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THE CONVERSE OF BADRIKIAN'S THEOREM

YOSHIAKI OKAZAKI and YASUJI TAKAHASHI

1. Introduction. Let E be a locally convex Hausdorff space and E' be the topological dual of E. Denote by E'_S the dual with the weak $*$ topology $\sigma(E', E)$. Let μ be a cylinder set measure on E'. Then μ is called a continuous cylinder set measure if the characteristic functional

$$\hat{\mu}(x) = \int_{E'} e^{i\langle x, a\rangle} d\mu(a), \ x \in E$$

is continuous on E.

A family $\{\mu_\alpha\}$ of cylinder set measures on E' is called equicontinuous if the characteristic functionals $\{\hat{\mu}_\alpha(x)\}$ are equicontinous on E. It is well-known that if E is a nuclear space, then each equicontinuous family $\{\mu_\alpha\}$ is tight on E', that is, for every $\varepsilon > 0$ there exists $\sigma(E', E)$-compact set K such that $\mu_\alpha(K) > 1 - \varepsilon$ for every α (Badrikian's theorem, see Badrikian [2], p. 177). We consider the converse problem.

Problem. Let E be a locally convex Hausdorff space. Suppose that each equicontinuous family $\{\mu_\alpha\}$ of cylinder set measures on E' is tight on E'_S. Then is E nuclear? We shall prove the following affirmative answer.

Theorem. *Let p be $0 < p < 2$ and E be a barrelled space. If each equicontinuous family of discrete p-stable cylinder set measures on E' is tight on E'_S, then E is nuclear.*

2. Preliminaries. The cylinder set measure μ on E' is called a continuous discrete p-stable cylinder set measure on E' if the characterictic functional $\hat{\mu}(x)$ is given by $\hat{\mu}(x) = \exp(-||T(x)||^p_{\ell_p})$, $x \in E$, where $T: E \to \ell_p$ is a continuous linear operator, $\ell_p = \{(t_n) : ||(t_n)||_{\ell_p} = (\Sigma_{n=1}^\infty |t_n|^p)^{1/p} < \infty\}$ and $0 < p \leqq 2$. See Linde [6].

Let F, G be normed spaces and $0 < p < \infty$. A linear operator $S : F' \to G'$ is called p-summing if for every $\{a_n\} \subset F'$ with $\Sigma_{n=1}^\infty |\langle x, a_n\rangle|^p < \infty$ for every $x \in F$, it holds that $\Sigma_{n=1}^\infty ||S(a_n)||^p_{G'} < \infty$. Each p-summing operator is q-summing for $p < q$. See Schwartz [11].

Let E be a locally convex Hausdorff space. For a closed absolutely convex neighborhood U of 0, we set $N(U) = \{x \in E : p_U(x) = 0\}$ where

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$p_U(x) = \inf\{t > 0 : x \in tU\}$. Denote by $x(U)$ the equivalence class corresponding to $x \in E$ in the quotient space $E(U) = E/N(U)$. $E(U)$ is a normed space with norm $p[x(U)] = p_U(x)$ for $x \in E$.

For a closed absolutely convex bounded subset A of E, we set $E(A) = \{x \in E : x \in tA$ for some $t > 0\}$. $E(A)$ is a linear subspace of E. We put the norm on $E(A)$ by $p_A(x) = \inf\{t > 0 : x \in tA\}$ for $x \in E(A)$.

For a neighborhoods U of 0 in E, the polar $U^\circ = \{a \in E' : |\langle x, a\rangle| \leqq 1$ for every $x \in U\}$ is weakly compact absolutely convex subset of E'_S. The normed space $E'(U^\circ)$ is a Banach space and $E(U)' = E'(U^\circ)$ by the duality $\langle x(U), a\rangle = \langle x, a\rangle$.

For zero neighborhoods U, V with $V \subset U$ we define a canonical mapping $E(V, U) : E(V) \to E(U)$ by associating $x(U)$ with $x(V)$. For two closed absolutely convex bounded subsets A and B with $A \subset B$, it holds that $E(A) \subset E(B)$ and the canonical mapping $E(A, B) : E(A) \to E(B)$ is defined by $E(A, B)(x) = x$ for $x \in E(A)$.

A locally convex Hausdorff space E is called nuclear if it has a fundamental system $U_F(E)$ of zero neighborhoods which has the following property (see Pietsch [8], 4.1.2):

> For each $U \in U_F(E)$ there exists $V \in U_F(E)$ with $V \subset U$ such that the canonical mapping $E'(U^\circ, V^\circ) : E'(U^\circ) \to E'(V^\circ)$ is 2-summing.

For the theory of locally convex space, we refer to Schaefer [10].

3. Converse of Badrikian's theorem. We consider the converse of Badrikian's theorem as formulated in Section 1.

Lemma 1. *Let E be a barrelled space, F be a Banach space and $0 < p \leqq 2$. Let $\pi : E \to F$ be a continuous linear mapping and $\pi' : F' \to E'$ be the adjoint of π. Suppose that for each equicontinous family of discrete p-stable cylinder set measures $\{\mu_\alpha\}$ on F', the image $\{\pi'(\mu_\alpha)\}$ is $\sigma(E', E)$-tight on E'_S. Then there is a zero neighborhood U of E such that for each continous discrete p-stable cylinder set measure μ on F', $\pi'(\mu)$ is $\sigma(E'(U^\circ), E(U))$-Radon on $E'(U^\circ)$.*

Remark 1. If the assumption of Lemma 1 is satisfied, then for every continous p-stable cylinder set measure on F', by the 0–1 law of the p-stable measure (Dudley and Kanter [3]), there exists a zero neighborhood

$V = V(\mu)$ which depends on each μ such that $\pi'(\mu)(E'(V^\circ)) = 1$. This lemma asserts that V is independent of each continous p-stable cylinder set measure μ.

Proof. Let $B = \{T : T$ is a continous linear operator from F to ℓ_p with norm $||T|| \leqq 1\}$. Consider the equicontinous family $\{\mu_T : T \in B\}$, where $\hat{\mu}_T(y) = \exp(-||T(y)||^p_{\ell_p})$, $y \in F$. Then the family $\{\pi'(\mu_T) : T \in B\}$ is $\sigma(E', E)$-tight. There exists a $\sigma(E', E)$-compact set K such that $\inf\{\pi'(\mu_T)(K) : T \in B\} > 0$. Since E is barrelled, there exists a zero neighborhood U such that $K \subset U^\circ$. By the 0–1 law (Dudley and Kanter [3]), it follows that $\pi'(\mu_T)(E'(U^\circ)) = 1$ for every $T \in B$. For every continous p-stable cylinder set measure μ on F' with $\hat{\mu}(y) = \exp(-||S(y)||^p_{\ell_p})$, where $S : F \to \ell_p$ is a continous linear operator, we have $\pi'(\mu)(E'(U^\circ)) = 1$. In fact, for μ_1 with $\hat{\mu}_1(y) = \exp(-||S_1(y)||^p_{\ell_p})$ where $S_1(y) = S(y)/||S||$, $||S_1|| = 1$, it holds that $\pi'(\mu_1)(E'(U^\circ)) = 1$. Since $\pi'(\mu))$ is the image of $\pi'(\mu_1)$ by the constant multiple: $a \to ||S||a$, we have $\pi'(\mu)(E'(U^\circ)) = 1$. This proves the lemma.

Lemma 2. *Let F, G be Banach spaces, $\psi : G \to F$ be a continous linear mapping and $\psi' : F' \to G'$ be the adjoint of ψ. Let $0 < p < 2$ and $(a_i) \subset F'$ be $\Sigma_{i=1}^\infty |\langle x, a_i\rangle|^p < \infty$ for every $x \in F$. Let μ be a continous discrete p-stable cylinder set measure on F' with $\hat{\mu}(x) = \exp(-\Sigma_{i=1}^\infty |\langle x, a_i\rangle|^p)$. Suppose that the image $\psi'(\mu)$ is $\sigma(G', G)$-Radon on G'_S. Then it holds that $\Sigma_{i=1}^\infty ||\psi'(a_i)||^p_{G'} < \infty$.*

Proof. We follow Linde [6], Cor. 6.5.2 and Maurey [7], Prop. 2b). For every N let λ_N, τ_N be the cylinder set measures on G' with

$$\hat{\lambda}_N(z) = \exp(-\Sigma_{n=1}^N |\langle z, \psi'(a_n)\rangle|^p)$$
$$\hat{\tau}_N(z) = \exp(-\Sigma_{n=N+1}^\infty |\langle z, \psi'(a_n)\rangle|^p), \ z \in G.$$

Then we have $\lambda_N * \tau_N = \psi'(\mu)$, where $*$ denotes the convolution. By the manner same to Hoffmann-Jørgensen [4], Th. 2.6, it holds that for every $q < p$

$$\int_{G'} ||a||^q_{G'} d\lambda_N(a) \leqq K \int_{G'} ||a||^q_{G'} d\psi'(\mu)(a),$$

where $K = 2^{1-q}$ for $0 < q < 1$ and $K = 1$ for $q \geqq 1$. Let $\{f_n(\omega)\}$ be a sequence of independent identically distributed symmetric p-stable random variables on a probability space (Ω, P) with the characteristic functional

$e^{-|t|^p}$. Let q be fixed such that $0 < q < p$. For every N, we set

$$S_N(\omega) = \Sigma_{n=1}^{N} \psi'(a_n) f_n(\omega).$$

S_N is a random variable which values in a finite-dimensional subspace of G' and the distribution of S_N is λ_N. If we set

$$H_N(\omega) = \max_{1 \leqq n \leqq N} ||\psi'(a_n) f_n(\omega)||_{G'}$$

then by Kwapien [5], Remark 1, it follows that

$$\begin{aligned}\int_\Omega H_N(\omega)^q dP(\omega) &\leqq 8C^2 \int_\Omega ||S_N(\omega)||_{G'}^q dP(\omega) \\ &= 8C^2 \int_{G'} ||a||_{G'}^q d\lambda_N(a),\end{aligned}$$

where $C = 1$ for $0 < q < 1$ and $C = 2^{q-1}$ for $q \geqq 1$. Consequently, we have

$$\int_\Omega H_N(\omega)^q dP(\omega) \leqq 8C^2 K \int_{G'} ||a||_{G'}^q d\psi'(\mu)(a).$$

Since $\psi'(\mu)$ is a p-stable $\sigma(G',\ G)$-Radon measure on G' and $0 < q < p$, we have

$$L = \int_{G'} ||a||_{G'}^q d\psi'(\mu)(a) < \infty$$

(see de Acosta [1], Linde [6], Cor. 6.7.5). Thus we have

$$\int_\Omega \max_{1 \leqq n \leqq N} ||\psi'(a_n) f_n(\omega)||_{G'}^q dP(\omega) \leqq 8C^2 KL < \infty$$

for every $N = 1,\ 2 \cdots$. Letting $N \to \infty$, we have

$$\int_\Omega \sup_n ||\psi'(a_n f_n(\omega))||_{G'}^q dP(\omega) < \infty.$$

Hence there exists $R > 0$ such that

$$\begin{aligned}&P(\omega :\ \sup_n ||\psi'(a_n f_n(\omega))||_{G'} \leqq R) \\ &= \prod_{n=1}^{\infty} \{1 - P(\omega :\ |f_n(\omega)| > R/||\psi'(a_n)||_{G'})\} > 0,\end{aligned}$$

where we have used the independence of $\{f_n(\omega)\}$. This implies that

$$\sum_{n=1}^{\infty} P(\omega :\ |f_n(\omega)| > R/||\psi'(a_n)||_{G'}) < \infty.$$

We remark that for every n,

$$\int_\Omega ||\psi'(a_n)f_n(\omega)||^q_{G'}dP(\omega) = ||\psi'(a_n)||^q_{G'}\int_\Omega |f_n(\omega)|^q dP(\omega) \leqq 8C^2KL,$$

that is, $\sup_n ||\psi'(a_n)||_{G'} < \infty$. Furthermore, it is known that $P(\omega : |f_n(\omega)| > t) \sim t^{-p}$ as $t \to \infty$, so we obtain for sufficiently large R,

$$P(\omega : |f_n(\omega)| > R/||\psi'(a_n)||_{G'}) \sim ||\psi'(a_n||^p_{G'}/R^p.$$

Hence it follows that $\Sigma_{n=1}^\infty ||\psi'(a_n||^p_{G'} < \infty$.

Lemma 3. *Let F, G be Banach spaces and* $0 < p < 2$. *Let* $\psi : G \to F$ *be a continous linear mapping and* $\psi' : F' \to G'$ *be the adjoint of* ψ. *Suppose that for every continous discrete p-stable cylinder set measure* μ *on* F', *the image* $\psi'(\mu)$ *is* $\sigma(G', G)$*-Radon. Then* $\psi' : F' \to G'$ *is p-summing.*

Proof. For each $(a_i) \subset F'$ with $\Sigma_{i=1}^\infty |\langle y, a_i\rangle|^p < \infty$ for every y in F, let μ be a continous p-stable cylinder set measure on F' satisfying $\hat{\mu}(y) = \exp(-\Sigma_{i=1}^\infty |\langle y, a_i\rangle|^p)$. Then $\psi'(\mu)$ is $\sigma(G', G)$-Radon. By Lemma 2, it follows that $\Sigma_{i=1}^\infty ||\psi'(a_i)||^p_{G'} < \infty$, which proves the assertion.

Theorem 1. *Let E be a barrelled space and p be* $0 < p < 2$. *Suppose that each equicontinous family of discrete p-stable cylinder set measures on* E'_S *is* $\sigma(E', E)$*-tight. Then E is nuclear.*

Proof. Let $\{U_\alpha\}$ be a fundamental system of zero neighborhoods of E. Let α be arbitrarily fixed. For every continous linear operator $T : E(U_\alpha) \to \ell_p$, let μ_T be the corresponding continous discrete p-stable cylinder set measure on $E'(U_\alpha^\circ)$ with $\hat{\mu}_T(x) = \exp(-||T(x)||^p_{\ell p})$, $x \in E(U_\alpha)$. Let $\tau_\alpha : E'(U_\alpha^\circ) \to E'$ be the canonical injection. Then by the assumption, image $\tau_\alpha(\mu_T)$ is $\sigma(E', E)$-Radon on E'_S. By Lemma 1, there exists $\beta = \beta(\alpha)$ with $U_\beta \subset U_\alpha$ (depends only on α, not on each T) such that for every continous p-stable cylinder set measure μ on $E'(U_\alpha^\circ)$, $\tau_\alpha(\mu)$ is Radon on $E'(U_\beta^\circ)$. By Lemma 3, the canonical injection $E'(U_\alpha^\circ, U_\beta^\circ)$ is p-summing and also 2-summing since $p < 2$. This completes the proof.

Remark 2. In Theorem 1, the barrelledness is essential. Let τ_S be the Sazonov topology on the infinite-dimensional Hilbert space H. Then

τ_S is not nuclear but each τ_S-equicontinous family of cylinder set measures is tight, see Sazonov [9].

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