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Some Generalizations of Boolean Rings

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SOME GENERALIZATIONS OF BOOLEAN RINGS

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Throughout, R will represent a ring with center C. Let N denote the set of nilpotents in R, and N^* the subset of N consisting of all elements in R which square to zero. Let E be the set of idempotents in R. If $E \subseteq C$ then R is called *normal*. In case R has 1, we denote by U the multiplicative group of units of R. Following [10], R is called *(E-N) representable*, if each $x \in R$ can be written uniquely in the form $x = e + a$, where $e \in E$ and $a \in N$. Given $x \in R$, we define inductively $x^{(1)} = x$, $x^{(k)} = x^{(k-1)} \circ x$, where $x \circ y = x + y + xy$. In [8], Hirano, Komatsu, Tominaga and Yaqub considered the following condition which arose, presummably, in connection with logic:
(*) for any $x, y \in R$, $(x+xy) \circ (y+yx) = 0$ if and only if $x = y$,
and proved that R satisfies (*) if and only if R is commutative, R/N is a Boolean ring and $a^{(2)} = 0$ for all $a \in N$ (see Theorem 1 below). Obviously, every Boolean ring satisfies the condition (*). If R has 1, then (*) becomes
(*)′ for any $x, y \in R$, $(1+x+xy)(1+y+yx) = 1$ if and only if $x = y$.
Recently, Grosen [4] gave a number of characterizations of a ring with 1 in which the condition (*)′ holds.

An element x in R is called *strongly regular*, if there exist $y, y' \in R$ such that $x^2y = x = y'x^2$. As is well-known, if x is strongly regular, there exists (uniquely) $z \in R$ such that $x^2z = x$, $z^2x = z$ and $xz = zx$; furthermore, z commutes with every element which commutes with x. We denote by S the set of strongly regular elements in R. A ring R is called a *B'-ring* if $S = E$. Obviously, every Boolean ring is a B'-ring.

A ring R is called *s-unital* if $x \in Rx \cap xR$ for all $x \in R$, or equivalently if for each finite subset F of R there exists $e \in R$ such that $ex = x = xe$ for all $x \in F$ (see [6]). Following [11], R is called an *s^*-unital ring* if for each $x \in R$ there exist $e', e'' \in E$ such that $xe' = x = e''x$, or equivalently if for each finite subset F of R there exists $e \in E$ such that $eFe = F$ (see [11, Corollary 7]). As is easily seen, every s-unital π-regular ring is s^*-unital. In what follows, we shall use freely this fact. A ring R is a *cs^*-unital ring* if for each $x \in R$ there exists a central idempotent e such that $ex = x$.

A ring R is called an *I-ring* (resp. *N-ring*) if every element of R is expressible as a product of elements in E (resp. N); R is called an *NI-ring* (or *I′-ring*) if every element of R is expressible as a product of elements in

$E \cup N$ (see [1] and [7]). Needless to say, every Boolean ring is an I-ring.

Our present objective is to improve several results of Grosen obtained in [4, § 5] and the main theorems of Abu-Khuzam [1] and reprove the main theorems of [2].

First, as preliminaries, we state the following lemmas.

Lemma 1 ([10, Theorem 4]). *The following are equivalent:*

1) *R is (E-N) representable.*

2) *R is normal, and every element of R can be written as a sum of an idempotent and a nilpotent element.*

3) *R is normal and $x-x^2 \in N$ for every $x \in R$.*

4) *R is normal, N is an ideal and R/N is a Boolean ring.*

Lemma 2 ([8, Lemma 5]). *Let $f(X) = k_1X+k_2X^2+\cdots+k_mX^m$ be a polynomial in $X\mathbf{Z}[X]$ with $(k_1, k_2) = 1$. If N satisfies the identity $f(X) = 0$, then N satisfies the identities $X^3 = 0 = k_1X+(k_2-k_1)X^2$.*

Lemma 3. *If N is closed under $\circ$ (in particular, if N is an ideal) and satisfies the identity $X^{(2)} = 0$, then N is commutative.*

Proof. For any $a, b \in N$, $a \circ b = a \circ (a \circ b)^{(2)} \circ b = a^{(2)} \circ (b \circ a) \circ b^{(2)} = b \circ a$, whence $ab = ba$ follows.

Lemma 4. (1) *If R satisfies the identity $(X+X^2)^{(2)} = 0$, then $8x = 0$, $x^5 = x^3$ and $x-x^2 \in N$ (or $x+x^2 \in N$) for all $x \in R$, and $a^3 = 0 = a^{(2)}$ for all $a \in N$.*

(2) *If N satisfies the identity $(X+X^2)^{(2)} = 0$, then $4a = 0$ and $a^3 = 0 = a^{(2)}$ for all $a \in N$.*

Proof. (1) Since $6x^2+2x^4 = (x+x^2)^{(2)}+(-x+(-x)^2)^{(2)} = 0$ and $4x+4x^3 = (x+x^2)^{(2)}-(-x+(-x)^2)^{(2)} = 0$, we get $8x = (4x+4x^3)(2+x^2)-2(6x^2+2x^4)x = 0$. Further, noting that $2x+3x^2+2x^3+x^4 = (x+x^2)^{(2)} = 0$, we can easily see that $a^3 = 0 = a^{(2)}$ for all $a \in N$ (Lemma 2). Since $(x+x^2)^6 = \{(x+x^2)^{(2)}-2(x+x^2)\}^3 = -8(x+x^2)^3 = 0$, we have $(x+x^2)^3 = 0$ (and $(x-x^2)^3 = 0$) by the above, and therefore $x^5-x^3 = (x+x^2)^3-(x+x^2)^{(2)}x^2 = 0$.

(2) By the proof of (1), we obtain $8a = 0$ and $a^3 = 0 = a^{(2)}$ for all $a \in N$. Hence $4a = -2a^2 = a^3 = 0$.

Lemma 5. *Let $x \in R$. If $2x \in N$ and $x^n - x^{n+2^k} \in N$ for some integers $n > 0$ and $k \geq 0$, then $x - x^2 \in N$.*

Proof. As is easily seen,

$$(x-x^2)^{2^k}x^n = (x^n - x^{n+2^k})x^{2^k} + \sum_{i=1}^{2^k-1}(-1)^i\binom{2^k}{i}x^{2^k+i+n} + 2x^{2^{k+1}+n} \in N.$$

Hence $x - x^2 \in N$.

Lemma 6. *The following are equivalent:*

1) *R is normal.*

2) *If $e, f \in E$ and $e - f \in N^*$, then $e = f$.*

In particular, if $(*)$ holds in $-E$, then R is normal.

Proof. If $e, f \in E$, $ef = fe$ and $e - f \in N^*$, then $e - f = (e-f)^3 = 0$. Conversely, suppose 2). Let $e \in E$, and $x \in R$. Then $f = e - ex(1-e) \in E$ and $e - f = ex(1-e) \in N^*$. Hence we have $ex = exe$; similarly, $xe = exe$. This proves that R is normal. Now, let $e, f \in E$. Then $(-e+(-e)(-f)) \circ (-f+(-f)(-e)) = ef + fe - e - f$. This enables us to see the latter assertion.

Corollary 1. *Suppose that $x^2y - y^2x \in N \cap C$ for all $x, y \in R \backslash N$. Then $x - x^2 \in N$ for all $x \in R$, and R is normal.*

Proof. If $x \in N$, clearly $x - x^2 \in N$. If $x \in R \backslash N$, then $(x-x^2)x^3 = x^2 \cdot x^2 - (x^2)^2 x \in N$. Thus $x - x^2 \in N$ for all $x \in R$. Now, let $e, f \in E$ and $e - f \in N^*$. Then $ef + fe = e + f$ and $ef - fe \in C$, and so $e = e(ef + fe - e - f)e + e = efe = efe + \{e(ef - fe) - (ef - fe)e\} = -efe + ef + fe = -e + e + f = f$. Hence R is normal, by Lemma 6.

Lemma 7. *Let R be a ring with 1. If $U \subseteq E + N$, then $2 \in N$. If, furthermore, R is normal and for each $x \in R \backslash U$ there exist integers $n > 0$ and $k \geq 0$ such that $x^n - x^{n+2^k} \in N$, then $x - x^2 \in N$ for all $x \in R$.*

Proof. Let $-1 = e + a$, $e \in E$ and $a \in N$. Then $-(1+a) = e = 1$, since $-(1+a) \in U$. Hence $2 = -a \in N$. If R is normal, then $u - u^2 \in N$ for any $u \in U$. Now, the latter assertion is clear, by Lemma 5.

We are now ready to complete the proof of our first theorem.

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Theorem 1. *The following are equivalent:*

1) *R satisfies* (*).

2) *R is commutative, $x-x^2 \in N$ for all $x \in R$ (or R/N is a Boolean ring), and $a^{(2)} = 0$ for all $a \in N$.*

3) *R is normal, $x-x^2 \in N$ for all $x \in R$, and $a^{(2)} = 0$ for all $a \in N$.*

4) *R is (E-N) representable and $a^{(2)} = 0$ for all $a \in N$.*

5) *R is normal and satisfies the identity $(X+X^2)^{(2)} = 0$.*

6) *R is normal, N satisfies the identity $(X+X^2)^{(2)} = 0$, and $x-x^2 \in N$ for all $x \in R$.*

7) *R is normal, $2R \subseteq N$, for each $x \in R$ there exist integers $n > 0$ and $k \geq 0$ such that $x^n - x^{n+2^k} \in N$, and $a^{(2)} = 0$ for all $a \in N$.*

8) *N satisfies the identity $(X+X^2)^{(2)} = 0$, and $x^2y-y^2x \in N \cap C$ for all $x, y \in R \backslash N$.*

Proof. By Lemma 6, 1) implies 5).

3) ⇨ 4) ⇨ 2). By Lemma 1, and Lemmas 1 and 3, respectively.

5) ⇨ 7) ⇨ 3). By Lemma 4 (1), and Lemma 5, respectively.

8) ⇨ 6) ⇨ 3). By Corollary 1, and Lemma 4 (2), respectively.

1) ⇨ 8). We have seen that 1) implies 3) and 2). Hence $x^2y-y^2x = (x^2-x)y-(y^2-y)x \in N$ for all $x, y \in R$.

2) ⇨ 1). Let $x, y \in R$, and put $a = x+xy$, $b = y+yx$. Obviously, $x+x^2 \in N$, and $(x+x^2)^{(2)} = 0$. Conversely, if $a \circ b = 0$ then $a^2+(a+a^2)b = a(a \circ b) = 0$, and so $a^2 = -(a+a^2)b \in N$. This implies that $a \in N$. Hence $y+xy = 0 \circ b = a^{(2)} \circ b = a \circ (a \circ b) = a \circ 0 = x+xy$, whence $y = x$ follows.

The next includes [4, Theorems 5.5, 5.6 and Corollaries 5.1, 5.3, 5.7] and improves [4, Theorems 5.14, 5.15 and Corollary 5.6].

Corollary 2. *Let R be a ring with 1. Then the following are equivalent:*

1) *R satisfies* (*).

2) *R is commutative, R/N is a Boolean ring, and $u^2 = 1$ for all $u \in U$ (or $(1+a)^2 = 1$ for all $a \in N$).*

3) *R is normal, $x-x^2 \in N$ for all $x \in R$, and $u^2 = 1$ for all $u \in U$.*

4) *R is (E-N) representable and $u^2 = 1$ for all $u \in U$.*

5) *R is normal and satisfies the identity $(X+X^2)^{(2)} = 0$.*

6) *R is normal, N satisfies the identity $(X+X^2)^{(2)} = 0$, and $x-x^2 \in N$ for all $x \in R$.*

7) *R is normal, $2 \in N$, and for each $x \in R$ there exist integers $n > 0$*

and $k \geq 0$ *such that* $x^n - x^{n+2^k} \in N$, *and* $u^2 = 1$ *for all* $u \in U$.

8) *N satisfies the identity* $(X+X^2)^{(2)} = 0$, *and* $x^2y - y^2x \in N \cap C$ *for all* $x, y \in R \backslash N$.

9) *R is normal, U satisfies the identity* $(X+X^2)^{(2)} = 0$, *and* $x - x^2 \in N$ *for all* $x \in R$.

10) *R is normal,* $2 \in N$, *and for each* $x \in R$ *there exists a positive integer n such that* $x^n - x^{n+2} = 0$.

11) *R is normal,* $2 \in N$, *for each* $x \in R$ *there exist integers* $n > 0$ *and* $k \geq 0$ *such that* $x^n - x^{n+2^k} \in N$, *and if* $u, v \in U$ *and* $u - v \in N$ *then* $u^2 = v^2$.

12) *R is normal,* $U \subseteq E + N$, *for each* $x \in R \backslash U$ *there exist integers* $n > 0$ *and* $k \geq 0$ *such that* $x^n - x^{n+2^k} \in N$, *and if* $u, v \in U$ *and* $u - v \in N$ *then* $u^2 = v^2$.

Proof. Obviously, 1) $\Rightarrow$ 11) and 12), and the equivalence of 1)$-$10) is clear by Lemma 4 (1) and Theorem 1.

11) (resp. 12)) $\Rightarrow$ 3). By Lemma 5 (resp. Lemma 7), $x - x^2 \in N$ for all $x \in R$. In particular, for each $u \in U$, we obtain $1 - u = u^{-1}(u - u^2) \in N$, and so $1 = u^2$.

Theorem 2. *The following are equivalent:*

1) *R satisfies* (*).

2) $2R \subseteq N$, *and there exists a subset A of R containing* $N \cup (-E)$ *such that* (*) *holds in A and* $R \backslash A \subseteq E + N$.

3) *R is normal, and there exists a subset A of R containing N and satisfying the identity* $(X+X^2)^{(2)} = 0$ *such that* $R \backslash -A \subseteq E + N$.

Proof. By Theorem 1, 1) $\Rightarrow$ 2) and 3).

2) $\Rightarrow$ 1). By Lemma 6, R is normal, and so $x - x^2 \in N$ for all $x \in R \backslash A$. Now, let $x \in A$. Then $(x - x^2)^2 = (x + x^2)^2 - 4x^3 = -2(x + x^2 + 2x^3) \in N$. Hence $x - x^2 \in N$ for all $x \in R$, and therefore R satisfies (*), by Theorem 1 6).

3) $\Rightarrow$ 1). In view of Theorem 1, it suffices to show that $x - x^2 \in N$ for all $x \in R$. First, we consider the case that $x \in A$. If $-x \notin A$, clearly $x - x^2 \in N$. If $-x \in A$ then, by the proof of Lemma 4 (1), $8x = 0$, and so $2x \in N$. Hence $(x - x^2)^2 = (x + x^2)^2 - 4x^3 = -2(x + x^2 + 2x^3) \in N$; $x - x^2 \in N$. Next, we consider the case that $x \notin A$: $a = x + x^2 \in N$. Since $2x \notin A$ forces a contradiction $2x = 4a - (2x + 4x^2) \in N \subseteq A$, we see that $2x \in A$. Then $(4a - 2x)^2 = (2x + 4x^2)^2 = -2(2x + 4x^2) = -2(4a$

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$-2x$), whence $4(x-x^2) \in N$ follows. Combining this with $x+x^2 = a \in N$, we obtain $8x \in N$, and so $2x^2 \in N$ and $x-x^2 = a-2x^2 \in N$.

Let R be a ring with 1. A subset A of R is called a *weakly normal subset* if for each $x \in R$, either $-x$ or $x-1$ is in A ; a weakly normal subset A of R is called a *normal subset* if $e, f \in E$ and $e-f \in N^*$ imply $-e, -f \in A$ or $-e, -f \notin A$. As is easily seen, if a weakly normal subset A of R satisfies the identity $(X+X^2)^{(2)} = 0$ then R satisfies the same identity ; if (*) holds in a normal subset A of R then R is normal. (Note that if $e, f \in E$, then $(-e+(-e)(-f)) \circ (-f+(-f)(-e)) = ef+fe-e-f$ and $(e-1+(e-1)(f-1)) \circ (f-1+(f-1)(e-1)) = ef+fe-e-f$.)

The next includes [4, Theorems 5.1, 5.2, 5.7, 5.12 and 5.13].

Corollary 3. *Let R be a ring with* 1. *Then the following are equivalent*:

1) *R satisfies* (*).

2) $2 \in N$, *and there exists a subset A of R containing $N \cup (-E)$ such that* (*) *holds in A and $R \backslash A \subseteq E+N$.*

3) *There exists a subset A of R containing $U \cup (-E)$ such that* (*) *holds in A and $R \backslash A \subseteq E+N$.*

4) *R is normal, and there exists a subset A of R containing N and satisfying the identity $(X+X^2)^{(2)} = 0$ such that $R \backslash -A \subseteq E+N$.*

5) *There exists a subset A of R satisfying the identity $(X+X^2)^{(2)} = 0$ such that $A \supseteq N$, $(-A) \cap E \subseteq \{0, 1\}$ and every element in $R \backslash -A$ is uniquely expressible as $e+a$ with $e \in E$ and $a \in N$.*

6) *R is normal, and there exists a weakly normal subset A of R satisfying the identity $(X+X^2)^{(2)}=0$.*

7) *There exists a normal subset A of R in which* (*) *holds.*

Proof. Obviously, 1) ⇨ 2)−7). By Theorem 2, each of 2) and 4) implies 1). Further, combining Corollary 2 with the remark stated just above, we readily see that each of 6) and 7) implies 1).

3) ⇨ 1). Obviously, $8 = (1+1^2)^2+2(1+1^2) = 0$, and so $2 \in N$. Furthermore, R is normal, by Lemma 6. Now, it is easy to see that $x-x^2 \in N$ for all $x \in R$. (See the proof of 2) ⇨ 1) of Theorem 2.) Hence R satisfies (*), by Corollary 2 9).

5) ⇨ 1). By Lemma 6 and Theorem 2.

The next proves the latter part of [2, Theorem 2] and improves [12, Theorem B].

Theorem 3. (1) *If for each $x \in R$ there exists a positive integer n such that $x^{n+1} = x^n$, then N is an ideal of R and R/N is a Boolean ring.*

(2) *Let R be an s-unital ring. If for each $x \in R$ there exists a positive integer n such that $x^{n+1} - x^n \in C$, then R is commutative.*

Proof. (1) In the complete matrix ring $M_t(D)$ over a division ring D with $t > 1$, $(1+e_{12})^{k+1} \neq (1+e_{12})^k$ for each positive integer k. Thus, in virtue of the structure theorem of primitive rings, we can easily see that any primitive homomorphic image of R is a division ring. This shows that R/J is a reduced ring, where J is the Jacobson radical of R. Since J is a nil ideal, we conclude that $J = N$ and R/N is a Boolean ring.

(2) In virtue of [6, Proposition 1], we may assume that R has 1. Let x be an arbitrary element in R. Then there exists a positive integer n such that $(1+x)^{n+1} - (1+x)^n \in C$. Since $(1+X)^{n+1} - (1+X)^n = X - X^2 f(X)$ with some $f(X) \in \mathbf{Z}[X]$, R is commutative by [5, Theorem 19].

Now, we shall reprove [2, Theorem 1].

Theorem 4. *The following are equivalent:*

1) *R is a Boolean ring.*

2) *R is an s-unital, π-regular B'-ring.*

3) *R is an s-unital B'-ring satisfying the identity $(X+X^2)^{(2)} = 0$.*

4) *R is a cs*-unital B'-ring and an NI-ring.*

5) *R is a B'-ring and an I-ring.*

6) *R is a semiprime I-ring and N^* is commutative.*

7) *R is a semiprime NI-ring and PI ring, and N^* is commutative.*

8) *R is an s-unital ring, and for each $x \in R$ there exists a positive integer n such that $x^{n+1} = x^n$.*

Proof. Obviously, 1) implies 3), 4), 7) and 8).

3) $\Rightarrow$ 2). By Lemma 4 (1).

4) $\Rightarrow$ 5). By [7, Lemma 1].

5) $\Rightarrow$ 1). Let $a \in N^*$, and choose $e \in E$ with $eae = a$ Then $e - a = (e-a)^2(e+a)$, and $e - a \in E$, whence $a = 0$ follows. Hence $N = 0$ and E is central.

2) $\Rightarrow$ 1). As above, we see that $N = 0$. Now, let $x \in R$. Then there exists $y \in R$ such that $x^n y x^n = x^n$ for some n. Since $x^n y$ and yx^n are central idempotents, we obtain $x^{2n} y = x^n = yx^{2n}$. As is well-known, there exists $z \in R$ such that $xz = zx$ and $x^{n+1}z = x^n$. Then $(x - x^2 z)^n =$

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$\sum_{i=0}^{n}(-1)^i\binom{n}{i}x^{n+i}z^i = \sum_{i=0}^{n}(-1)^i\binom{n}{i}x^n = (x-x)^n = 0$, whence $x = x^2z$. This proves that R is strongly regular, and consequently Boolean.

6) ⇒ 1). Let $e \in E$. Then $(1-e)ReR(1-e)Re = (1-e)R\{eR(1-e)\cdot(1-e)Re\}e = (1-e)R\{(1-e)Re\cdot eR(1-e)\}e = 0$, whence $(1-e)Re = 0$ follows ; similarly, $eR(1-e) = 0$. Hence E is central and R is Boolean.

7) ⇒ 6). By [9, Theorem 3], $N = 0$.

8) ⇒ 1). By Theorem 3 (1), it suffices to show that $N = 0$. Suppose, to the contrary, that $N \neq 0$, and choose a non-zero a in N^*. Then there exists an idempotent e such that $ea = ae = a$. By hypothesis, there exists a positive integer n such that $(e+a)^{n+1} = (e+a)^n$. But this forces a contradiction $a = 0$.

Corollary 4 (cf. [2, Lemma 1 (3) and Theorem 2]). *If R is a π-regular B'-ring, then for each $x \in R$ there exists a positive integer n such that $x^{n+1} = x^n$.*

Proof. There exists $y \in R$ such that $x^m y x^m = x^m$ for some m. Then $e' = x^m y$ is an idempotent and $e'Re'$ is a Boolean ring by Theorem 4 2). Hence $x^{2m}y = e'x^m e'$ is an idempotent, and so $x^{2m} = x^{2m}yx^m = (x^{2m}y)^2x^m = x^{3m}$. This proves that $e = x^{2m}$ is in E. Again by Theorem 4 2), eRe is a Boolean ring, and therefore $x^{2m+2} = ex^2 = (exe)^2 = exe = x^{2m+1}$.

Finally, we state the following which includes [1, Theorems 1, 2 and 3]

Theorem 5. *Let R be an NI-ring.*

(1) *If R is Artinian, then N is a nilpotent ideal of R and R/N is the finite direct sum of copies of* GF(2).

(2) *If R is a π-regular PI ring, then N coincides with the prime radical of R and R/N is a Boolean ring.*

(3) *If N is commutative, then N is a commutative ideal of R and R/N is a Boolean ring.*

Proof. (1) As is well-known, the Jacobson radical J of R is nilpotent and R/J is a finite direct sum of matrix rings over division rings. Then, by [7, Lemma 1], R/J is a Boolean ring and $J = N$.

(2) This is [7, Corollary 1].

(3) By [3, Theorem 2], N is a commutative ideal of R. Since R/N is

a reduced I-ring, it is normal and Boolean.

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