RADIAL HEAT OPERATORS ON JACOBI-LIKE FORMS

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Abstract

We consider a differential operator D^X_λ associated to an integer λ acting on the space of formal power series, which may be regarded as the heat operator with respect to the radial coordinate in the 2λ-dimensional space for $\lambda > 0$. We show that D^X_λ carries Jacobilike forms of weight λ to ones of weight $\lambda+2$ and obtain the formula for the m-fold composite $(D^X_\lambda)^{[m]}$ of such operators. We then determine the corresponding operators on modular series and as well as on automorphic pseudodifferential operators.

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ABSTRACT. We consider a differential operator $\mathcal{D}_\lambda^X$ associated to an integer λ acting on the space of formal power series, which may be regarded as the heat operator with respect to the radial coordinate in the 2λ-dimensional space for $\lambda > 0$. We show that $\mathcal{D}_\lambda^X$ carries Jacobi-like forms of weight λ to ones of weight $\lambda + 2$ and obtain the formula for the m-fold composite $(\mathcal{D}_\lambda^X)^{[m]}$ of such operators. We then determine the corresponding operators on modular series and as well as on automorphic pseudodifferential operators.

1. INTRODUCTION

Jacobi forms were systematically introduced by Eichler and Zagier in [4], and, since then, they have been studied extensively in connection with various topics in number theory. Jacobi-like forms are formal power series, whose coefficients are holomorphic functions on the Poincaré upper half plane $\mathcal{H}$, satisfying a certain transformation formula with respect to an action of a discrete subgroup Γ of $SL(2, \mathbb{R})$. This formula is essentially one of the two equations that must be satisfied by Jacobi forms. The coefficients of a Jacobi-like form are closely linked to modular forms for Γ. For example, they may be regarded as special types of quasimodular forms, which generalize modular forms (see e.g. [6, Section 17.1]). Furthermore, there is a one-to-one correspondence between the space of Jacobi-like forms and the space of certain sequences of modular forms. This isomorphism is obtained by expressing each coefficient of a Jacobi-like form as a linear combination of derivatives of modular forms in the corresponding sequence (see [2] and [7]).

Modular series are certain formal power series whose coefficients are modular forms, and they can be defined by slightly modifying the transformation formula for Jacobi-like forms. To each sequence of modular forms corresponding to a Jacobi-like form, a modular series is naturally associated by using the terms of the sequence as the coefficients of a formal power series. Then the above-mentioned correspondence between Jacobi-like forms and sequences of modular forms can be interpreted as an isomorphism between the space of Jacobi-like forms and that of modular series.

M.H. LEE

Pseudodifferential operators are formal Laurent series in the formal inverse ∂^{-1} of the derivative operator ∂ whose coefficients are complex-valued functions, and they have been studied in numerous papers over the years in connection with a wide variety of problems in pure and applied mathematics. For example, they play a crucial role in the theory of nonlinear integrable partial differential equations, also known as soliton equations (see e.g. [3]). When the coefficients are holomorphic functions on $\mathcal{H}$, the usual linear fractional operation of $SL(2,\mathbb{R})$ on $\mathcal{H}$ induces an action of $SL(2,\mathbb{R})$ on the space of pseudodifferential operators. Automorphic pseudodifferential operators are elements of this space that are invariant under the action of a discrete subgroup Γ of $SL(2,\mathbb{R})$. It is known (cf. [2]) that there is a one-to-one correspondence between automorphic pseudodifferential operators and Jacobi-like forms.

The above descriptions suggest that the spaces of Jacobi-like forms, modular series, and automorphic pseudodifferential operators are isomorphic to one another. In this paper we consider a differential operator $\mathcal{D}_\lambda^X$ associated to an integer λ acting on the space of formal power series, which may be regarded as the heat operator with respect to the radial coordinate in the 2λ-dimensional space for $\lambda > 0$. We show that $\mathcal{D}_\lambda^X$ carries Jacobi-like forms of weight λ to ones of weight $\lambda + 2$ and obtain the formula for the m-fold composite $(\mathcal{D}_\lambda^X)^{[m]}$ of such operators. We then determine the operators on the spaces of modular series and automorphic pseudodifferential operators corresponding to $(\mathcal{D}_\lambda^X)^{[m]}$.

2. Jacobi-like forms and modular series

In this section we review some basic properties of Jacobi-like forms and modular series. In particular, we describe an isomorphism between the space of Jacobi-like forms and and that of modular series.

Let $\mathcal{H}$ be the Poincaré upper half plane on which the group $SL(2,\mathbb{R})$ acts as usual by linear fractional transformations. Thus we may write

$$\gamma z = \frac{az+b}{cz+d}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$. For the same z and γ, we set

$$(2.1) \qquad \mathfrak{J}(\gamma, z) = cz + d, \quad \mathfrak{K}(\gamma, z) = c\mathfrak{J}(\gamma, z)^{-1} = \frac{c}{cz+d}.$$

If $f : \mathcal{H} \to \mathbb{C}$ is a function, for $\gamma \in SL(2,\mathbb{R})$ and $w \in \mathbb{Z}$ we denote by $f \mid_w \gamma$ the function on $\mathcal{H}$ defined by

$$(f \mid_w \gamma)(z) = \mathfrak{J}(\gamma, z)^{-w} f(\gamma z)$$

for all $z \in \mathcal{H}$. Throughout this paper we fix a discrete subgroup Γ of $SL(2, \mathbb{R})$.

Definition 2.1. Given an integer w, a *modular form of weight w for* Γ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying

$$f \mid_w \gamma = f$$

for all $\gamma \in \Gamma$. We denote by $M_w(\Gamma)$ the space of modular forms of weight w for Γ.

Remark 2.2. We have modified the usual definition of modular forms by suppressing the finiteness condition at the cusps. This allows, for example, the consideration of modular forms of negative weight.

Let R denote the ring of holomorphic functions on $\mathcal{H}$, and let $R[[X]]$ be the complex algebra of formal power series in X with coefficients in R. Given a formal power series $\Phi(z, X) \in R[[X]]$, an integer λ, and an element $\gamma \in SL(2, \mathbb{R})$, we set

$$(2.2) \qquad (\Phi \mid_\lambda^J \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} e^{-\mathfrak{K}(\gamma, z)X} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2} X),$$

$$(2.3) \qquad (\Phi \mid_\lambda^M \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2} X)$$

for all $z \in \mathcal{H}$. If γ' is another element of Γ, it can be shown that

$$\Phi \mid_\lambda^J (\gamma \gamma') = (\Phi \mid_\lambda^J \gamma) \mid_\lambda^J \gamma', \quad \Phi \mid_\lambda^M (\gamma \gamma') = (\Phi \mid_\lambda^M \gamma) \mid_\lambda^M \gamma';$$

hence the operations $\mid_\lambda^J$ and $\mid_\lambda^M$ determine right actions of $SL(2, \mathbb{R})$ on $R[[X]]$.

Definition 2.3. Let $\Phi(z, X)$ be a formal power series belonging to $R[[X]]$, and let λ be an integer.

(i) $\Phi(z, X)$ is a *Jacobi-like form for* Γ *of weight* λ if it satisfies

$$(\Phi \mid_\lambda^J \gamma)(z, X) = \Phi(z, X)$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$. We denote by $\mathcal{J}_\lambda(\Gamma)$ the space of all Jacobi-like forms for Γ of weight λ

(ii) $\Phi(z, X)$ is a *modular series for* Γ *of weight* λ if it satisfies

$$(2.4) \qquad (\Phi \mid_\lambda^M \gamma)(z, X) = \Phi(z, X)$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$. We denote by $\mathcal{M}_\lambda(\Gamma)$ the space of all modular series for Γ of weight λ.

Remark 2.4. Modular series may be regarded as special types of Jacobi-like forms in the following sense. In addition to the weight, we may also introduce an index to a Jacobi-like form, so that a Jacobi-like form of weight λ and index $\mu \in \mathbb{C}$ is defined by using the operation

$$(\Phi \mid_{\lambda, \mu}^J \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} e^{-\mu \mathfrak{K}(\gamma, z)X} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2} X)$$

for $\gamma \in \Gamma$ instead of (2.3). Then Jacobi-like forms of index 1 are simply Jacobi-like forms in the sense of Definition 2.3. On the other hand, modular series for Γ of weight λ are Jacobi-like forms of weight λ and index 0.

If δ is a nonnegative integer, we set

$$R[[X]]_\delta = X^\delta R[[X]],$$

so that an element $\Phi(z, X) \in R[[X]]_\delta$ can be written in the form

$$(2.5) \qquad \Phi(z, X) = \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta}$$

with $\phi_k \in R$ for each $k \geq 0$. Similarly, we denote by $\mathcal{J}_\lambda(\Gamma)_\delta$ and $\mathcal{M}_\lambda(\Gamma)_\delta$ the subspaces of $\mathcal{J}_\lambda(\Gamma)$ and $\mathcal{M}_\lambda(\Gamma)$, respectively, defined by

$$(2.6) \qquad \mathcal{J}_\lambda(\Gamma)_\delta = \mathcal{J}_\lambda(\Gamma) \cap R[[X]]_\delta, \quad \mathcal{M}_\lambda(\Gamma)_\delta = \mathcal{M}_\lambda(\Gamma) \cap R[[X]]_\delta.$$

Lemma 2.5. *The formal power series $\Phi(z, X)$ in (2.5) is a modular series belonging to $\mathcal{M}_\lambda(\Gamma)_\delta$ if and only if*

$$\phi_k \in M_{2k+2\delta+\lambda}(\Gamma)$$

for each $k \geq 0$.

Proof. This follows easily from (2.3), (2.4) and Definition 2.1. $\qquad\square$

Proposition 2.6. *Let δ and λ be integers with $\delta \geq 0$, and consider the formal power series*

$$(2.7) \qquad \Phi(z, X) = \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta} \in R[[X]]_\delta.$$

Then the following conditions are equivalent:

(i) *The formal power series $\Phi(z, X)$ is a Jacobi-like form belonging to $\mathcal{J}_\lambda(\Gamma)_\delta$.*

(ii) *The coefficients of $\Phi(z, X)$ satisfy*

$$(\phi_k \mid_{2k+2\delta+\lambda} \gamma)(z) = \sum_{r=0}^{k} \frac{1}{r!} \mathfrak{K}(\gamma, z)^r \phi_{k-r}(z)$$

for all $k \geq 0$ and $\gamma \in \Gamma$.

(iii) *Each coefficient of $\Phi(z, X)$ can be written in the form*

$$(2.8) \qquad \phi_k = \sum_{r=0}^{k} \frac{1}{r!(2k + 2\delta + \lambda - r - 1)!} f_{k+\delta-r}^{(r)}$$

for $k \geq 0$, where f_ℓ is a modular form belonging to $M_{2\ell+\lambda}(\Gamma)$ for each $\ell \geq \delta$.

Proof. This is a slight generalization of a part of Proposition 2 in [2] and can be proved by modifying the proof of that proposition. For example, the proof of similar statements for a more general case of vector-valued Jacobi-like forms can be found in Theorem 2.6 in [5]. $\square$

Lemma 2.7. *The system of relations* (2.8) *between the coefficients of the Jacobi-like form* $\Phi(z, X)$ *and the corresponding modular forms can be written in the form*

$$(2.9) \quad f_k = (2k + \lambda - 1) \sum_{r=0}^{k-\delta} (-1)^r \frac{(2k + \lambda - r - 2)!}{r!} \phi_{k-\delta-r}^{(r)} \in M_{2k+\lambda}(\Gamma)$$

for all $k \geq \delta$.

Proof. This also extends another result contained in Proposition 2 in [2] and can be proved in a manner similar to the proof of that proposition. See, for example, [5, Theorem 2.8] for a detailed proof for the case of vector-valued Jacobi-like forms. $\square$

Corollary 2.8. *There is a canonical isomorphism between the complex vector spaces* $\mathcal{J}_\lambda(\Gamma)_\delta$ *and* $\mathcal{M}_\lambda(\Gamma)_\delta$.

Proof. If $\Phi(z, X)$ is a Jacobi-like form in $\mathcal{J}_\lambda(\Gamma)_\delta$ given by (2.7), we set

$$\Xi_\lambda(\Phi(z, X)) = \sum_{k=\delta}^{\infty} f_k(z) X^k = \sum_{\ell=0}^{\infty} f_{\ell+\delta}(z) X^{\ell+\delta},$$

where f_k is as in (2.9) for all $k \geq \delta$. Then $\Xi_\lambda(\Phi(z, X)) \in \mathcal{M}_\lambda(\Gamma)_\delta$ by Lemma 2.5, and it follows from Proposition 2.6 and Lemma 2.7 that the resulting linear map

$$(2.10) \qquad\qquad \Xi_\lambda : \mathcal{J}_\lambda(\Gamma)_\delta \to \mathcal{M}_\lambda(\Gamma)_\delta$$

is an isomorphism. $\square$

3. Radial heat operators

In this section we introduce radial heat operators on the space $R[[X]]$ of formal power series over R. We show that these operators carry Jacobi-like forms to Jacobi-like forms and determine the image of a Jacobi-like form under the composite of a finite number of such operators. Similar operators were considered for Jacobi forms by Eichler and Zagier in [4].

Given a positive integer m, let $x_1, \ldots, x_m$ be the coordinate functions on the Euclidean space $\mathbb{R}^m$. Then we recall that the Laplace operator Δ on $\mathbb{R}^m$ is given by

$$(3.1) \qquad\qquad \Delta = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}.$$

If an additional coordinate function t is introduced, then the associated heat operator $\mathcal{D}$ can be written as

$$(3.2) \qquad \mathcal{D} = \frac{\partial}{\partial t} - \Delta = \frac{\partial}{\partial t} - \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}.$$

Remark 3.1. If t represents time and $T = T(t, x_1, \ldots, x_m)$ denotes the temperature as a function of time and space for heat propagation in an isotropic and homogeneous medium in the m-dimensional space $\mathbb{R}^m$, the function T satisfies the heat equation

$$\frac{\partial T}{\partial t} - \kappa \sum_{i=1}^{m} \frac{\partial^2 T}{\partial x_i^2} = 0,$$

where the constant κ depends on the thermal conductivity, the density and the heat capacity of the medium. By rescaling t we see that the above heat equation can be written as

$$\mathcal{D}T = 0,$$

where $\mathcal{D}$ is as in (3.2).

Lemma 3.2. *Let $r = (x_1^2 + \cdots + x_m^2)^{1/2}$ be the radial coordinate function on the Euclidean space $\mathbb{R}^m$. Then the radial part $\widetilde{\Delta}$ of the Laplace operator Δ in (3.1) can be written in the form*

$$(3.3) \qquad \widetilde{\Delta} = \frac{m-1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}.$$

Proof. This can be proved easily by a simple change of variables. $\qquad\square$

The operator $\widetilde{\Delta}$ in (3.3) may be called the radial Laplace operator on the m-dimensional space $\mathbb{R}^m$. Similarly, the associated differential operator

$$(3.4) \qquad \widetilde{\mathcal{D}} = \frac{\partial}{\partial t} - \widetilde{\Delta} = \frac{\partial}{\partial t} - \frac{m-1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}$$

is the radial heat operator on $\mathbb{R}^m$.

We are now interested in such operators on $R[[X]]$, where $R[[X]]$ be the complex algebra of formal power series in X with coefficients in the space R of holomorphic functions on $\mathcal{H}$ as in Section 2.

Definition 3.3. Given an integer λ, the associated *radial heat operator* on $R[[X]]$ is the formal differential operator $\mathcal{D}_\lambda^X$ given by

$$(3.5) \qquad \mathcal{D}_\lambda^X = \frac{\partial}{\partial z} - \lambda \frac{\partial}{\partial X} - X \frac{\partial^2}{\partial X^2}.$$

Remark 3.4. (i) Let f be a modular form belonging to $M_{2\delta+\lambda}(\Gamma)$, so that $F_f(z, X) = f(z)X^\delta$ is a modular series belonging to $\mathcal{M}_\lambda(\Gamma)$. If Ξ_λ is the isomorphism in (2.10), it can be shown that

$$\Xi_\lambda^{-1}(F_f(z, X)) = \sum_{k=0}^\infty \frac{f^{(k)}(z)}{k!(k + \lambda + 2\delta - 1)!} X^{k+\delta} \in \mathcal{J}_\lambda(\Gamma)_\delta.$$

It was pointed out by Zagier in [7] that this Jacobi-like form satisfies the differential equation $\mathcal{D}_\lambda^X \Phi = 0$.

(ii) The name, radial heat operator, of $\mathcal{D}_\lambda^X$ in Definition 3.3 can be justified as follows. We consider the variable $w = 2\sqrt{X}$, so that we have

$$\frac{\partial}{\partial w} = \frac{1}{\partial w/\partial X} \frac{\partial}{\partial X} = \sqrt{X} \frac{\partial}{\partial X},$$

$$\frac{\partial^2}{\partial w^2} = \sqrt{X} \frac{\partial}{\partial X} \left(\sqrt{X} \frac{\partial}{\partial X} \right) = \frac{1}{2} \frac{\partial}{\partial X} + X \frac{\partial^2}{\partial X^2}.$$

Hence we obtain

$$\mathcal{D}_\lambda^X = \frac{\partial}{\partial z} - \frac{2\lambda}{w} \frac{\partial}{\partial w} - \left(\frac{\partial^2}{\partial w^2} - \frac{1}{w} \frac{\partial}{\partial w} \right)$$

$$= \frac{\partial}{\partial z} - \frac{2\lambda - 1}{w} \frac{\partial}{\partial w} - \frac{\partial^2}{\partial w^2}.$$

By comparing this with $\widetilde{\mathcal{D}}$ in (3.4) we see that $\mathcal{D}_\lambda^X$ is the usual radial heat operator on the 2λ-dimensional space, assuming that $\lambda > 0$.

Proposition 3.5. *Given an integer λ and a formal power series $\Phi(z, X) \in R[[X]]$, we have*

$$(3.6) \qquad \mathcal{D}_\lambda^X(\Phi \mid_\lambda^J \gamma)(z, X) = (\mathcal{D}_\lambda^X(\Phi) \mid_{\lambda+2}^J \gamma)(z, X)$$

for all $\gamma \in SL(2, \mathbb{R})$ and $z \in \mathcal{H}$, where $\Phi \mid_\lambda^J \gamma$ is as in (2.2). In particular, we have

$$(3.7) \qquad \mathcal{D}_\lambda^X(\mathcal{J}_\lambda(\Gamma)) \subset \mathcal{J}_{\lambda+2}(\Gamma),$$

and therefore $\mathcal{D}_\lambda^X$ induces the complex linear map

$$(3.8) \qquad \mathcal{D}_\lambda^X : \mathcal{J}_\lambda(\Gamma) \to \mathcal{J}_{\lambda+2}(\Gamma)$$

of Jacobi-like forms.

Proof. Let γ be an element of $SL(2, \mathbb{R})$ whose $(2, 1)$-entry is c, so that

$$\frac{\partial}{\partial z} \mathfrak{J}(\gamma, z) = c = \mathfrak{J}(\gamma, z)\mathfrak{K}(\gamma, z),$$

where $\mathfrak{J}(\gamma, z)$ and $\mathfrak{K}(\gamma, z)$ are as in (2.1). Given a formal power series $\Phi(z, X) \in R[[X]]$, using (2.2), we see that

$$
\frac{\partial}{\partial z}(\Phi \mid_\lambda^J \gamma)(z, X)
$$

$$
= -\lambda c\mathfrak{J}(\gamma, z)^{-\lambda-1} e^{-\mathfrak{K}(\gamma, z)X} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X)
$$

$$
+ \mathfrak{J}(\gamma, z)^{-\lambda} \mathfrak{K}(\gamma, z)^2 X e^{-\mathfrak{K}(\gamma, z)X} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X)
$$

$$
+ \mathfrak{J}(\gamma, z)^{-\lambda} e^{-\mathfrak{K}(\gamma, z)X} \mathfrak{J}(\gamma, z)^{-2} \frac{\partial \Phi}{\partial z}(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X)
$$

$$
+ \mathfrak{J}(\gamma, z)^{-\lambda} e^{-\mathfrak{K}(\gamma, z)X}(-2c)\mathfrak{J}(\gamma, z)^{-3} X \frac{\partial \Phi}{\partial z}(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X),
$$

$$
\frac{\partial}{\partial X}(\Phi \mid_\lambda^J \gamma)(z, X)
$$

$$
= \mathfrak{J}(\gamma, z)^{-\lambda-2} e^{-\mathfrak{K}(\gamma, z)X}
$$

$$
\times \left(-c\mathfrak{J}(\gamma, z)\Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X) + \frac{\partial \Phi}{\partial X}(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X) \right),
$$

$$
\frac{\partial^2}{\partial X^2}(\Phi \mid_\lambda^J \gamma)(z, X)
$$

$$
= \mathfrak{J}(\gamma, z)^{-\lambda-2} e^{-\mathfrak{K}(\gamma, z)X} \left(c^2 \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X) \right.
$$

$$
- 2\mathfrak{K}(\gamma, z)\frac{\partial \Phi}{\partial X}(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X)
$$

$$
\left. + \mathfrak{J}(\gamma, z)^{-2} \frac{\partial^2 \Phi}{\partial X^2}(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X) \right).
$$

From these relations and (3.5) we obtain

$$
\mathcal{D}_\lambda^X(\Phi \mid_\lambda^J \gamma)(z, X)
$$

$$
= \frac{\partial}{\partial z}(\Phi \mid_\lambda^J \gamma)(z, X) - \lambda\frac{\partial}{\partial X}(\Phi \mid_\lambda^J \gamma)(z, X) - X\frac{\partial^2}{\partial X^2}(\Phi \mid_\lambda^J \gamma)(z, X)
$$

$$
= \mathfrak{J}(\gamma, z)^{-\lambda-2} e^{-\mathfrak{K}(\gamma, z)X} \left(\frac{\partial \Phi}{\partial z}(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X) - \lambda\frac{\partial \Phi}{\partial X}(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X) \right.
$$

$$
\left. - \mathfrak{J}(\gamma, z)^{-2} X \frac{\partial^2 \Phi}{\partial X^2}(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X) \right).
$$

Thus it follows that

$$
(\mathcal{D}_\lambda^X(\Phi \mid_\lambda^J \gamma))(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda-2} e^{-\mathfrak{K}(\gamma, z)X}(\mathcal{D}_\lambda^X \Phi)(\gamma z, \mathfrak{J}(\gamma, z)^{-2}X)
$$

$$
= (\mathcal{D}_\lambda^X(\Phi) \mid_{\lambda+2}^J \gamma)(z, X),
$$

which verifies (3.6). On the other hand, if $\Phi(z, X) \in R[[X]]$ is a Jacobi-like form belonging to $\mathcal{J}_\lambda(\Gamma)$, then from (3.6) we obtain

$$\mathcal{D}_\lambda^X(\Phi)(z, X) = \mathcal{D}_\lambda^X(\Phi \mid_\lambda^J \gamma)(z, X) = (\mathcal{D}_\lambda^X(\Phi) \mid_{\lambda+2}^J \gamma)(z, X)$$

for all $\gamma \in \Gamma$; hence (3.7) follows.   □

Given a positive integer m, we denote by

$$(3.9) \qquad (\mathcal{D}_\lambda^X)^{[m]} = \mathcal{D}_{\gamma+2m-2}^X \circ \cdots \circ \mathcal{D}_{\lambda+2}^X \circ \mathcal{D}_\lambda^X : \mathcal{J}_\lambda(\Gamma) \to \mathcal{J}_{\lambda+2m}(\Gamma)$$

the composite of m linear maps of the form (3.8). The next theorem determines an explicit formula for this map.

Theorem 3.6. *Let* $\Phi(z, X) = \sum_{k=0}^\infty \phi_k(z) X^{k+\delta}$ *be a Jacobi-like form belonging to* $\mathcal{J}_\lambda(\Gamma)_\delta$ *with* $\delta \geq 0$. *Then its image under the map* $(\mathcal{D}_\lambda^X)^{[m]}$ *in (3.9) can be written in the form*

$$(3.10) \qquad (\mathcal{D}_\lambda^X)^{[m]}(\Phi(z, X))) = \sum_{k=0}^\infty \phi_{m,k}(z) X^{k+\delta-m},$$

where $\phi_{m,k} = 0$ *for* $k < m - \delta$ *and*

$$(3.11) \quad \phi_{m,k} = \sum_{j=0}^m (-1)^j \binom{m}{j} \frac{(k+\delta-m+j)!}{(k+\delta-m)!} \frac{(k+\delta+\lambda+j-2)!}{(k+\delta+\lambda-2)!} \phi_{k-m+j}^{(m-j)}$$

for $k \geq m - \delta$; *here we assume that* $\phi_\ell = 0$ *if* $\ell < 0$.

Proof. We shall verify that the relation (3.10) with $\phi_{m,k}$ as in (3.11) by using induction on m. Let $\Phi(z, X) = \sum_{k=0}^\infty \phi_k(z) X^{k+\delta} \in \mathcal{J}_\lambda(\Gamma)_\delta$ be as given. Then from (3.5) and (3.11) we obtain

$$(\mathcal{D}_\lambda^X \Phi)(z, X) = \sum_{k=0}^\infty \phi_k'(z) X^{k+\delta} - \lambda \sum_{k=0}^\infty (k+\delta) \phi_k(z) X^{k+\delta-1}$$
$$- \sum_{k=0}^\infty (k+\delta)(k+\delta-1) \phi_k(z) X^{k+\delta-1}$$
$$= \sum_{k=0}^\infty \Big(\phi_{k-1}'(z) - (k+\delta)(k+\delta+\lambda-1) \phi_k(z) \Big) X^{k+\delta-1}$$
$$= \sum_{k=0}^\infty \phi_{1,k}(z) X^{k+\delta-1},$$

which proves the case for $m = 1$. We now assume that (3.10) and (3.11) hold for a positive integer m. Then by applying $\mathcal{D}_\lambda^X$ to (3.10) we see that

the coefficient $\phi_{m+1,k}$ of the series

$$(\mathcal{D}_\lambda^X)^{[m+1]}(\Phi(z,X)) = \sum_{k=0}^\infty \phi_{m+1,k}(z) X^{k+\delta-m-1} \in \mathcal{J}_{\lambda+2m+2}(\Gamma)$$

can be written in the form

$$\phi_{m+1,k} = \phi'_{m,k-1} - (k+\delta-m)(k+\delta-m+(\lambda+2m)-1)\phi_{m,k}$$
$$= \phi'_{m,k-1} - (k+\delta-m)(k+\delta+\lambda+m-1)\phi_{m,k}$$

for each $k \geq 0$. From this and (3.11) we obtain

$$\phi_{m+1,k} = \sum_{j=0}^m (-1)^j \binom{m}{j} \frac{(k+\delta-m+j-1)!}{(k+\delta-m-1)!}$$
$$\times \frac{(k+\delta+\lambda+j-3)!}{(k+\delta+\lambda-3)!} \phi_{k-m-1+j}^{(m-j+1)}$$
$$- (k+\delta-m)(k+\delta+\lambda+m-1)$$
$$\times \sum_{j=0}^m (-1)^j \binom{m}{j} \frac{(k+\delta-m+j)!}{(k+\delta-m)!}$$
$$\times \frac{(k+\delta+\lambda+j-2)!}{(k+\delta+\lambda-2)!} \phi_{k-m+j}^{(m-j)}$$

$$= \sum_{j=0}^m (-1)^j \binom{m}{j} \frac{(k+\delta-m+j-1)!}{(k+\delta-m-1)!}$$
$$\times \frac{(k+\delta+\lambda+j-3)!}{(k+\delta+\lambda-3)!} \phi_{k-m-1+j}^{(m-j+1)}$$
$$+ \sum_{j=1}^{m+1} (-1)^j \binom{m}{j-1} \frac{(k+\delta-m+j-1)!}{(k+\delta-m-1)!}$$
$$\times \frac{(k+\delta+\lambda+j-3)!}{(k+\delta+\lambda-3)!} \phi_{k-m-1+j}^{(m-j+1)}$$
$$+ (m+1) \sum_{j=1}^{m+1} (-1)^j \binom{m}{j-1} \frac{(k+\delta-m+j-1)!}{(k+\delta-m-1)!}$$
$$\times \frac{(k+\delta+\lambda+j-3)!}{(k+\delta+\lambda-2)!} \phi_{k-m-1+j}^{(m-j+1)}.$$

Using this and the identities

$$\binom{m}{j} + \binom{m}{j-1} = \binom{m+1}{j}, \quad (m+1)\binom{m}{j-1} = j\binom{m+1}{j},$$

we see that

$$
\begin{aligned}
\phi_{m+1,k} &= \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \frac{(k+\delta-m+j-1)!}{(k+\delta-m-1)!} \big((k+\delta+\lambda-2)+j\big) \\
&\qquad\qquad \times \frac{(k+\delta+\lambda+j-3)!}{(k+\delta+\lambda-2)!} \phi_{k-m-1+j}^{(m-j+1)} \\
&= \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} \frac{(k+\delta-m+j-1)!}{(k+\delta-m-1)!} \\
&\qquad\qquad \times \frac{(k+\delta+\lambda+j-2)!}{(k+\delta+\lambda-2)!} \phi_{k-m-1+j}^{(m-j+1)},
\end{aligned}
$$

which is simply (3.11) with m replaced by $m+1$; hence the theorem follows.
□

Let $\Phi(z,X) = \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta} \in \mathcal{J}_\lambda(\Gamma)_\delta$ and assume that $m \geq \delta$, so that (3.10) can be written in the form

$$(3.12) \qquad (\mathcal{D}_\lambda^X)^{[m]}(\Phi(z,X)) = \sum_{k=m-\delta}^{\infty} \phi_{m,k}(z) X^{k+\delta-m}.$$

Since $(\mathcal{D}_\lambda^X)^{[m]}$ belongs to $\mathcal{J}_{\lambda+2m}(\Gamma)$, we see that its initial coefficient $\phi_{m,m-\delta}$ is a modular form belonging to $M_{\lambda+2m}(\Gamma)$. On the other hand, if $\Xi_\lambda : \mathcal{J}_\lambda(\Gamma)_\delta \to \mathcal{M}_\lambda(\Gamma)_\delta$ is the isomorphism in (2.10) and

$$(3.13) \qquad \Xi_\lambda(\Phi(z,X)) = \sum_{\ell=\delta}^{\infty} f_{\Phi,\ell}(z) X^\ell,$$

then for each $m \geq \delta$ the coefficient $f_{\Phi,m}$ is also a modular form belonging to $M_{2m+\lambda}(\Gamma)$. The next corollary shows that it is a constant multiple of $\phi_{m,m-\delta}$.

Corollary 3.7. *Let $\phi_{m,k}$ and $f_{\Phi,\ell}$ be as in (3.12) and (3.13), respectively. Then we have*

$$\phi_{m,m-\delta} = \frac{(-1)^m m!}{(m+\lambda-2)!(2m+\lambda-1)} f_{\Phi,m}$$

for each positive integer $m \geq \delta$.

Proof. Given $m \geq \delta$, from (2.9) we see that the modular form $f_{\Phi,m} \in M_{2m+\lambda}(\Gamma)$ can be written as

$$(3.14) \qquad f_{\Phi,m} = (2m + \lambda - 1) \sum_{r=0}^{m-\delta} (-1)^r \frac{(2m + \lambda - r - 2)!}{r!} \phi_{m-\delta-r}^{(r)}$$

On the other hand, by (3.11) the initial term $\phi_{m,m-\delta}$ of $(\mathcal{D}_\lambda^X)^{[m]}(\Phi(z, X))$ in (3.12) is given by

$$\phi_{m,m-\delta} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{j!(m + \lambda + j - 2)!}{(m + \lambda - 2)!} \phi_{j-\delta}^{(m-j)}$$

$$= \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} \frac{(m - r)!(2m + \lambda - r - 2)!}{(m + \lambda - 2)!} \phi_{m-\delta-r}^{(r)},$$

where we changed the index from j to $r = m - j$. Using this and the fact that $\phi_{m-\delta-r} = 0$ for $r > m - \delta$, we obtain

$$\phi_{m,m-\delta} = \frac{(-1)^m m!}{(m + \lambda - 2)!} \sum_{r=0}^{m-\delta} (-1)^r \frac{(2m + \lambda - r - 2)!}{r!} \phi_{m-\delta-r}^{(r)}.$$

Hence the corollary follows by comparing this with (3.14). $\qquad\qquad \square$

4. Modular series

Let $\mathcal{M}_\lambda(\Gamma)$ be the space of modular series in Definition 2.3, which is isomorphic to the space $\mathcal{J}_\lambda(\Gamma)$ of Jacobi-like forms. In this section we discuss operators on $\mathcal{M}_\lambda(\Gamma)$ that are compatible with the composite of a finite number of heat operators on $\mathcal{J}_\lambda(\Gamma)$ studied in Section 3.

Let $F(z, X) \in R[[X]]_\delta$ with $\delta \geq 0$ be a formal power series of the form

$$F(z, X) = \sum_{k=\delta}^{\infty} f_k(z) X^k$$

with $f_k \in R$ for each $k \geq \delta$. Given a positive integer m, let

$$(\mathcal{D}_\lambda^X)^{[m]} : \mathcal{J}_\lambda(\Gamma) \to \mathcal{J}_{\lambda+2m}(\Gamma)$$

be as in (3.9), and set

$$(4.1) \qquad (\mathcal{D}_\lambda^M)^{[m]}(F(z, X)) = \sum_{k=\delta-m}^{\infty} \widehat{f_k}(z) X^k$$

for all $z \in \mathcal{H}$, where $\widehat{f_k}$ is an element of R such that

(4.2)

$$\frac{\widehat{f_k}}{(2k + \lambda + 2m - 1)} = \sum_{j=0}^{m} \sum_{u=\delta-m}^{k} \sum_{\ell=0}^{u-\delta+j} (-1)^{j+k-u} \binom{m}{j} \frac{(u+j)!}{\ell! u! (k-u)!}$$

$$\times \frac{(k+u+2m+\lambda-2)!}{(2u+2j+\lambda-\ell-1)!}$$

$$\times \frac{(u+m+\lambda+j-2)!}{(u+m+\lambda-2)!} f_{u+j-\ell}^{(k+m-u-j+\ell)}$$

for each $k \geq \delta - m$. Thus the formula (4.1) determines a linear endomorphism

$$(\mathcal{D}_\lambda^M)^{[m]} : R[[X]]_\delta \to R[[X]]_\delta$$

of the space of formal power series over R.

Theorem 4.1. *If $F(z, X)$ is a modular series belonging to $\mathcal{M}_\lambda(\Gamma)_\delta$, then we have*

$$(\mathcal{D}_\lambda^M)^{[m]}(F(z, X)) \in \mathcal{M}_{2m+\lambda}(\Gamma)_\delta$$

for each $m \geq 1$. Furthermore, if $(\mathcal{D}_\lambda^M)^{[m]} : \mathcal{M}_\lambda(\Gamma)_\delta \to \mathcal{M}_{2m+\lambda}(\Gamma)$ is the induced linear map on modular series, the diagram

(4.3)
$$
\begin{array}{ccc}
\mathcal{J}_\lambda(\Gamma)_\delta & \xrightarrow{(\mathcal{D}_\lambda^X)^{[m]}} & \mathcal{J}_{2m+\lambda}(\Gamma) \\
\Xi_\lambda \downarrow & & \downarrow \Xi_{2m+\lambda} \\
\mathcal{M}_\lambda(\Gamma)_\delta & \xrightarrow{(\mathcal{D}_\lambda^M)^{[m]}} & \mathcal{M}_{2m+\lambda}(\Gamma)
\end{array}
$$

commutes, where Ξ_λ and $\Xi_{2m+\lambda}$ are the isomorphisms in (2.10) and $(\mathcal{D}_\lambda^X)^{[m]}$ is as in (3.9).

Proof. Given a modular series $F(z, X) = \sum_{k=\delta}^{\infty} f_k(z) X^k$ belonging to $\mathcal{M}_\lambda(\Gamma)_\delta$, the corresponding Jacobi-like form in $\mathcal{J}_\lambda(\Gamma)_\delta$ is given by

$$\Xi_\lambda^{-1}(F(z, X)) = \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta},$$

where ϕ_k is as in (2.8) for each $k \geq 0$. Then we have

$$((\mathcal{D}_\lambda^X)^{[m]} \circ \Xi_\lambda^{-1})(F(z, X)) = \sum_{k=0}^{\infty} \phi_{m,k}(z) X^{k+\delta-m},$$

where $\phi_{m,k}$ is given by (3.11) for $k \geq \delta - m$ and $\phi_{m,k} = 0$ for $k < \delta - m$. If we set

$$(\Xi_{2m+\lambda} \circ (\mathcal{D}_\lambda^X)^{[m]} \circ \Xi_\lambda^{-1})(F(z, X)) = \sum_{k=\delta-m}^{\infty} \widetilde{f}_k(z) X^k,$$

then from (2.9) and (3.11), for each $k \geq \delta - m$, we obtain

$$\frac{\widetilde{f}_k}{(2k + 2m + \lambda - 1)}$$

$$= \sum_{r=0}^{k-\delta+m} (-1)^r \frac{(2k + \lambda + 2m - r - 2)!}{r!} \phi_{m,k-\delta+m-r}^{(r)}$$

$$= \sum_{r=0}^{k-\delta+m} \sum_{j=0}^{m} (-1)^{j+r} \frac{(2k + 2m + \lambda - r - 2)!(k - r + j)!}{r!(k - r)!}$$

$$\times \binom{m}{j} \frac{(k + m - r + \lambda + j - 2)!}{(k + m - r + \lambda - 2)!} \phi_{k-\delta-r+j}^{(m+r-j)},$$

which is a modular form belonging to $M_{2k+2m+\lambda}(\Gamma)$. We now set

$$(\mathcal{D}_\lambda^M)^{[m]}(F(z, X)) = (\Xi_{2m+\lambda} \circ (\mathcal{D}_\lambda^X)^{[m]} \circ \Xi_\lambda^{-1})(F(z, X)).$$

Then clearly the formal power series $(\mathcal{D}_\lambda^M)^{[m]}(F(z, X))$ is a modular series belonging to $\mathcal{M}_{2m+\lambda}(\Gamma)$ and the diagram (4.3) commutes. On the other hand, by using (2.8) we have

$$\phi_{k-\delta-r+j}^{(m+r-j)} = \sum_{\ell=0}^{k-\delta-r+j} \frac{1}{\ell!(2k - 2r + 2j + \lambda - \ell - 1)!} f_{k+j-r-\ell}^{(m-j+r+\ell)}.$$

Hence we obtain

$$\frac{\widetilde{f}_k}{(2k + 2m + \lambda - 1)} = \sum_{j=0}^{m} \sum_{r=0}^{k-\delta+m} \sum_{\ell=0}^{k-\delta-r+j} \binom{m}{j}$$

$$\times \frac{(-1)^{j+r}(2k + 2m + \lambda - r - 2)!(k - r + j)!}{\ell!(2k - 2r + 2j + \lambda - \ell - 1)!r!(k - r)!}$$

$$\times \frac{(k + m - r + \lambda + j - 2)!}{(k + m - r + \lambda - 2)!} f_{k+j-r-\ell}^{(m-j+r+\ell)}.$$

Changing the index r to $u = k - r$, we obtain the formula (4.2); hence the proof of the theorem is complete. $\qquad\square$

5. Pseudodifferential operators

As was discussed in the previous sections, the spaces $\mathcal{J}_\lambda(\Gamma)$ and $\mathcal{M}_\lambda(\Gamma)$ of Jacobi-like forms and modular series are isomorphic to each other. In this section we consider another space isomorphic to each of those spaces consisting of some pseudodifferential operators. We determine the operator on this new space corresponding to the linear maps $(\mathcal{D}_\lambda^X)^{[m]}$ and $(\mathcal{D}_\lambda^M)^{[m]}$ in (4.3).

If R is the ring of holomorphic functions on $\mathcal{H}$ as above, a pseudodifferential operator over R is a formal Laurent series in the formal inverse ∂^{-1} of ∂ with coefficients in R of the form $\sum_{k=-\infty}^{u} h_k(z)\partial^k$ with $u \in \mathbb{Z}$ and $h_k \in R$. We denote by $\Psi\mathrm{DO}$ the set of all pseudodifferential operators over R. Then the group $SL(2,\mathbb{R})$ acts on $\Psi\mathrm{DO}$ on the right by

$$\left(\sum_{k=-\infty}^{u} h_k(z)\partial^k \right) \cdot \gamma = \sum_{k=-\infty}^{u} h_k(\gamma z)(\mathfrak{J}(\gamma,z)^2\partial)^k$$

for all $\gamma \in SL(2,\mathbb{R})$, where $\mathfrak{J}(\gamma,z)$ is as in (2.1).

Definition 5.1. Given a discrete subgroup Γ of $SL(2,\mathbb{R})$, an *automorphic pseudodifferential operator for* Γ is a pseudodifferential operator that is invariant under the action of each element of Γ. We denote by $\Psi\mathrm{DO}^\Gamma$ the subspace of $\Psi\mathrm{DO}$ consisting of all automorphic pseudodifferential operators for Γ.

Given an integer α we denote by $\Psi\mathrm{DO}_\alpha$ the subspace of $\Psi\mathrm{DO}$ consisting of the pseudodifferential operators of the form

$$\sum_{k=0}^{\infty} \psi_k(z)\partial^{\alpha-k}$$

with $\psi_k \in R$. We also set

$$\Psi\mathrm{DO}_\alpha^\Gamma = \Psi\mathrm{DO}^\Gamma \cap \Psi\mathrm{DO}_\alpha.$$

Proposition 5.2. *Let* $\Phi(z,X) = \sum_{k=0}^{\infty} \phi_k(z)X^{k+\delta} \in R[[X]]_\delta$ *for some nonnegative integer* δ. *Then, given an integer* η, *the given formal power series* $\Phi(z,X)$ *is a Jacobi-like form belonging to* $\mathcal{J}_{2\eta}(\Gamma)_\delta$ *if and only if the pseudodifferential operator* $\Psi(z,\partial) \in \Psi\mathrm{DO}$ *given by*

$$\Psi(z,\partial) = \sum_{k=0}^{\infty} (-1)^{k+\delta+\eta}(k+\delta+\eta)!(k+\delta+\eta-1)!\phi_k(z)\partial^{-k-\delta-\eta}$$

is an automorphic pseudodifferential operator belonging to $\Psi\mathrm{DO}_{-\delta-\eta}^\Gamma$.

Proof. This can be proved by modifying the proof of one of the results contained in [2, Proposition 2]. □

The above proposition implies the existence of an isomorphism between the space of Jacobi-like forms and that of automorphic pseudodifferential operators as described below. Given a formal power series

$$\Phi(z, X) = \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta} \in R[[X]]_\delta$$

and a pseudodifferential operator

$$\Psi(z, \partial) = \sum_{k=0}^{\infty} \psi_k(z) \partial^{-k-\varepsilon} \in \Psi\mathrm{DO}_{-\varepsilon}$$

with $\delta, \varepsilon \geq 0$, we set

$$(5.1) \qquad \mathcal{L}_\eta^\partial(\Phi(z, X)) = \sum_{k=0}^{\infty} (-1)^{k+\delta+\eta} (k+\delta+\eta)!$$
$$\times (k+\delta+\eta-1)! \phi_k(z) \partial^{-k-\delta-\eta},$$

$$(5.2) \qquad \mathcal{L}_\eta^X(\Psi(z, \partial)) = \sum_{k=0}^{\infty} \frac{(-1)^{k+\varepsilon} \psi_k(z)}{(k+\varepsilon)!(k+\varepsilon-1)!} X^{k+\varepsilon-\eta}.$$

Then it can be easily seen that

$$\mathcal{L}_\eta^X(\mathcal{L}_\eta^\partial(\Phi(z, X))) = \Phi(z, X), \quad \mathcal{L}_\eta^\partial(\mathcal{L}_\eta^X(\Psi(z, \partial))) = \Psi(z, \partial).$$

Thus, using Proposition 5.2, we see that the resulting linear maps

$$\mathcal{L}_\eta^\partial : \mathcal{J}_{2\eta}(\Gamma)_\delta \to \Psi\mathrm{DO}^\Gamma_{-\delta-\eta}, \quad \mathcal{L}_\eta^X : \Psi\mathrm{DO}^\Gamma_{-\varepsilon} \to \mathcal{J}_{2\eta}(\Gamma)_{\varepsilon-\eta}$$

are isomorphisms.

If $\varepsilon, \eta \in \mathbb{Z}$ with $\eta \geq 0$, we define the linear map $\mathcal{D}_{2\eta}^\partial : \Psi\mathrm{DO}_{-\varepsilon} \to \Psi\mathrm{DO}_{-\varepsilon}$ by

$$(5.3) \qquad \mathcal{D}_{2\eta}^\partial(\Psi(z, \partial)) = -\sum_{k=0}^{\infty} \Big((k+\varepsilon)(k+\varepsilon-1)\psi'_{k-1}(z)$$
$$+ (k+\varepsilon-\eta)(k+\varepsilon+\eta-1)\psi_k \Big) \partial^{-k-\varepsilon}$$

for $\Psi(z, \partial) = \sum_{k=0}^{\infty} \psi_k(z) \partial^{-k-\varepsilon} \in \Psi\mathrm{DO}_{-\varepsilon}$. If m is a positive integer, we also denote by

$$(\mathcal{D}_{2\eta}^\partial)^{[m]} = \mathcal{D}_{2\eta}^\partial \circ \cdots \circ \mathcal{D}_{2\eta}^\partial : \Psi\mathrm{DO}_{-\varepsilon} \to \Psi\mathrm{DO}_{-\varepsilon}$$

the composite of m copies of $\mathcal{D}_{2\eta}^\partial$.

Theorem 5.3. *Let* $\mathcal{D}_{2\eta}^X : \mathcal{J}_{2\eta}(\Gamma)_\delta \to \mathcal{J}_{2\eta+2}(\Gamma)_{\delta-1}$ *be the radial heat operator given by* (3.5)*, and let* $\Phi(z, X) \in \mathcal{J}_{2\eta}(\Gamma)$*. Then we have*

$$(5.4) \qquad \mathcal{D}_{2\eta}^\partial(\mathcal{L}_\eta^\partial(\Phi(z, X))) = \mathcal{L}_{\eta+1}^\partial(\mathcal{D}_{2\eta}^X(\Phi(z, X))).$$

More generally, we have

$$(5.5) \qquad (\mathcal{D}_{2\eta}^\partial)^{[m]}(\mathcal{L}_\eta^\partial(\Phi(z, X))) = \mathcal{L}_{\eta+m}^\partial((\mathcal{D}_{2\eta}^X)^{[m]}(\Phi(z, X)))$$

for each positive integer m*, where* $(\mathcal{D}_{2\eta}^X)^{[m]}$ *is as in* (3.9)*.*

Proof. Let $\Phi(z, X) = \sum_{k=0}^\infty \phi_k(z) X^{k+\delta} \in \mathcal{J}_{2\eta}(\Gamma)_\delta$, so that

$$\mathcal{D}_{2\eta}^X(\Phi(z, X)) = \sum_{k=0}^\infty \Big(\phi_{k-1}'(z) - (k+\delta)(k+\delta+2\eta-1)\phi_k(z)\Big) X^{k+\delta-1}$$

with $\phi_{-1} = 0$ is a Jacobi-like form belonging to $\mathcal{J}_{2\eta+2}(\Gamma)$. From this and (5.1) we obtain

$$(\mathcal{L}_{\eta+1}^\partial \circ \mathcal{D}_{2\eta}^X)(\Phi(z, X))$$
$$= \sum_{k=0}^\infty (-1)^{k+\delta+\eta}(k+\delta+\eta)!(k+\delta+\eta-1)!$$
$$\times \Big(\phi_{k-1}'(z) - (k+\delta)(k+\delta+2\eta-1)\phi_k(z)\Big)\partial^{-k-\delta-\eta}.$$

On the other hand, we have

$$\mathcal{L}_\eta^\partial(\Phi(z, X)) = \sum_{k=0}^\infty (-1)^{k+\delta+\eta}(k+\delta+\eta)!(k+\delta+\eta-1)!\phi_k(z)\partial^{-k-\delta-\eta}.$$

Combining this with (5.3), we see that

$$(\mathcal{D}_{2\eta}^\partial \circ \mathcal{L}_\eta^\partial)(\Phi(z, X))$$
$$= -\sum_{k=0}^\infty \Big((-1)^{k+\delta+\eta-1}(k+\delta+\eta)!(k+\delta+\eta-1)!\phi_{k-1}'(z)$$
$$+ (-1)^{k+\delta+\eta}(k+\delta)(k+\delta+2\eta-1)$$
$$\times (k+\delta+\eta)!(k+\delta+\eta-1)!\phi_k(z)\Big)\partial^{-k-\delta-\eta}$$
$$= \sum_{k=0}^\infty (-1)^{k+\delta+\eta}(k+\delta+\eta)!(k+\delta+\eta-1)!$$
$$\times \Big(\phi_{k-1}'(z) - (k+\delta)(k+\delta+2\eta-1)\phi_k(z)\Big)\partial^{-k-\delta-\eta};$$

hence we obtain (5.4). Then (5.5) follows easily from this by induction. $\square$

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Remark 5.4. In [1] the heat operator $\mathcal{D}_\lambda^X$ for $\lambda = 1/2$ was studied. Note that this operator does not send Jacobi-like forms to Jacobi-like forms. In the same paper, an isomorphism $\mathcal{L}_\partial : R[[X]] \to \Psi\mathrm{DO}$ was considered whose formula is the same as the one for $\mathcal{L}_\lambda^\partial$ in (5.1) with $\lambda = 0$, and a linear endomorphism of $\Psi\mathrm{DO}$ compatible with $\mathcal{D}_{1/2}^X$ under this isomorphism was constructed. Such a map can also be obtained for an integer λ by setting

$$\widetilde{\mathcal{D}}_\lambda^\partial = \partial - \lambda\mathcal{I},$$

where $\mathcal{I} : \Psi\mathrm{DO} \to \Psi\mathrm{DO}$ be the formal integration operator with respect to the symbol ∂, that is, an operator given by

$$\mathcal{I}\left(\sum_{k=0}^\infty \phi_k(z)\partial^{-k-\delta}\right) = \sum_{k=0}^\infty \frac{\phi_k(z)}{1-k-\delta}\partial^{-k-\delta+1}.$$

Then it can be shown that

$$\mathcal{L}_\partial \circ \mathcal{D}_\lambda^X = \widetilde{\mathcal{D}}_\lambda^\partial \circ \mathcal{L}_\partial$$

for each nonnegative integer λ. Given a positive integer m and a pseudodifferential operator $\Psi(z,\partial) = \sum_{k=0}^\infty \psi_k(z)\partial^{-k-\delta}$, it can also be shown that

$$(5.6) \qquad (\widetilde{\mathcal{D}}_\lambda^\partial)^{[m]}(\Psi(z,\partial)) = \sum_{k=0}^\infty \psi_{m,k}(z)\partial^{-k-\delta+m},$$

where $(\widetilde{\mathcal{D}}_\lambda^\partial)^{[m]}$ is the m-fold composition of $\widetilde{\mathcal{D}}_\lambda^\partial$ and

$$\psi_{m,k} = \sum_{j=0}^m \binom{m}{j} \frac{(k+\delta+\lambda+j-2)!}{(k+\delta+\lambda-2)!} \frac{(k+\delta-m-1)!}{(k+\delta-m+j-1)!} \psi_{k-m+j}^{(m-j)}$$

for each $k \geq 0$.

Theorem 5.5. *Let $\Psi(z,\partial)$ be an automorphic pseudodifferential operator of the form*

$$\Psi(z,\partial) = \sum_{k=0}^\infty \psi_k(z)\partial^{-k-\varepsilon} \in \Psi\mathrm{DO}_{-\varepsilon}^\Gamma$$

with $\varepsilon \geq 0$. Given integers $m \geq 1$ and $\eta \geq 0$, we have

$$(\mathcal{D}_{2\eta}^\partial)^{[m]}(\Psi(z,\partial)) = \sum_{k=0}^\infty \psi_{m,k}(z)\partial^{k+\varepsilon-\eta-m},$$

where $\psi_{m,k} = 0$ for $k < m - \varepsilon + \eta$ and

$$\psi_{m,k} = \sum_{j=0}^{m} \binom{m}{j} \frac{(k + \varepsilon - \eta - m + j)!}{(k + \varepsilon - \eta - m)!} \frac{(k + \varepsilon + \eta + j - 2)!}{(k + \varepsilon + \eta - 2)!}$$

$$\times \frac{(-1)^m \varepsilon! (\varepsilon - 1)! \psi_{k-m+j}^{(m-j)}}{(k - m + j + \varepsilon)!(k - m + j + \varepsilon - 1)!}$$

for $k \geq m - \delta$ with $\psi_\ell = 0$ for $\ell < 0$.

Proof. By (5.2) the Jacobi-like form $\mathcal{L}_\eta^X(\Psi(z, \partial)) \in \mathcal{J}_{2\eta}(\Gamma)_{\varepsilon - \eta}$ is given by

$$\mathcal{L}_\eta^X(\Psi(z, \partial)) = \sum_{k=0}^{\infty} \frac{(-1)^{k+\varepsilon} \psi_k(z)}{(k + \varepsilon)!(k + \varepsilon - 1)!} X^{k+\varepsilon-\eta}.$$

Using (3.10) and (3.11), we see that

$$(\mathcal{D}_\lambda^X)^{[m]}(\mathcal{L}_\eta^X(\Psi(z, \partial))) = \sum_{k=0}^{\infty} \psi_{m,k}(z) \partial^{-k-\varepsilon},$$

where $\psi_{m,k} = 0$ for $k < m - \varepsilon + \eta$ and

$$\psi_{m,k} = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \frac{(k + \varepsilon - \eta - m + j)!}{(k + \varepsilon - \eta - m)!} \frac{(k + \varepsilon + \eta + j - 2)!}{(k + \varepsilon + \eta - 2)!}$$

$$\times \frac{(-1)^{k-m+j+\varepsilon} \psi_{k-m+j}^{(m-j)}}{(k - m + j + \varepsilon)!(k - m + j + \varepsilon - 1)!}$$

for $k \geq m - \delta$, assuming that $\psi_\ell = 0$ for $\ell < 0$. From this and (5.1) we obtain

$$(\mathcal{D}_{2\eta}^\partial)^{[m]}(\Psi(z, \partial))) = (\mathcal{L}_{\eta+m}^\partial \circ (\mathcal{D}_\lambda^X)^{[m]} \circ \mathcal{L}_\eta^X)(\Psi(z, \partial))$$

$$= \sum_{k=0}^{\infty} (-1)^{k+\varepsilon} \varepsilon! (\varepsilon - 1)! \psi_{m,k}(z) \partial^{-k-\varepsilon};$$

hence the theorem follows. $\square$

References

[1] Y. Choie and M. H. Lee, *Rankin-Cohen brackets on pseudodifferential operators*, J. Math. Anal. Appl. **326** (2007), 882–895.

[2] P. B. Cohen, Y. Manin, and D. Zagier, *Automorphic pseudodifferential operators*, Algebraic Aspects of Nonlinear Systems, Birkhäuser, Boston, 1997, pp. 17–47.

[3] L. Dickey, *Soliton equations and Hamiltonian systems*, World Scientific, Singapore, 1991.

[4] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Math., vol. 55, Birkhäuser, Boston, 1985.

[5] M. H. Lee, *Vector-valued Jacobi-like forms*, Monatsh. Math. **152** (2007), 321–336.

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[6] F. Martin and E. Royer, *Formes modulaires et transcendance*, Formes modulaires et périodes (S. Fischler, E. Gaudron, and S. Khémira, eds.), Soc. Math. de France, 2005, pp. 1–117.

[7] D. Zagier, *Modular forms and differential operators*, Proc. Indian Acad. Sci. Math. Sci. **104** (1994), 57–75.

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