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Yasuyuki Hirano* Hiroaki Komatsu† Isao Mogami‡

*Okayama University
†Okayama University
‡Tsuyama College Of Technology

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ON JACOBSON'S CONJECTURE

YASUYUKI HIRANO, HIROAKI KOMATSU and ISAO MOGAMI

Throughout, R will represent a ring (not necessarily with unity), and J the Jacobson radical of R. R is called *right* (resp. *left*) *weakly Noetherian* if for any ideal A of R and for any $x \in R$ there exists a positive integer n such that $x^{-n}A = x^{-m}A$ (resp. $Ax^{-n} = Ax^{-m}$) for all $m \geq n$, where $x^{-n}A = \{r \in R \mid x^n r \in A\}$ (resp. $Ax^{-n} = \{r \in R \mid rx^n \in A\}$); R is called *weakly Noetherian* if R is both right and left weakly Noetherian. R is called a *right* (resp. *left*) *duo ring* if every right (resp. left) ideal of R is two-sided; R is called a *duo ring* if R is a right and left duo ring. Finally, R is called *right bounded* if every essential right ideal of R contains a (two-sided) ideal of R which is essential as a right ideal; R is called *right fully bounded* if each prime factor ring of R is right bounded. Given $x \in R$, we denote by I_x the ideal $(1-x)R + R(1-x) + R(1-x)R$, where 1 is used formally. An ideal A of R is called a (*right*) *WAR-ideal* if for each $a \in A$ there exist positive integers m, n such that $A^m \cap I_a^n \subset I_a^n A$. Needless to say, every ideal of R having the right AR-property is a WAR-ideal.

We say that R satisfies the Jacobson's conjecture if $\bigcap_{n=1}^{\infty} J^n = 0$. All weakly Noetherian, duo rings with unity satisfy the Jacobson's conjecture ([4, Corollary 2]), and all (right and left) Noetherian, right fully bounded rings with unity also satisfy Jacobson's conjecture ([3, Theorem 3.7], see also [2, Theorem 7.5]). In the present paper, we shall prove two theorems which deduce the results mentioned just above.

Now, we shall begin our study with the next easy lemma.

Lemma 1. *Suppose that $RI = IR = I$ for every ideal I of R. Let A and B be ideals of R with $A+B = R$, and m a positive integer. Then the following are equivalent*:

1) $A^m B \subset BA$ (*resp.* $AB^m \subset BA$).
2) $A^m B \subset BA^m$ (*resp.* $AB^m \subset B^m A$).
3) $A^m \cap B = BA^m$ (*resp.* $B^m \cap A = B^m A$).
4) $A^m \cap B \subset BA$ (*resp.* $B^m \cap A \subset BA$).

Proof. Obviously, 2) ⇒ 3) ⇒ 4) ⇒ 1).

1) ⇒ 2). Since $R = A^{m^2} + B$ and $A^{(m+1)m}B \subset BA^{m+1}$, we get $A^m B = (A^{m^2}+B)A^m B = A^{(m+1)m}B + BA^m B \subset BA^{m+1} + BA^m B = BA^m(A+B) = BA^m$.

Corollary 1. *Suppose that $RI = IR = I$ for every ideal I of R. If A is an ideal of R, then the following are equivalent:*

1) *For each $a \in A$ there exist positive integers m, n such that $A^m I_a^n \subset I_a^n A$.*

2) *For each $a \in A$ there exists a positive integer m such that $A^m I_a^m \subset I_a^m A^m$.*

3) *For each $a \in A$ there exists a positive integer m such that $A^m \cap I_a^m = I_a^m A^m$.*

4) *A is a WAR-ideal.*

Lemma 2. *Let R be a right (resp. left) weakly Noetherian, right (resp. left) duo ring. Then, for any finitely generated ideal A and for any ideal B of R, there exists a positive integer n such that $A^n \cap B \subset AB$ (resp. $A^n \cap B \subset BA$).*

Proof. As is easily seen, there exists an ideal V of R which is maximal with respect to the property that $V \cap B \subset AB$. Noting that $(AB+V) \cap B = AB$, we see that $AB+V = V$, namely $AB \subset V$. Let a be an arbitrary element of A. Then there exists a positive integer k such that $a^{-k}V = a^{-h}V$ for all $h \geq k$. We set $W = a^kR+V$, and choose an arbitrary $b \in W \cap B$. Then $b = a^kx+v$ with some $x \in R$ and $v \in V$. Since $a^{k+1}x+av = ab \in AB \subset V$, we have $a^{k+1}x \in V$, and therefore $a^kx \in V$. Hence b is in V, whence it follows that $b \in V \cap B \subset AB$, namely $W \cap B \subset AB$. Now, by the maximality of V we get $W = V$, and hence $a^{k+1} \in V$. Then, as is easily seen, there exists a positive integer n such that $A^n \subset V$, and hence $A^n \cap B \subset AB$.

Corollary 2. *Let R be a ring with a.c.c. for ideals such that $RI = IR = I$ for every ideal I of R. If R is a right or left duo ring, then every ideal A of R is a WAR-ideal.*

Proof. Let B be an ideal of R with $A+B = R$. If R is a right (resp. left) duo ring, then there exists a positive integer m such that $AB^m \subset B^m \cap A \subset BA$ (resp. $A^mB \subset A^m \cap B \subset BA$), by Lemma 2. Hence, by Lemma 1, $AB^m \subset B^mA$ (resp. $A^mB \subset BA^m$).

Lemma 3. *Let R be a ring with unity in which every (right) primitive ideal is maximal, and M a unital right R-module of finite length. If A is a WAR-ideal of R with $MA = M$, then $M(1-c) = 0$ for some $c \in A$.*

Proof. By hypothesis, M has a composition series $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_t = M$, and the primitive ideal $P_i = \mathrm{Ann}_R(M_i/M_{i-1})$ is maximal ($i = 1, 2, \ldots, t$). Let us set $Q = P_1 \cap P_2 \cap \cdots \cap P_t$. Then $Q^t \subset \mathrm{Ann}_R(M)$. Since R/Q is a finite direct sum of simple rings, there exists an ideal B of R such that $A+B = R$ and $A \cap B \subset Q$. Write $1 = a+b$, $a \in A$ and $b \in B$. Then $I_a \subset B$. Since A is a WAR-ideal, there exists a positive integer m such that $A^m I_a^m \subset I_a^m A^m$ (Corollary 1). Hence $MI_a^{mt} = MA^{mt}I_a^{mt} \subset M(I_a^m A^m)^t \subset MQ^t = 0$, whence $I_a^{mt} \subset \mathrm{Ann}_R(M)$ follows. Since $A+I_a^{mt} = R$, this implies that $A+\mathrm{Ann}_R(M) = R$. Hence $M(1-c) = 0$ for some $c \in A$.

Theorem 1 (cf. [4, Corollary 1]). *Let R be a weakly Noetherian duo ring. If A is a finitely generated ideal of R, then $\bigcap_{n=1}^{\infty} A^n = \{r \in R \mid r(1-c) = 0$ for some $c \in A\} = \{r \in R \mid (1-c)r = 0$ for some $c \in A\}$; in particular, $(\bigcap_{n=1}^{\infty} A^n)A = A(\bigcap_{n=1}^{\infty} A^n) = \bigcap_{n=1}^{\infty} A^n$* and $\bigcap_{n=1}^{\infty} J^n = 0$.

Proof. If $r = rc$ (or $r = cr$) for some $c \in A$ then $r \in \bigcap_{n=1}^{\infty} A^n$. Conversely, suppose that $r \in \bigcap_{n=1}^{\infty} A^n$. By Lemma 2, there exists a positive integer m such that $r \in A^m \cap (r) \subset A\cdot(r) \cap (r)\cdot A = Ar \cap rA$.

Theorem 2. *Let R be a Noetherian, right fully bounded ring with unity, and M a finitely generated unital right R-module. If A is a WAR-ideal of R then $\bigcap_{n=1}^{\infty} MA^n = \{u \in M \mid u(1-c) = 0$ for some $c \in A\}$; in particular, $(\bigcap_{n=1}^{\infty} MA^n)A = \bigcap_{n=1}^{\infty} MA^n$ and $\bigcap_{n=1}^{\infty} MJ^n = 0$.*

Proof. It suffices to show that $u \in uA$ for every $u \in \bigcap_{n=1}^{\infty} MA^n$. In view of [5, Remark, p.329], every primitive factor ring of R is Artinian simple, and so every primitive ideal of R is maximal. As is well known, there exists a family $\{M_\lambda\}_{\lambda \in \Lambda}$ of submodules of M with M/M_λ subdirectly irreducible such that $\bigcap_{\lambda \in \Lambda} M_\lambda = uA$. Since M/M_λ is Artinian by [2, Theorem 7.10], there exists a positive integer m_λ such that $(MA^{m_\lambda}+M_\lambda)/M_\lambda = (MA^{m_\lambda+1}+M_\lambda)/M_\lambda$. Now, by Lemma 3, we can find $c_\lambda \in A$ such that $u(1-c_\lambda) \in MA^{m_\lambda}(1-c_\lambda) \subset M_\lambda$. Since $uc_\lambda \in uA \subset M_\lambda$, this proves that $u \in M_\lambda$ for all λ, and hence $u \in uA$.

Combining Theorem 2 with [1, Theorem 9], we readily obtain

Corollary 3. *Let R be a Noetherian PI-ring with unity, and M a finitely generated unital right R-module. Then, for any WAR-ideal A of R, $\bigcap_{n=1}^{\infty} MA^n = \{u \in M \mid u(1-c) = 0$ for some $c \in A\}$.*

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Corollary 4. *Let R be a left Noetherian, right duo ring with unity, and M a finitely generated unital right R-module. Then, for any ideal A of R, $\bigcap_{n=1}^{\infty} MA^n = \{u \in M \mid u(1-c) = 0$ for some $c \in A\}$.*

Proof. Obviously, R is Noetherian and right fully bounded. Since A is a WAR-ideal by Corollary 2, the assertion is clear by Theorem 2.

Remark 1. Let A be a polycentral ideal of a ring R with unity, namely suppose that there exists a finite number of elements $c_1, c_2, \dots, c_n$ in A such that $A = Rc_1 + Rc_2 + \cdots + Rc_n$, c_1 is central, and for any $r \in R$, $rc_i - c_i r \in Rc_1 + \cdots + Rc_{i-1}$ $(i = 2, 3, \dots, n)$. For any ideal B of R, we have $AB \subset BA + Rc_1 + \cdots + Rc_{n-1}$, whence $AB^2 \subset BA + Rc_1 + \cdots + Rc_{n-2}$ follows, and eventually $AB^n \subset BA$. If $A + B = R$, then $AB^n \subset B^n A$ by Lemma 1. This proves that every polycentral ideal of R is a WAR-ideal. In case R is right Noetherian, every polycentral ideal of R has the right AR-property, by [2, Theorem 11.7].

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OKAYAMA UNIVERSITY

OKAYAMA UNIVERSITY

TSUYAMA COLLEGE OF TECHNOLOGY

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