. Bull. **26** (1983), 341–346.

[2] L. O. CHUNG and J. LUH: Nilpotency of derivations on an ideal, Proc. Amer. Math. Soc. **90** (1984), 211–214.

[3] Y. HIRANO and H. TOMINAGA: Some commutativity theorems for prime rings with derivations and differentially semiprime rings, Math. J. Okayama Univ. **26** (1984), 101–108.

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