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# On coprimary decomposition theory for modules

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# ON COPRIMARY DECOMPOSITION THEORY FOR MODULES

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Recently, in his paper [2], D. Kirby introduced the notion of coprimary modules over a commutative ring, and obtained several results on coprimary decompositions for Artinian modules. In this note, by making use of the technique employed in [1] and [3], we shall investigate the $s$-coprimary decomposition theory for modules over non-commutative rings.

**1. Preliminaries.** Throughout, $R$ will represent a ring, and $M$ a non-zero left $R$-module. Given an ideal $\mathfrak{a}$ of $R$, $M^{\mathfrak{a}}$ is defined to be the intersection $\cap\ \mathfrak{b}M$, where $\mathfrak{b}$ runs over all the finite products of ideals of $R$ not contained in $\mathfrak{a}$. ($M^R = M$ by definition.) As in [3], $p(M)$ will denote the prime radical of $l(M) = \{x \in R \mid xM = 0\}$. If $l(M') \subseteq p(M)$, or equivalently $p(M') = p(M)$, for every non-zero submodule $M'$ of $M$, 0 is defined to be a primary submodule of $M$ (cf. [1]). Now, dualizing the notion, $M$ is defined to be *coprimary* if $l(M/M') \subseteq p(M)$, or equivalently $p(M/M') = p(M)$, for every proper submodule $M'$ of $M$. In case $M$ is coprimary and $\mathfrak{p} = p(M)$, $M$ will be called a $\mathfrak{p}$-*coprimary* module. If $M$ is coprimary and $p(M)$ is nilpotent modulo $l(M)$, $M$ is defined to be $s$-*coprimary.*

The next is easy, and will be freely used without mention.

**Proposition 1.** *The following conditions are equivalent:*

(1) *$M$ is coprimary.*

(2) *$\mathfrak{a}M = M$ for every ideal $\mathfrak{a}$ of $R$ not contained in $p(M)$.*

(3) *$M^{p(M)} = M$.*

An idea $\mathfrak{p}$ of $R$ is called a *coassociated ideal* of $M$ if there exists a proper submodule $M'$ such that $M/M'$ is $\mathfrak{p}$-coprimary. The set of all coassociated ideals of $M$ will be denoted by $P^*(M)$. ($P^*(0) = \emptyset$ by definition.) If there exists an ideal $\mathfrak{p}$ in $R$ such that $P^*(M/M') = \{\mathfrak{p}\}$ for every proper submodule $M'$ of $M$ then $M$ is called a $P^*$-*module.*

**Proposition 2.** (1) *If $M$ is coprimary, and $M'$ a proper submodule of $M$, then $l(M/M')$ is a right-primary ideal.*

(2) *Let $N$ and $M'$ be submodules of $M$. If $N$ is $\mathfrak{p}$-coprimary and not contained in $M'$ then $N+M'/M'$ is $\mathfrak{p}$-coprimary.*



 

(3) *If $N$ and $N'$ are $\mathfrak{p}$-coprimary submodules of $M$, then so is $N+N'$.*

*Proof.* (1) Assume that there exist ideals $\mathfrak{a}$, $\mathfrak{b}$ of $R$ such that $\mathfrak{ab}\subseteq l(M/M')$ and $\mathfrak{b}\not\subseteq p(M/M')$. Then, $M'\supseteq \mathfrak{ab}M=\mathfrak{a}M$, namely, $\mathfrak{a}\subseteq l(M/M')$.

(2) This is obvious by $N+M'/M'\cong N/N\cap M'$.

(3) Since $l(N+N')=l(N)\cap l(N')$, we have $p(N+N')=p(N)\cap p(N')=\mathfrak{p}$. If $\mathfrak{a}$ is an ideal of $R$ not contained in $\mathfrak{p}$, then $\mathfrak{a}N=N$ and $\mathfrak{a}N'=N'$, and hence $\mathfrak{a}(N+N')=N+N'$.

**Proposition 3.** (1) *If $M$ is $\mathfrak{p}$-s-coprimary then $\mathfrak{p}$ is prime and $\mathfrak{a}M\neq M$ for every ideal $\mathfrak{a}$ of $R$ contained in $\mathfrak{p}$.*

(2) *Let $N$ and $M'$ be submodules of $M$. If $N$ is $\mathfrak{p}$-s-coprimary and is not contained in $M'$ then $M+M'/M'$ is $\mathfrak{p}$-s-coprimary.*

(3) *If $N$ and $N'$ are $\mathfrak{p}$-s-coprimary, then so is $N+N'$.*

*Proof.* (2) and (3) are easy by Prop. 2 (2) and (3).

(1) If $\mathfrak{a}$ is an ideal of $R$ contained in $\mathfrak{p}$ then there exists a positive integer $h$ such that $\mathfrak{a}^hM=0$, which means $\mathfrak{a}M\neq M$. Next, we shall prove that $\mathfrak{p}$ is prime. Let $\mathfrak{b}$, $\mathfrak{c}$ be ideals of $R$ such that $\mathfrak{bc}\subseteq\mathfrak{p}$. As was shown just above, there holds $\mathfrak{bc}M\neq M$. If $\mathfrak{c}\not\subseteq\mathfrak{p}$, then $M\supset\mathfrak{bc}M=\mathfrak{b}M$, and hence $\mathfrak{b}\subseteq\mathfrak{p}$.

**Proposition 4.** *If $N$ is a submodule of $M$ then $P^*(M/N)\subseteq P^*(M)\subseteq P^*(N)\cup P^*(M/N)$.*

*Proof. Let* $S$ be a proper submodule of $M$ such that $M/S$ is $\mathfrak{p}$-coprimary. If $S+N\neq M$ then $M/S+N$ *is* $\mathfrak{p}$-coprimary and $\mathfrak{p}\in P^*(M/N)$. On the other hand, if $S+N=M$ then $N/N\cap S\cong M/S$ is $\mathfrak{p}$-coprimary and $\mathfrak{p}\in P^*(N)$. The inclusion $P^*(M/N)\subseteq P^*(M)$ is almost evident.

**2. Coprimary decompositions.** A finite set $\{M_i\,|\,i\in I\}$ of coprimary (resp. $s$-coprimary) submodules of $M$ is called a *coprimary* (resp. *s-coprimary*) *decomposition* of $M$ if $M=\sum_{i\in I}M_i$, $M\neq\sum_{i\in I'}M_i$ for every proper subset $I'$ of $I$, and $p(M_i)\neq p(M_j)$ for every $i\neq j$. If $\{N_j\,|\,j\in J\}$ is a finite set of coprimary (resp. $s$-coprimary) submodules of $M$ with $M=\sum_{j\in J}N_j$, then Prop. 2 (3) (resp. Prop. 3 (3)) secures the existence of a coprimary (resp. $s$-coprimary) decomposition of $M$.

**Proposition 5.** *Let $\{M_i\,|\,i=1,\cdots,k\}$ be an s-coprimary decomposition of $M$, and $\mathfrak{p}_i=p(M_i)$ $(i=1,\cdots,k)$. Let $\mathfrak{a}$ be an ideal of $R$.*

(1) *If $\mathfrak{a}M=M$ then $\mathfrak{a}\not\subseteq\mathfrak{p}_i$ $(i=1,\cdots,k)$, and conversely.*

(2) $M^{\mathfrak{a}}=\sum_{\mathfrak{p}_i\subseteq\mathfrak{a}}M_i$. *If $\mathfrak{a}$ does not contain all $\mathfrak{p}_i$'s then $M^{\mathfrak{a}}=\mathfrak{b}M$ with a finite product $\mathfrak{b}$ of ideals of $R$ not contained in $\mathfrak{a}$.*

*Proof.* (1) If $\mathfrak{a}$ is contained in some $\mathfrak{p}_i$ then $\mathfrak{a}^h M_i = 0$ for some positive integer $h$. Accordingly, we have $\mathfrak{a}^h M \neq M$, whence it follows $\mathfrak{a}M \neq M$. The converse is obvious.

(2) Without loss of generality, we may assume that $\mathfrak{p}_1, \cdots, \mathfrak{p}_l \subseteq \mathfrak{a}$ and $\mathfrak{p}_{l+1}, \cdots, \mathfrak{p}_k \not\subseteq \mathfrak{a}$. In case $l=k$, our assertion is evident by (1). Henceforth, we assume that $(0 \leqslant)\ l < k$. There exists a positive integer $h$ such that $\mathfrak{p}_j^h M_j = 0$ $(j = l+1, \cdots, k)$. Since every $\mathfrak{p}_i$ is prime by Prop. 3 (1), $\mathfrak{b} = (\mathfrak{p}_{l+1} \cdots \mathfrak{p}_k)^h \not\subseteq \mathfrak{p}_i$ $(i=1, \cdots, l)$. There holds then $M^{\mathfrak{a}} \supseteq \sum_{i=1}^l M_i^{\mathfrak{a}} \supseteq \sum_{i=1}^l M_i^{\mathfrak{p}_i} = \sum_{i=1}^l M_i = \mathfrak{b}M \supseteq M^{\mathfrak{a}}$, namely, $M^{\mathfrak{a}} = \sum_{i=1}^l M_i = \mathfrak{b}M$.

**Theorem 1.** *Let $\{M_i \mid i=1, \cdots, k\}$ be an s-coprimary decomposition of $M$, and $\mathfrak{p}_i = p(M_i)$ $(i=1, \cdots, k)$. Then there holds the following*:

(1) $P^*(M) = \{\mathfrak{p}, \cdots, \mathfrak{p}_k\}$.

(2) *A prime divisor $\mathfrak{p}$ of $l(M)$ is contained in $P^*(M)$ if and only if $\mathfrak{p}M^{\mathfrak{p}} \neq M^{\mathfrak{p}}$. Every minimal prime divisor of $l(M)$ is contained in $P^*(M)$, and if $\mathfrak{p}_i$ is minimal in $P^*(M)$ then $M^{\mathfrak{p}_i} = M_i$.*

*Proof.* (1) Evidently, $p(M)$ is nilpotent modulo $l(M)$. Next, we claim that if $M$ is $s$-coprimary then $k=1$. Since $\mathfrak{p} = p(M) = \bigcap_{i=1}^k \mathfrak{p}_i$ is prime by Prop. 3 (1), without loss of generality, we may assume that $\mathfrak{p}_1, \cdots, \mathfrak{p}_m \subseteq \mathfrak{p}$ $(m \geqslant 1)$ and $\mathfrak{p}_{m+1}, \cdots, \mathfrak{p}_k \not\subseteq \mathfrak{p}$. Then, by Prop. 5, $M = M^{\mathfrak{p}} = \sum_{i=1}^m M_i$, whence it follows $m=k$. Combining this with $\mathfrak{p} = \bigcap_{i=1}^k \mathfrak{p}_i$, we obtain $k=1$. Now, we shall proceed into the proof of (1). Obviously, $M/\sum_{j \neq i} M_j$ is $\mathfrak{p}_i$-coprimary as a non-zero homomorphic image of $M_i$, and so $P^*(M) \supseteq \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$. Conversely, assume that $M/N$ is $\mathfrak{p}$-coprimary. Then, $M/N = \sum_{i=1}^k (M_i + N)/N$, where $(M_i+N)/N$ is either 0 or $\mathfrak{p}_i$-$s$-coprimary by Prop. 3 (2). Accordingly, by Prop. 3 (3), $M/N$ has an $s$-coprimary decomposition $\{M_j'/N \mid j=1. \cdots, l\}$ such that $\{\mathfrak{p}(M_j'/N) \mid j=1, \cdots, l\} \subseteq \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$. Then, as was mentioned above, we obtain $l=1$ and $\mathfrak{p} \in \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$.

(2) If $\mathfrak{p}$ is contained in $P^*(M) = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$, then we may assume that $\mathfrak{p}_1, \cdots, \mathfrak{p}_{m-1} \subseteq \mathfrak{p} = \mathfrak{p}_m$ and $\mathfrak{p}_{m+1}, \cdots, \mathfrak{p}_k \not\subseteq \mathfrak{p}$. Then, $M^{\mathfrak{p}} = M_1 + \cdots + M_m \neq \mathfrak{p}M^{\mathfrak{p}}$ by Prop. 5. Next, we shall prove the converse. Since $\mathfrak{p} \supseteq \bigcap_{i=1}^k \mathfrak{p}_i$, we may assume that $\mathfrak{p}_1, \cdots, \mathfrak{p}_m \subseteq \mathfrak{p}$ $(m \geqslant 1)$ and $\mathfrak{p}_{m+1}, \cdots, \mathfrak{p}_k \not\subseteq \mathfrak{p}$. (If $\mathfrak{p}$ is a minimal prime divisor of $l(M)$ then it is obviously in $P^*(M)$.) Since $M^{\mathfrak{p}} \neq \mathfrak{p}M^{\mathfrak{p}}$, we obtain $\sum_{i=1}^m M_i \neq \mathfrak{p}(\sum_{i=1}^m M_i)$ by Prop. 5 (2), and hence $\mathfrak{p} \subseteq \mathfrak{p}_i$, namely, $\mathfrak{p} = \mathfrak{p}_i$, for some $i \leqslant m$ (Prop. 5 (1)). The final assertion is evident by Prop. 5 (2).

Now, let $\{M_i \mid i=1, \cdots, k\}$ be an $s$-coprimary decomposition of $M$. A subset $P^*$ of $\{\mathfrak{p}_i = p(M_i) \mid i=1, \cdots, k\}$ is called an *isolated subset* of $\{\mathfrak{p}_i \mid i=1, \cdots, k\}$ if every $\mathfrak{p}_i$ contained in one of the members of $P^*$ belongs to $P^*$. For an isolated subset $P^*$ of $\{\mathfrak{p}_i \mid i=1, \cdots, k\}$ we set $M^{P^*} = \sum_{\mathfrak{p}_i \in P^*} M_i$.

which coincides with $\sum_{\mathfrak{p}\in P^*} M^{\mathfrak{p}}$ by Prop. 5 (2) and is called a *coisolated component* of $M$. By Th. 1, we readily obtain the following:

**Theorem 2.** *Suppose that $M$ has an $s$-coprimary decomposition. Then, the set of coisolated components of $M$ does not depend on the choice of $s$-coprimary decompositions of $M$.*

Finally, we shall examine cases in which every $s$-coprimary decomposition is direct.

**Theorem 3.** *Suppose $R$ contains 1 and $M$ is unital. Let $\{M_i \mid i=1, \cdots, k\}$ be a finite set of $s$-coprimary submodules of $M$ such that $M=\sum_{i=1}^k M_i$ and $(R\neq)\ \mathfrak{p}_i = p(M_i)$ $(i=1, \cdots, k)$. If $\mathfrak{p}_i$'s are pairwise comaximal, then $M=\bigoplus_{i=1}^k M_i$ and this is the unique $s$-coprimary decompsition of $M$.*

*Proof.* Since $\mathfrak{p}_i$'s are comaximal, so are $l(M_i)$'s, and so $l(M_i)+l(\sum_{j\neq i} M_j)=R$. Hence, $M_i \cap \sum_{j\neq i} M_j = (l(M_j)+l(\sum_{j\neq i} M_j))(M_i \cap \sum_{j\neq i} M_j) =0$, which means $M=\bigoplus_{i=1}^k M_i$. Obviously, the last is an $s$-coprimary decomposition of $M$ and $P^*(M)=\{\mathfrak{p}_1, \cdots, \mathfrak{p}_k\}$ by Th. 1. Further, every $\mathfrak{p}_i$ is minimal in $P^*(M)$ and $M_i = M^{\mathfrak{p}_i}$ by Th. 1 (2), which means the uniqueness of the $s$-coprimary decompositions.

**Corollary.** *Let $R$ be a left Artinian ring with 1. If $M$ is a completely reducible module with a finite number of homogeneous components, then the idealistic decomposition of $M$ is the unique $s$-coprimary decomposition of $M$.*

*Proof.* If $N$ is an arbitrary irreducible submodule of $M$ then $l(N)=p(N)$ is a maximal ideal of $R$ and $N$ is isomorphic to a minimal left ideal of $R/l(N)$. We have seen therefore that if $N'$ is another irreducible submodule of $M$ non-isomorphic to $N$ then $p(N')\neq p(N)$. Further, to be easily seen, the homogeneous component of $M$ containing $N$ is $p(N)$-$s$-coprimary. Now, our assertion is a consequence of Th. 3.

**3. Coprimary decomposition theory and $AR^*$-modules.** When every non-zero submodule of $M$ has a coprimary (resp. $s$-coprimary) decomposition, $M$ is said to have the *coprimary* (resp. *$s$-coprimary*) *decomposition theory*. In case $M$ has the coprimary (resp. $s$-coprimary) decomposition theory, every non-zero factor submodule of $M$ has a coprimary (resp. $s$-coprimary) decomposition by Prop. 2 (resp. Prop. 3), and if $N$ is a primary submodule of $M$ then $M/N$ is coprimary. Conversely, in case $M$ has the primary decomposition theory, if $M/N$ is coprimary then $N$ is primary. (Cf. [3].)

If $M$ satisfies one of the following equivalent conditions (I) and (II), it is called an *$AR^*$-module*:

(I) For each submodule $N$ of $M$ and each ideal $\mathfrak{a}$ of $R$, there exists a positive integer $h$ such that $N+\mathfrak{a}^{-h}0\supseteq\mathfrak{a}^{-1}N\ (=\{x\in M\,|\,\mathfrak{a}x\subseteq N\})$.

(II) For each submodule $N$ of $M$ and each ideal $\mathfrak{a}$ of $R$, there exists a positive integer $h$ such that $\mathfrak{a}N+(\mathfrak{a}^{-h}0\cap N)=N$.

One may remark here that if $M$ is an $AR^*$-module, then so is every non-zero factor submodule of $M$. Finally, $M$ is said to be *$p^*$-worthy* if $P^*(M^*)$ is finite and non-empty for every non-zero factor submodule $M^*$ of $M$.

**Proposition 6.** *If $M$ has the s-coprimary decomposition theory, then there holds the following*:

(1) *$M$ is an s-module, that is, $p(M^*)$ is nilpotent modulo $l(M^*)$ for every non-zero factor submodule $M^*$ of $M$.*

(2) *For every submodule $N$ of $M$, if $N^{\mathfrak{a}_1}\supset(N^{\mathfrak{a}_1})^{\mathfrak{a}_2}\supset\cdots\supset(\cdots(N^{\mathfrak{a}_1})\cdots)^{\mathfrak{a}_n}$* then $n< s(N)$ with a positive integer $s(N)$ depending solely on $N$.

(3) *$M$ is $p^*$-worthy.*

(4) *$M$ is an $AR^*$-module.*

Proof. (1)–(3) are easy by Props. 3 and 5 and Th. 1.

(4) It suffices to consider non-zero $N$. Let $\{N_i \mid i=1,\cdots,k\}$ be an $s$-coprimary decomposition of $N$. We may assume then $\mathfrak{a}\subseteq p(N_1),\cdots,p(N_l)$ and $\mathfrak{a}\not\subseteq p(N_{l+1}),\cdots,p(N_k)$. There exists a positive integer $h$ such that $\mathfrak{a}^hN_i=0$ $(i=1,\cdots,l)$. Since $N_1+\cdots+N_l\subseteq\mathfrak{a}^{-h}0\cap N$ and $N_{l+1}+\cdots+N_k\subseteq\mathfrak{a}N$, it follows $\mathfrak{a}N+(\mathfrak{a}^{-h}0\cap N)=N$.

**Proposition 7.** *Let $M$ be an $AR^*$-module and an s-module.*

(1) *If $N$ is a $P^*$-submodule of $M$ then $N$ is s-coprimary.*

(2) *If $M$ is Artinian, then $M$ has the s-coprimary decomposition theory.*

*Proof.* (1) Let $N'$ be an arbitrary proper submodule of $N$. Since $P^*(N/N')=P^*(N)=\{\mathfrak{p}\}$, there exists a proper submodule $N''$ of $N$ containing $N'$ such that $N/N''$ is $\mathfrak{p}$-$s$-coprimary. Now, let $W$ be an arbitrary proper submodule of $N$, and choose a proper submodule $W'$ of $N$ containing $W$ such that $N/W'$ is $\mathfrak{p}$-$s$-coprimary. Since $l(N/N')\subseteq l(N/N'')\subseteq\mathfrak{p}$, Prop. 3 (1) yields $l(N/N')N+W'\subset N$, which means that $l(N/N')N$ is small in $N$. By the condition (II), there exists a positive integer $h$ such that $l(N/N')N+((l(N/N'))^{-h}0\cap N)=N$. It follows then $(l(N/N'))^{-h}0\cap N=N$, namely, $l(N/N')^hN=0$. This means that $l(N/N')\subseteq p(N)$, that is, $N$ is $s$-coprimary.

(2) Since every non-zero submodule of $M$ is a finite sum of sum-

irreducible submodules, it remains only to show that if a non-zero submodule $N$ of $M$ is not $s$-coprimary then $N$ is not sum-irreducible. There exists a proper submodule $N'$ of $N$ such that $\mathfrak{a}=l(N/N')\not\subseteq p(N)$, or $\mathfrak{a}^n N \neq 0$ for every positive integer $n$. By the condition (II), there exists a positive integer $h$ such that $\mathfrak{a}N+(\mathfrak{a}^{-h}0\cap N)=N$. It is obvious that $\mathfrak{a}N\subseteq N'\subset N$ and $\mathfrak{a}^{-h}0\cap N\subset N$.

Combining Prop. 6 with Prop. 7, we obtain at once

**Theorem 4.** *Let M be an Artinian module. In order that M have the s-coprimary decomposition theory, it is necessary and sufficient that M be an $AR^*$-module and an s-module.*

In [2], D. Kirby has proved that every unital Artinian $s$-module over a commutative ring with 1 has the $s$-coprimary decomposition theory. However, the following example will show that it is not the case for non-commutative rings.

**Example.** Let $R=\begin{pmatrix} F & 0 & 0 \\ F & F & 0 \\ F & F & F \end{pmatrix}$, where $F$ is a field, and $M$ the left $R$-module $R$. To be easily seen, $\mathfrak{a}=\begin{pmatrix} 0 & 0 & 0 \\ F & 0 & 0 \\ F & F & F \end{pmatrix}$ is an ideal of $R$ and $\mathfrak{a}\supset\mathfrak{a}^2=\mathfrak{a}^3=\cdots=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ F & F & F \end{pmatrix}$. Moreover, $\mathfrak{a}^{-2}0\cap\mathfrak{a}=0$, and we have $\mathfrak{a}\cdot\mathfrak{a}+(\mathfrak{a}^{-2}0\cap\mathfrak{a})\neq\mathfrak{a}$, which means that $M$ is not an $AR^*$-module.

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