

# *Mathematical Journal of Okayama University*

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*Volume 41, Issue 1*

1999

*Article 9*

JANUARY 1999

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## Multi Local Invariants on Real Unit Balls

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Math. J. Okayama Univ. **41** (1999), 149–179

## MULTI LOCAL INVARIANTS ON REAL UNIT BALLS

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### 1. INTRODUCTION

Quantities depending on several points of a space have been considered from early geometry on as there is the description of action at distance, two point functions, distance functions, bitangent curves et cetera. Functions which depend on tangent vectors or on higher order tangent vectors at distinct points of the manifold although typical differential geometric objects, also appear in analysis as for example invariant kernel functions and recently also in geometric analysis of feature detection in computer and human vision [12] [10] [4]. A classical example of a differential invariant depending on distinct points is the complex Poisson kernel [11] which is invariant with respect to complex Möbius transformations and depends as well on an interior point of the hyperbolic unit ball in  $\mathbb{C}^n$  as on the rotation invariant volume form on the boundary. In this paper we consider invariants for the real Möbius transformations acting on the real hyperbolic unit ball in  $\mathbb{R}^n$  and construct differential invariants, which we call multi local invariants, at several distinct points in the interior and/or at the boundary. In particular we find the real Möbius invariant Poisson kernel. The proof that it satisfies the invariant Laplace equation is based upon the Lie algebra structure of invariant vector fields which depend on extra boundary points. Most attention is given to invariants which characterize motions of points in  $B^n$  or in  $\partial B^n$  depending parametrically on sets of extra points.

The general case consists of two parts. First we consider only points in the interior of  $B^n$  and secondly we treat invariants depending as well on interior points as on boundary points. Dimension two and three are special and hence are treated as particular cases. For dimension two, which is the classical well known example [1], the calculations are carried out in full detail.

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1991 *Mathematics Subject Classification.* 53A20, 53C40, 57S25.

*Key words and phrases.* Differential Invariants, Multi local geometry, Unit Balls, Hyperbolic spaces, Invariant Poisson operator.

## 2. BASIC DEFINITIONS AND PROPERTIES

**2.1. Generating sets of invariants.** Let  $G$  be a Lie group acting on a manifold by diffeomorphisms. A function is called invariant if it is invariant under the induced action of the diffeomorphisms. A general question in geometry is to construct all invariant functions. In order to tackle this vast problem we consider sheafs of germs of real functions which are defined on the manifold or on an appropriate jet bundles over it, thus giving invariants of a certain weight.

The germ of functions at  $x \in M$  is the equivalence class of functions agreeing upon open neighborhoods of  $x$ . The set of all germs of functions at a given point  $p$  is called the stack of germs at  $x$  and the union of all stacks is the sheaf of germs of functions on  $M$ . The main purpose of this approach is to avoid discussions on the specification of domains of functions and to allow a local analysis. Because we are interested in sheafs of invariant functions we like to be able to produce a finite set of germs with the property that the set contains enough information to construct all invariant functions on open dense subsets. We call a set of functions on an open neighborhood  $U$  a generating set if every germ in the invariant sheaf over  $U$  can be reconstructed from the germs of the given functions by means of functional composition. A key notion here is functional independence of the germs, which means that the exterior product of the differentials of the germs is the germ of a non zero  $n$ -form [10]. We consider  $C^\infty$ -functions. Because we often take norms of germs of vector fields as generators as well as scalar products or inverses of functions, a set of generating invariants will be a set of invariant germs of maximal functional rank at an open dense subset which we call a regular subset. Extension of the germs of the generators onto a larger subset then often requires taking inverses or square roots. In order to keep the formulation transparent such extensions will be taken without mentioning.

**2.2. The tangent bundle and higher order jet bundles.** We assume some familiarity with the geometry of tangent bundles to manifolds [14]. Let  $M$  be a manifold. The tangent bundle,  $TM$ , is a bundle with projection map  $\pi : TM \rightarrow M$ . Let  $(x^i)$  be local coordinates on  $U \subset M$ , a saturated neighborhood  $\pi^{-1}U$  carries the coordinates  $((x^i), (\dot{x}^i))$ . A one form  $\omega$  on  $M$  is a function on  $TM$  in a natural way which is denoted by  $\hat{\omega}$ . The following lifts are defined as natural operations on  $TM$ . Let  $f$  be a function then  $f^c = \hat{d}f$  is the complete lift and  $f^v = f \circ \pi$  is the vertical lift. Let  $X$  be a vector field on  $M$ , the complete, or first order lift  $X^c \equiv X^{(1)}$ , is defined by  $X^c(f^c) = (X(f))^c$ . Lifts of one forms then are defined by  $\omega^c(X^c) = (\omega(X))^c$ .

The  $k^{th}$ -order jet bundle of curves in  $M$  is defined as follows [13][3]. Let  $J^k(\mathbb{R}, M)$  be the set of  $k$ -jets of maps of  $\mathbb{R}$  into  $M$ . This space is equipped with the source map  $\alpha$  which sends the  $k$ -jet of a map  $\gamma(t)$  into its source point. We call the space  $J^k(M) = \alpha^{-1}(0)$  the  $k^{th}$ -order jet bundle over  $M$ . The space  $J^1(M)$  identifies in a natural manner with the tangent space  $TM$ . More generally we have a natural embedding  $j : J^{k+1}(M) \rightarrow TJ^k(M)$  for each  $k$ . The bundle of infinite jets  $J(M) = \varprojlim J^k(M)$  possesses the total derivative operator,  $T$ , which is the total derivative with respect to  $t$ , as natural operator on functions on  $J^k(M)$ . Functions on  $J^k(M)$  are included into the set of functions on  $J(M)$  for this matter. We will always identify  $J^1(M)$  with  $TM$  and hence use the operator  $T$  for functions, one forms and covariant tensor fields on  $M$ , which results in entities on  $TM$ . It is easily verified that for any differential one form  $\omega$  on  $M$  we have  $\mathcal{L}_T\omega = \omega^c$ , where  $\mathcal{L}_T$  stands for the Lie derivative with respect to  $T$ .

Moreover let  $g$  be a Riemannian metric on  $M$ , then  $g^c = \mathcal{L}_Tg$ , is the complete lift of the metric  $g$  to  $TM$ . Let  $g^c$  be given in local coordinates as  $g_{ij}dx^i dx^j$ , the complete lift  $g^c$  is given by  $2g_{ij}\dot{x}^i dx^j + 2\Gamma_{ij}^k g_{kl}\dot{x}^i dx^l dx^j$ , where  $\Gamma_{ij}^k$  are the connection coefficients defined by  $g$ . Let  $G$  be a transformation group acting on  $M$ , the action extends to the space  $TM$  and to any  $J^k(M)$ , which is called prolongation of the group action [8]. If the metric  $g$  is invariant under the induced action of  $G$ , the complete lift  $g^c$  will be invariant under the prolonged action on  $TM$ . This result is immediate because the operator  $T$  is by construction equivariant for the induced action of the group on the ring of functions. The generalization to higher order is immediate. The metric  $j^*(\mathcal{L}_T)^k g = g^{(k)}$ , which is the complete lift of  $g$  onto  $J^k(M)$ , is also invariant under the induced action of the group  $G$ . We will use the notation  $g^c$  for  $g^{(1)}$ .

On  $TM$  one has some extra structure. The vector field  $I = \dot{x}^i \partial_{\dot{x}^i}$  is the canonical vertical vector field satisfying  $I(f^c) = f^c$  for any function  $f$  on  $M$ . When  $g$  is a pseudo Riemannian metric on  $M$  then  $\theta = g^c(I)$  is the canonical one form associated with  $g$ . In local coordinates one has  $\theta = g_{ij}\dot{x}^i dx^j$ . The spray  $\Gamma$  is by definition the vector field on  $TM$  such that for each geodesic  $\gamma(t)$  one has  $\Gamma(\dot{\gamma}^c(t)) = \dot{\gamma}^c(\dot{\gamma}^c(t))$  citekn:Yan. The vector field  $\Gamma$  satisfies  $\Gamma = g^{c-1}(d\hat{g})$ , which in local coordinates yields  $\Gamma = \dot{x}^i \partial_{\dot{x}^i} - \Gamma_{kl}^i \dot{x}^k \dot{x}^l \partial_{\dot{x}^i}$ .

Following properties are easily verified.

**Properties 2.1.** *i)*  $\Gamma(\hat{g}) = 0$ , *ii)*  $g^c(\Gamma, \Gamma) = 0$ , *iii)*  $g^c(I, \Gamma) = \hat{g}$ .

**Properties 2.2.** *Let  $\gamma(t)$  be a curve in  $M$ , then*

$$g^c(\dot{\gamma}^c(t), \dot{\gamma}^c(t))(\gamma(t), \dot{\gamma}(t)) = g(\nabla_{\dot{\gamma}(t)} \dot{\gamma}(t), \dot{\gamma}(t)).$$

**2.3. Invariant vector fields and motions.** Let  $G$  be any Lie group acting to the right on the manifold  $M$  by  $\phi : G \times M \rightarrow M$ ,  $\tilde{x} = \phi(g, x) \equiv x.g$ . It is well known that for given coordinates,  $(g^i)$ , at a neighborhood of the identity in  $G$ , the vector fields  $X_{g^i} = D_{g^i}\phi(g^j=0, x)$  form a complete set of generating vector fields of the action [6]. Now if  $G$  acts simply transitively on  $M$  there exists, with respect to a given fixed point  $x^o \in M$ , a local diffeomorphism at the identity in  $G$  with a neighborhood of  $x^o$  in  $M$ . This local diffeomorphism is given by  $\phi_o(g) = x^o.g$  on a neighborhood. Let  $(x^i)$  be a set of coordinates covering a neighborhood of  $x^o$  in  $M$  such that  $D\phi_o(e) = Id$  and consider the vector fields

$$(1) \quad Y_{x^i} = \phi_o^{-1} * D_{x^i}\phi(g, x^o).$$

Let  $\mathcal{L}(M)$  be the Lie algebra of the fundamental vector fields  $(X_{g^i})$  and  $\mathcal{R}(M)$  the Lie algebra of the vector fields  $(Y_{x^i})$ . Next proposition is a classical result in Lie transformation groups [6].

**Proposition 2.3.**

- (1) *The vector fields  $Y_{x^i}$  are invariant for the right action of  $G$  on  $M$ .*
- (2) *There exists an anti isomorphism of the Lie algebra  $\mathcal{R}(M)$  with the Lie algebra  $\mathcal{L}(M)$ .*

Let  $M_1$  and  $M_2$  be two  $G$ -manifolds, with  $G$  acting on both manifolds at the right and  $\xi = (\xi_1, \xi_2)$  coordinates on  $M_1 \times M_2$ . For  $g \in G$  we denote the product action as  $\xi.g = (\xi_1.g, \xi_2.g)$ .

**Definition 2.4.**

- (1) *Let  $\gamma(t)$  be a regular curve in the  $G$ -manifold  $M$ , with  $\xi^o = \gamma(t=0)$ . The curve  $\gamma(t)$  is called a motion (or  $G$ -motion) if  $\exists g_t$ , one parameter subgroup of  $G$ , such that  $\gamma(t) = \xi^o.g_t$ .*
- (2) *Let  $\gamma(t)$  be an  $M_1$ -regular curve in  $M_1 \times M_2$ , both  $G$ -manifolds, with  $\gamma(t=0) = (\xi_1^o, \xi_2^o)$ . Then  $\gamma(t)$  is a motion ( $G$ -motion) depending parametrically on  $M_2$  if  $\exists g_t$  one parameter subgroup of  $G$  and  $\exists f(t)$  a curve in  $\mathcal{I}_{\xi_1^o}$ , isotropy group at  $\xi_1^o$ , such that  $\gamma(t) = (\xi_1^o.g_t, \xi_2^o.f(t).g_t)$ .*

Next proposition provides a criterion for a curve to be a motion in the above sense. The proof is a direct consequence of the isomorphism of the vector space of generators over an orbit with the vector space of left invariant vector fields on the Lie group.

**Proposition 2.5.**

- (1) *Let  $X_{g^k}$  be a complete set of generators of the right action of  $G$  on  $M$ . Then a curve  $\gamma(t)$  in  $M$  is a motion iff  $\dot{\gamma}(t) = \sum a^i X_{g^i}(\gamma(t))$ , with  $a^i$  constants.*

(2) Let  $X_{g^k}$  be a complete set of generators of the right product action on  $M_1 \times M_2$ . Then a  $M_1$ -regular curve in  $M_1 \times M_2$  is a motion depending parametrically on  $M_2$  iff  $\pi_{1*}\dot{\gamma}(t) = \sum a^i \pi_{1*}X_{g^i}(\gamma(t))$ ,  $\pi_1$  denotes the projection on  $M_1$ .

**Corollary 2.6.** *The above proposition remains true if the set  $X_{g^k}$  is replaced by a complete set of invariant vector fields.*

**Corollary 2.7.** *Let  $G$  be a semi simple transformation group acting on the manifold  $M$  such that the dimensions of the orbits equals the dimension of the group. Let  $\gamma(t)$  be a regular curve in  $M$ , then  $\gamma(t)$  is a motion in  $M$  iff (a)  $\gamma(t)$  is lying in an orbit of  $G$ , (b) a generating set of first order invariants is constant along  $\gamma(t)$ .*

*Proof.* Since  $G$  is semi simple the Killing metric has maximal rank and each orbit, which being locally diffeomorphic to  $G$ , carries an invariant metric which we denote by  $K$ . Let  $\gamma(t)$  be lying in an orbit. Because the first order invariants along the orbit are generated by the invariant one forms  $K(X_i)$ , with  $X_i$  a complete set of invariant vector fields tangent to the orbit, constancy of the first order invariants along  $\gamma(t)$  implies that along  $\gamma$  the tangent vector  $\dot{\gamma}(t)$  is a constant combination of the right invariant vector fields. Conditions are sufficient because the curve is supposed to be regular. Necessity is immediate.  $\square$

### 3. MÖBIUS TRANSFORMATIONS ON $B^n$

**3.1. Structure and generators.** In this section we review the structure of the Möbius transformations more closely and derive some properties needed to achieve our main theorems. We utilize here a construction and some results of Beardon [2].

Let  $\widehat{\mathbb{R}}^n$  be the compactified  $\mathbb{R}^n$  space. A Möbius transformation on  $\widehat{\mathbb{R}}^n$  is defined as a finite composition of reflections in spheres or planes. The general Möbius group, which is the transformation group generated by all reflections, is denoted by  $GM(n)$  and has the proper subset  $M(n)$  of orientation preserving transformations. Furthermore one has the Lie group isomorphism :  $GM(n) \cong O^+(1, n+1)$  [2]. Notice that  $\dim GM(n) = \frac{1}{2}(n+2)(n+1)$ .

On  $\mathbb{R}^n$  we will make use of the Euclidean metric  $\langle ., . \rangle$  and corresponding norm function  $\|.\|$ . The Möbius transformations which preserve  $B^n$  then are compositions of rotations with center 0 and reflections with respect to the spheres or planes which are orthogonal to the boundary of  $B^n$  [2]. This group is  $GM(n-1)$  which can be deduced from the Poincaré extension and the Cayley map [2]. Reflection of  $B^n$  with respect to the sphere  $S(a, 1 - \|a\|^2)$ , with  $\|a\| > 1$  is given by :

$$(2) \quad \phi_a(x) = a - (1 - \|a\|^2) \frac{x - a}{\|x - a\|^2}.$$

We define  $\xi_0 \in B^n$  as  $\xi_0 = \frac{a}{\|a\|^2}$ . Then with  $\xi \in B_n$

$$(3) \quad \phi_{\xi_0}(\xi) = \frac{1}{\Omega} \left[ \frac{\|\xi_0\|^2(\|\xi\|^2 + 1) - 2 \langle \xi_0, \xi \rangle}{\|\xi_0\|^2} \xi_0 - (\|\xi_0\|^2 - 1)\xi \right],$$

where  $\Omega = \|\xi_0\|^2\|\xi\|^2 + 1 - 2 \langle \xi_0, \xi \rangle$ .

The mapping  $\phi_{\xi_0}$  is a diffeomorphism of  $B^n$  and maps the boundary,  $S^{n-1}$ , diffeomorphically into itself. One also has  $\phi_{\xi_0}(0) = \xi_0$ ,  $\phi_{\xi_0}(\xi_0) = 0$  and  $\phi_{\xi_0}(\phi_{\xi_0}(\xi)) = \xi$ , which means that  $\phi_{\xi_0}$  is an involution.

The Lie algebra of generating vector fields of the action is found by taking the generators of the mappings :

$$(4) \quad \psi_{tu} = \phi_u \circ \phi_{tu} : B^n \rightarrow B^n,$$

for  $u$  unit vector in  $\mathbb{R}^n$ . The flow of this map is given by

$$(5) \quad \psi_{tu}(\xi) = \frac{1}{\Omega} [(\|\xi\|^2 + 1)t u - 2 \langle \xi, u \rangle u - (t^2 - 1)\xi],$$

with

$$(6) \quad \Omega = t^2\|\xi\|^2 + 1 - 2t \langle \xi, u \rangle.$$

We then find  $\psi(t = 0) = Id$  and

$$(7) \quad D_t(\psi)(\xi)(t = 0) = [2 \langle \xi, u \rangle - (\|\xi\|^2 + 1)] \partial_\xi.$$

We define the generating vector field determined by  $u$ , unit vector in  $\mathbb{R}^n$ , as  $X_u = -\frac{1}{2}D_t(\psi)(\xi)(t = 0)$  or

$$(8) \quad X_u = \left[ \frac{1}{2}(\|\xi\|^2 + 1)u - \langle \xi, u \rangle \xi \right] \partial_\xi.$$

Remark that the only zeros of the vector field  $X_u$  are  $u$  and  $-u$  in  $\partial B^n$ . The point  $u$  is attractive while  $-u$  is repulsive.

Let  $u, v$  be two orthonormal vectors, then

$$(9) \quad [X_u, X_v] = (\langle \xi, u \rangle v - \langle \xi, v \rangle u) \partial_\xi.$$

Let  $(u_i)$  be an orthonormal base we find that  $(X_{u_i})$  generates the Lie algebra  $\mathfrak{so}(1, n)$ . Verification follows from the calculation of the brackets.

Let  $X_{uv} = [X_u, X_v]$ , we take  $\{Z_{u_i}, Z_{u_i u_j}\}$  as base. Additional to (9) we find

$$(10) \quad [Z_{u_i u_j}, Z_{u_k u_l}] = \delta_{jk} Z_{u_i u_l} - \delta_{ik} Z_{u_j u_l} - \delta_{jl} Z_{u_i u_k} + \delta_{il} Z_{u_j u_k}$$

and

$$(11) \quad [Z_{u_i}, Z_{u_j, u_k}] = \delta_{ij} Z_{u_k} - \delta_{ik} Z_{u_j}.$$

As a corollary from the above construction we find that invariance under the set  $X_{u_i}$  implies infinitesimal invariance.

**Proposition 3.1.** *The Killing form  $\mathcal{B}$  with respect to this basis is given by*

$$(12) \quad \begin{aligned} \mathcal{B}(Z_{u^i}, Z_{u^j}) &= 2(n-1) \delta_{ij}, \quad \mathcal{B}(Z_{u^i}, Z_{u^j u^k}) = 0, \\ \mathcal{B}(Z_{u^i u^j}, Z_{u^k u^l}) &= -2(n-1) (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}). \end{aligned}$$

**3.2. The fundamental invariants.** For any  $\{x, y, u, v\}$ , distinct points in  $\mathbb{R}^n$ , we define their cross ratio as  $\mathcal{Q}(x, y; u, v) = \{\|x - u\| \cdot \|y - v\|\} / \{\|x - v\| \cdot \|y - u\|\}$ , which is invariant under the action of  $GM(n)$ . It is a classical result that the cross-ratio can be extended for points in  $\widehat{\mathbb{R}}^n$ . We recall the theorem of Beardon asserting that a map,  $\phi : \widehat{\mathbb{R}}^n \rightarrow \widehat{\mathbb{R}}^n$ , is a Möbius transformation if and only if it preserves cross-ratios. In the real case the cross ratio is the basic invariant from which all other ones are derived.

Let  $(\xi_1, \xi_2)$  be two points in  $B^n$ . Then under the action of  $\phi_{\xi_1}$  the points are send into  $(0, \phi_{\xi_1}(\xi_2))$ . Taking the line through both images gives two intersection points  $(\varrho, -\varrho)$  with the boundary  $S^{n-1}$  of  $B^n$ . Then  $\varrho = \phi_{\xi_1}(\xi_2) / \|\phi_{\xi_1}(\xi_2)\|$ . Hence the function

$$(13) \quad \mathcal{Q}(\xi_1, \xi_2) = \mathcal{Q}(\xi_1, \phi_{\xi_1}(\varrho); \xi_2, \phi_{\xi_1}(-\varrho))$$

is invariant. Then

$$\mathcal{Q}(\xi_1, \xi_2) = \frac{\|\xi_1 - \xi_2\| \|\phi_{\xi_1}(\varrho) - \phi_{\xi_1}(-\varrho)\|}{\|\xi_1 - \phi_{\xi_1}(-\varrho)\| \|\phi_{\xi_1}(\varrho) - \xi_2\|}.$$

Using invariance this becomes

$$= \frac{\|\phi_{\xi_1}(\xi_2)\| \|2\varrho\|}{\|\varrho\| \|\varrho - \phi_{\xi_1}(\xi_2)\|},$$

which by simplification gives



$$(14) \quad = 2 \frac{\|\phi_{\xi_1}(\xi_2)\|}{1 - \|\phi_{\xi_1}(\xi_2)\|}.$$

This function is clearly symmetric in  $\xi_i$  and  $\xi_j$ . We remark that  $\|\phi_{\xi_i}(\xi_j)\|$  measures the invariant distance,  $\tilde{d}$ , between the two points in  $B^n$ , because

$$(15) \quad \tilde{d}(\xi_i, \xi_j) = \|\phi_{\xi_i}(\xi_i) - \phi_{\xi_i}(\xi_j)\| = \|\phi_{\xi_i}(\xi_j)\|.$$

**Proposition 3.2.** [2]

$$(16) \quad 1 - \|\phi_{\xi_i}(\xi_j)\|^2 = \frac{1}{1 + \frac{\|\xi_i - \xi_j\|^2}{(\|\xi_i\|^2 - 1)(\|\xi_j\|^2 - 1)}}.$$

Hence following function is invariant.

**Definition 3.3.**

$$(17) \quad \mathcal{Q}_{ij} = \frac{\|\xi_i - \xi_j\|^2}{(\|\xi_i\|^2 - 1)(\|\xi_j\|^2 - 1)}.$$

The most important geometric invariant derived from  $\mathcal{Q}_{ij}$  is the hyperbolic Riemannian metric. Consider  $B^n \times B^n$  equipped with the coordinates  $(\xi_1, \xi_2)$ . The diagonal  $\Delta = \{\xi_1 = \xi_2\}$  of  $B^n \times B^n$  is an invariant subspace. The function  $\mathcal{Q}_{12}$  is zero on  $\Delta$  with zero first order total derivatives with respect to  $\xi_1$  and  $\xi_2$  at  $\Delta$ . Hence the second order total derivatives of  $\mathcal{Q}_{12}$  at  $\Delta$  determine an invariant quadratic form. Set  $\xi = \xi_1 = \xi_2$  at  $\Delta$ , then

$$(18) \quad T_{\xi_1} T_{\xi_1} \mathcal{Q}_{12}|_{\Delta} = 2 \frac{\langle \dot{\xi}, \dot{\xi} \rangle}{(1 - \|\xi\|^2)^2}.$$

which determine the invariant metric. The normalized metric with curvature  $K = -1$  equals

$$(19) \quad g_o : ds^2 = 4 \frac{d\xi d\xi}{(1 - \|\xi\|^2)^2}.$$

#### 4. MULTI LOCAL INVARIANTS ON $B^n$

**4.1. Invariants depending on points in the interior.** In this section we examine invariants depending exclusively on points in  $B^n$ . Therefore consider the  $n$ -fold product of the unit ball with itself,  $\mathcal{E}_n = \prod_{i=0}^{n-1} B_i$ ,  $n$  is called the rank of the space.

The rank of  $\mathcal{E}_n$  is taken equal to  $n$  because generic orbits are of maximal dimension, for smaller rank dimension of the generic orbits drops. We

denote the  $i^{th}$  factor by  $B_i$  and call this the  $i^{th}$  layer for  $i = 0, 1, 2, \dots, n-1$ . It is the zeroth layer which will be of most importance to us.

4.1.1. *At zero order.*

**(a) Invariant functions** Let  $(\xi_i), i = 0, \dots, n-1$ , be the natural coordinates on  $\mathcal{E}_n$  and  $\mathcal{W}_0$  the set of all  $n$ -tuples of points in  $B^n$  of rank  $n-1$ , considered as subset of  $\mathcal{E}_n$ .

**Theorem 4.1.** *The set*

$$(20) \quad \mathcal{I}_0 = \{Q_{ij}, i < j\}$$

*is a set of generating invariant functions on  $\mathcal{W}_0$ .*

*Proof.* The set  $\mathcal{I}_0$  consists of  $n(n-1)/2$  functions. The differentials of their logarithm equal

$$(21) \quad d \log Q_{ij} = 2 \left[ \frac{\xi_i - \xi_j}{\|\xi_i - \xi_j\|^2} - \frac{\xi_i}{\|\xi_i\|^2 - 1} \right] d\xi_i - 2 \left[ \frac{\xi_i - \xi_j}{\|\xi_i - \xi_j\|^2} + \frac{\xi_j}{\|\xi_j\|^2 - 1} \right] d\xi_j.$$

The exterior product is a form of maximal rank on  $\mathcal{W}_0$  as can be seen from examining the form at the points where  $\xi_0 = 0$ . This is sufficient because  $GM(n-1)$  acts transitively on the zeroth-layer. Call  $\mathcal{V}_o$  the set  $\xi_0 = 0$ . The differentials  $d \log Q_{0i}$  at  $\mathcal{V}_o$  restricted to the zeroth layer equal  $(-2/(1 - \|\xi_i\|^2))\xi_i d\xi_0$ , which are  $n-1$  independent differentials. The differentials restricted to the first layer of  $Q_{1i}$  are given by a linear combination of  $\xi_1 d\xi_1$  and  $\xi_i d\xi_1$  at each point of the set  $\xi_0 = 0$  in  $\mathcal{W}_0$  and yield a set of  $n-2$  linear independent differentials. These differentials are clearly independent of the former ones because they belong to a different vector space. This construction repeats for each  $i$  which gives  $n-i-1$  independent one forms independent of the former ones. But this proves the theorem, the rank equals  $n(n-1)/2$ . □

**(b) Invariant vector fields** We construct a set of invariant vector fields which lie in the first layer of  $\mathcal{E}_n$ . Let  $g_o$  denote the invariant metric on the first layer we then define the invariant vector fields  $Y_i = g_o^{-1} d \log Q_{0i}$ ,  $i = 1, \dots, n-1$ . We find

$$(22) \quad Y_i = \frac{1}{2} \left[ \frac{(1 - \|\xi_0\|^2)^2}{\|\xi_0 - \xi_i\|^2} (\xi_0 - \xi_i) + (1 - \|\xi_0\|^2) \xi_0 \right] \partial_{\xi_0}.$$

Next define an extra invariant vector field  $Y_n$  by selecting an extra point  $\varrho$  in  $B^n$  such that  $Q_0(\xi_\alpha, \varrho) = 1$  for each  $\xi_\alpha$ ,  $\alpha = 0, \dots, n-1$ . Then  $\varrho$  belongs to  $B^n$  because (1)  $Q_0(\xi_0, \varrho) = 1$  implies  $\|\varrho\|^2 = 1/2$  at  $\xi_0 = 0$  and

(2)  $\mathcal{Q}_0(\xi_i, \varrho) = 1$  implies  $\varrho_i = 3/4\|\xi_i\|^2\xi_i$ . The set  $(\xi_i)$  being a set of rank  $n - 1$  for  $\xi_0 = 0$ . Hence the vector field

$$(23) \quad Y_n = \frac{1}{2} \left[ \frac{(1 - \|\xi_0\|^2)^2}{\|\xi_0 - \varrho\|^2} (\xi_0 - \varrho) + (1 - \|\xi_0\|^2)\xi_0 \right] \partial_{\xi_0}$$

is invariant. Let  $Y_\alpha = (Y_i, Y_n)$  denote this set of vector fields with  $\alpha = 1 \cdots, n$ .

**Proposition 4.2.** *The set  $Y_\alpha$  has rank  $n$  at each point in  $\mathcal{W}_0$ .*

*Proof.* In order to show linear independence of the set  $Y_i$  it is sufficient to consider the vectors at  $\mathcal{V}_0$ . The vectors

$$(24) \quad Y_\alpha(\xi_0 = 0) = -\frac{1}{2} \frac{\xi_\alpha}{\|\xi_\alpha\|^2} \partial_{\xi_0},$$

with  $\xi_n = \varrho$ , are linear independent at all points  $(\xi_0 = 0, \xi_1, \dots, \xi_{n-1})$  in  $\mathcal{W}_0$  and hence on  $\mathcal{W}_0$ . □

4.1.2. *At first order.*

**(a) Invariant functions** The first order invariants we are interested in are those which depend on derivatives of curves in the zeroth-layer,  $B_0$ , of  $\mathcal{E}_n$ . We therefore consider the space  $\mathcal{J}^{1,0,\dots,0}\mathcal{E}_n = \mathcal{J}^1 B_0 \times B_1 \times \dots \times B_{n-1}$  on which we define the following coordinates  $(\xi_0, \dot{\xi}_0, \xi_1, \dots, \xi_{n-1})$ .

The  $n$  invariant vector fields  $Y_\alpha$  determine an independent set of invariant one forms  $\mu_\alpha = g_o(Y_\alpha)$ . These forms lift to invariant functions on the space  $\mathcal{J}^{1,0,\dots,0}\mathcal{E}_n$ . Let  $\mathcal{W}_0^{(1)}$  be the subset  $\pi_o^{-1}\mathcal{W}_0 \setminus \{\dot{\xi}_0 = 0\}$ , with  $\pi_o : \mathcal{J}^{1,0,\dots,0}\mathcal{E}_n \rightarrow \mathcal{E}_n$ .

**Theorem 4.3.** *The set*

$$(25) \quad \mathcal{I}_0^{(1)} = \pi_o^* \mathcal{I}_0 \cup \{\tilde{\mu}_\alpha\}$$

*is a generating set of invariants on  $\mathcal{W}_0^{(1)}$ .*

Applying corollary (2.7) we find :

**Theorem 4.4.** *Let  $\gamma(t)$  be a curve in  $\mathcal{E}_n$ . Then  $\gamma(t)$  is a motion in  $B_0$  depending parametrically on  $\xi_i, i = 1, \dots, n - 1$  iff (1) all  $\mathcal{Q}_{ij}$ , for  $i, j = 0, 1, \dots, n - 1$  are constant along  $\gamma(t)$  and (2) the one form  $\mu_n$  is constant along  $\gamma(t)$ .*

**Corollary 4.5.** *In the above theorem constancy of the one form  $\mu_n$  can be replaced by constancy of the square of the length  $g_o(\dot{\gamma}(t), \dot{\gamma}(t))$  of the tangent vector to the curve.*

Next consider the space

$$(26) \quad \mathcal{J}^{1,0,\dots,0}\mathcal{E}_{n-1} = J^1B_0 \times \prod_{i=1}^{n-2} B_i$$

which is equipped with the coordinates  $(\xi_0, \dot{\xi}_0, \xi_1, \dots, \xi_{n-2})$ . Consider the subset  $\mathcal{W}_1$  which is defined by the conditions : (1)  $\dot{\xi}_0 \neq 0$  and (2)  $rk \{ \xi_0, \dot{\xi}_0, \xi_1, \dots, \xi_{n-2} \} = n - 1$ . The set of fundamental vector fields has maximal rank on  $\mathcal{W}_1$  because : (1) the rotation vector fields lift linearly onto the tangent bundle to  $B_0$  and (2) the vector fields  $X_u$  lift to  $[\frac{1}{2}(\langle \xi, \xi \rangle + 1)u > - \langle \xi, u \rangle \xi] \partial_\xi + [\langle \dot{\xi}, \xi \rangle u - \langle \dot{\xi}, u \rangle \xi - \langle \xi, u \rangle \dot{\xi}] \partial_{\dot{\xi}}$ .

**Theorem 4.6.** *The following set is a generating set of invariants on  $\mathcal{W}_1$  :*

$$(27) \quad \mathcal{I}_1 = \{ \mathcal{Q}_{0i}, \mathcal{Q}_{ij}, T_0\mathcal{Q}_{0,i}, \tilde{g}_o \}$$

$i, j = 1, \dots, n - 2.$

*Proof.*  $rk \{ \mathcal{Q}_{0i}, \mathcal{Q}_{ij} \} = (n - 1)(n - 2)/2$  by the same arguments as used in theorem (4.3), while the remaining subset of functions are first order in the variables  $\dot{\xi}_0$  and has rank  $n - 1$ . Hence the rank of  $\mathcal{I}_1$  on  $\mathcal{W}_1$  equals  $\frac{n(n-1)}{2}$ , while the space has dimension  $n^2$ . It follows that the level surfaces have same dimension as the group  $GM(n - 1)$ . □

The invariant sheaf  $\mathcal{I}_0^{(1)}$  on  $\mathcal{W}_0^{(1)}$  extends uniquely onto the invariant diagonal  $\xi_{n-1} = \xi_{n-2}$ . We use the same symbol for the extension. Consider the embedding

$$(28) \quad \Delta : \mathcal{J}^{1,0,\dots,0}\mathcal{E}_{n-1} \rightarrow \mathcal{J}^{1,0,\dots,0}\mathcal{E}_n$$

which takes  $\mathcal{J}^{1,0,\dots,0}\mathcal{E}_{n-1}$  into the diagonal  $\xi_{n-1} = \xi_{n-2}$ . This embedding is equivariant and hence we find equivalence of two sets of invariant generators, namely  $\Delta^*\mathcal{I}_0^{(1)}$  and  $\mathcal{I}_1$ . Remark that taking the embedding  $\Delta$  means taking the limit for  $\xi_n \rightarrow \xi_{n-1}$ , which results into restriction to the  $n - 1$  first points in  $B$ .

**(b) Invariant vector fields** We now construct invariant vector fields tangent to the first layer of  $\mathcal{J}^{1,0,\dots,0}\mathcal{E}_{n-1}$ . Let  $L_\alpha = Y_\alpha^{(1)}$ , where  $\alpha$  runs from 1 to  $n - 1$ , be the complete lifts of the  $n - 1$  vector fields  $Y_\alpha$  onto  $\mathcal{J}^{1,0,\dots,0}\mathcal{E}_{n-2}$ . They constitute a linear independent set. Define  $N_\alpha = g_o^c{}^{-1}\mu_\alpha$  and let further  $\Gamma$  be the geodesic spray and  $Z$  the vector field on  $J^1B_0$  orthonormal to all the former ones with respect to the metric  $g_o^c$ .

**Proposition 4.7.** *The set*

$$(29) \quad Z_\alpha = \{L_\alpha, N_\alpha, \Gamma, Z\}$$

*is a linear independent set of  $2n$  invariant vector fields at each point of  $\mathcal{W}_1$ .*

*Proof.* To show linear independence of the vector fields it is sufficient to look at points with  $\xi_0 = 0$ . We find with  $\alpha = 1, \dots, n - 1$  :

$$(30) \quad Y_\alpha^{(1)}(\xi_0 = 0) = -\frac{1}{2} \frac{\xi_\alpha}{\|\xi_\alpha\|^2} \partial_{\xi_0} + \left[ \frac{1}{2} \left( \frac{1}{\|\xi_\alpha\|^2} + 1 \right) \dot{\xi}_0 - \frac{\langle \dot{\xi}_0, \xi_\alpha \rangle}{\|\xi_\alpha\|^4} \xi_\alpha \right] \partial_{\dot{\xi}_0}$$

$$N_\alpha = g_o^{c-1} \mu_\alpha(\xi_0 = 0) = -\frac{1}{2} \frac{\xi_\alpha}{\|\xi_\alpha\|^2} \partial_{\dot{\xi}_0}$$

$$\Gamma(\xi_0 = 0) = \dot{\xi}_0 \partial_{\xi_0}$$

from which linear independence is easily determined. Invariance is obvious.  $\square$

Let  $\gamma(t)$  be a curve in the product  $\prod_{i=0}^{n-2} B_i$  then  $\gamma_0^{(1)}(t)$  denotes the prolongation of this curve in  $J^1 B_0 \times B_1 \times \dots \times B_{n-2}$ . Using theorem (2.7) we find :

**Theorem 4.8.** *Let  $\gamma(t)$  be a curve in  $B_0 \times B_1 \times \dots \times B_{n-2}$  which is regular in the zeroth layer. Then  $\gamma(t)$  is a motion depending parametrically on  $B_1 \times \dots \times B_{n-2}$  iff (1)  $\gamma_0^{(1)}(t)$  is lying in an orbit and (2)  $\dot{\gamma}_0^{(1)}$  is a constant combination of the set  $Z_\alpha$ .*

The construction of the invariants by means of invariant vector fields is not the most suitable one at higher order, we give a different construction based upon the Killing form.

Consider again the space  $\mathcal{J}^{1,0,\dots,0} \mathcal{E}_{n-1}$ . Each layer has an invariant metric :  $g_o^c$  on the first layer and  $g_i$  on the  $i^{th}$ -layer, with  $i = 1, \dots, n - 2$ . Let  $g = g_o^c + \sum_{i=1}^{n-2} g_i$  be the metric  $\mathcal{J}^{1,0,\dots,0} \mathcal{E}_{n-1}$ .

By means of the invariant set  $\mathcal{I}_1$  we construct following set of invariant vector fields which are transversal to the orbits nl.,  $N_i = g^{-1}(d\mathcal{Q}_{0i})$ ,  $L_i = g^{-1}(dT_0 \mathcal{Q}_{0i})$ ,  $M_{ij} = g^{-1}(d\mathcal{Q}_{ij})$ ,  $i < j$ ,  $I = g_o^{c-1}(\theta)$ .

Because the set  $\mathcal{I}_1$  is linear independent at each point the above set of vector fields is a linearly independent set at each point of  $\mathcal{W}_1$ . Moreover the set spans an invariant normal space to all orbits of the group action. Taking the Killing metric in the tangent space to each orbit, which is well defined because the generic orbits have maximal dimension, and taking the above vector fields as orthonormal vector fields normal to the orbits one obtains an invariant metric on  $\mathcal{W}_1$ , which we denote by  $K_2$ .

We will have to show that the differential of  $K_2$  is independent of the differentials of the other generators. Let  $p \in J^1B_0 \times B_1 \times \cdots \times B_{n-2}$  be such that  $\xi_0 = 0$  and the  $n - 1$  points  $\{\dot{\xi}_0, \xi_1, \dots, \xi_{n-2}\}$  are linear independent. Choose two unit vectors  $u, v \in \mathbb{R}^n$  such that  $u$  is orthogonal to the set  $\{\dot{\xi}_0, \xi_1, \dots, \xi_{n-2}\}$  and  $v$  orthogonal to  $\{u, \xi_1, \dots, \xi_{n-2}\}$ . Then

$$(31) \quad X_{uv}(p) = - \langle \dot{\xi}_0, v \rangle u \partial_{\dot{\xi}_0}.$$

But from the definition of the Killing form we have

$$(32) \quad K_2(\langle \xi_0, v \rangle u \partial_{\dot{\xi}_0}, \langle \xi_0, v \rangle u \partial_{\dot{\xi}_0})(p) = -2(n - 1).$$

Then

$$(33) \quad K_2(u \partial_{\dot{\xi}_0}, u \partial_{\dot{\xi}_0}) = - \frac{2(n - 1)}{(\langle \dot{\xi}_0, v \rangle)^2},$$

which proves independence with respect to the set of functions  $\{\mathcal{Q}_{0i}, T_0 \mathcal{Q}_{0i} \mathcal{Q}_{ij}, \tilde{g}_o\}$  at each point of the regular subset.

Together with theorem (2.7) we now have following theorem.

**Theorem 4.9.** *Let  $\gamma(t)$  be a  $B_0$ -regular curve in  $\Pi_{i=0}^{n-2} B_i$ . Then  $\gamma(t)$  is a motion depending parametrically on a point in  $\Pi_{i=1}^{n-2} B_i$  iff the set  $\{\mathcal{Q}_{0i}, \mathcal{Q}_{ij}, g_o, K_2|^\circ\}$  is constant along  $\gamma(t)$ .*

Remark that because  $\Gamma(\tilde{g}_o) = 0$  and  $g^c(\Gamma, \Gamma) = 0$  the vector field  $\Gamma$  is tangent to invariant null surfaces in  $J^1B_0$  with respect to the metric  $g_o^c$ . The null surfaces are given by  $\tilde{g}_o = \text{constant}$  for  $\dot{\xi}_0 \neq 0$  and hence are orbits which have signature  $(0, n - 1, n - 1)$ . Now if  $\gamma(t)$  is constant in the metric  $g_o$  the tangent vector field  $\dot{\gamma}_0^c(t)$  is a null vector such that  $g_o^c(\Gamma, \dot{\gamma}_0^c(t))(\gamma(t), \dot{\gamma}(t)) = 0$  (2.2). Hence  $\dot{\gamma}_0^c(t)$  is lying in an  $n$  dimensional invariant totally isotropic null space containing  $\Gamma$ . If  $\gamma(t)$  is a motion then  $\dot{\gamma}_0^c(t)$  is a fixed point in this space determined by  $T_0 T_0 \mathcal{Q}_{0i}, T_0 \tilde{g}_o$  and  $\tilde{K}_2|^\circ$ .

4.1.3. *Second order invariants.* The construction given in former paragraph yields the derivation of generating second order invariants. Consider  $\mathcal{J}^{2,0,\dots,0} \mathcal{E}_{n-1} = J^2 B_0 \times \Pi_{i=1}^{n-2} B_i$ .

**Theorem 4.10.** *The following sets are equivalent generating sets of invariants*

$$(34) \quad \mathcal{I}_2 = \langle \mathcal{Q}_{ij}, \mathcal{Q}_{0i}, T_0 \mathcal{Q}_{0i}, T_0 T_0 \mathcal{Q}_{0i}, \tilde{g}_o, T_0 \tilde{g}_o, \tilde{K}_2 \rangle$$

$$(35) \quad \mathcal{J}_2 = \langle \mathcal{Q}_{ij}, \mathcal{Q}_{0i}, T_0 \mathcal{Q}_{0i}, \tilde{g}_o, \widetilde{g_o^c L_\alpha}, \widetilde{g_o^c Z} \rangle$$

where  $L_\alpha = Y_\alpha^{(1)}$  and  $Y_\alpha = (Y_i, Y_n)$  and  $Z$  the vector field defined in (29).

Construct following embeddings :

$$(36) \quad J^2 B_0 \times \prod_{i=1}^{n-3} B_i \xrightarrow{\Delta} J^2 B_0 \times \prod_{i=1}^{n-2} B_i \xrightarrow{j} T J^1 B_0 \times \prod_{i=1}^{n-2} B_i,$$

where  $\Delta$  is the diagonal embedding  $\xi_{n-2} = \xi_{n-3}$  and  $j$  the canonical embedding of the jet bundle into the tangent space. Define the subset  $\mathcal{W}_2$  of  $J^2 B_0 \times \prod_{i=1}^{n-3} B_i$  as the set of points  $\{\xi_0, \dot{\xi}_0, \ddot{\xi}_0, \xi_1, \dots, \xi_{n-3}\}$  of rank  $n - 1$  with  $\dot{\xi}_0 \neq 0, \ddot{\xi}_0 \neq 0$ .

**Theorem 4.11.** *The set*

$$(37) \quad \mathcal{I}_{\mathcal{W}_2} = \{ \mathcal{Q}_{oi}, T_0 \mathcal{Q}_{oi}, \mathcal{Q}_{ij}, \tilde{g}_o, T_0 T_0 \mathcal{Q}_{oi}, \tilde{g}_o^c, \widetilde{K}^{|o} \},$$

where all functions are pulled back via the above defined mapping  $j \circ \Delta$ , is a generating set of invariants on  $\mathcal{W}_2$ .

4.1.4. *Higher order invariants.* At higher order we summarize the results. Verification follows from the same considerations as the ones at second order. Consider the space  $\mathcal{J}^{(k,0,\dots,0)} \mathcal{E}_{n-k} = J^k B_0 \times \prod_{i=1}^{n-(k+1)} B_i$ .

**Theorem 4.12.** *The set*

$$(38) \quad \mathcal{I}_{n-k} = \left\{ \begin{array}{cccc} \mathcal{Q}_{ij} & \mathcal{Q}_{oi} & & \\ & T_0 \mathcal{Q}_{oi} & \tilde{g}_o & \\ & T_0^2 \mathcal{Q}_{oi} & T_0 \tilde{g}_o & \widetilde{K}_2 \\ & \vdots & \vdots & \vdots \\ T_0^k \mathcal{Q}_{oi} & T_0^{k-1} \tilde{g}_o & T_0^{k-2} \widetilde{K}_2 & \dots \quad \widetilde{K}_k \end{array} \right\}$$

is a generating set of invariants at order  $k$ .

Given a generating set at order  $k$ , a set at order  $k + 1$  is found by prolongation of the former set together with  $\widetilde{K}_{k+1}$ , which is constructed in the same way as  $K_2$ . The embedding

$$(39) \quad \Delta : J^{n-1} B_0 \rightarrow J^{n-1} B_0 \times B_1$$

then allows the construction of a generating set at order  $n - 1$ , nl.

$$(40) \quad \mathcal{I}_{n-1} = \left\{ \begin{array}{ccc} \tilde{g}_o & & \\ T_0 \tilde{g}_o & \widetilde{K}_2 & \\ \vdots & \vdots & \\ T_0^{n-2} \tilde{g}_o & T_0^{n-3} \widetilde{K}_2 & \dots \quad \widetilde{K}_{n-1} \end{array} \right\}$$

Then at order  $n$  we have

**Theorem 4.13.** *The set*

$$(41) \quad \mathcal{I}_{n-1} \cup \{T_0^{n-1}\tilde{g}_0, T_0^{n-2}\tilde{K}_2, \dots, T_0\tilde{K}_{n-1}, \tilde{K}_n\}$$

*is a generating set of invariants of order  $n$ .*

Motions in  $B^n$  then are characterized by following theorem.

**Theorem 4.14.** *Let  $\gamma(t)$  be a regular curve in  $J^n B^n$ , lift of a curve in  $B^n$ , then  $\gamma(t)$  is a motion iff the functions  $\{\tilde{g}_0, \tilde{K}_2, \dots, \tilde{K}_n\}$  are constant along  $\gamma^c(t) = (\gamma(t), \dot{\gamma}(t), \dots, \gamma^{(n+1)}(t))$ .*

**4.2. Invariants depending on boundary and interior points.** Consider the space

$$(42) \quad \mathcal{E}_n = B_0 \times \underbrace{S^{n-1} \times \dots \times S^{n-1}}_{n-1 \text{ times}}.$$

On the first factor  $B_0$  we use the standard coordinates  $(\xi_0)$  from  $\mathbb{R}^n$ . Occasionally we use on each of the spheres,  $S^{n-1}$ , spherical coordinates induced from the standard coordinates on  $\mathbb{R}^n$ , but mostly when not mentioned we use the external coordinates which are denoted by  $(\xi_i)$ ,  $i = 1, \dots, n-1$ ,  $i$  stand for the layer. The generating vector fields of the action in this case are easily written down as

$$(43) \quad X_u = \sum_{j=0}^{n-1} \left( - \langle \xi_j, u \rangle \xi_j + \frac{1}{2} (\langle \xi_j, \xi_j \rangle + 1) u \right) \partial_{\xi_j}$$

$$X_{(u,v)} = \sum_{j=0}^{n-1} (- \langle \xi_j, v \rangle u \partial_{\xi_j} + \langle \xi_j, u \rangle v \partial_{\xi_j}),$$

where it is understood that each  $\xi_i$ , for  $i \neq 0$ , is a point in the boundary  $S^{n-1}$  of  $B^n$ .

**4.2.1. At zero order.** Let  $\xi_0$  be a point in the interior of  $B_0$  and  $\xi_i$  a point at the boundary  $S^{n-1}$ . Through  $\xi_0$  and  $\xi_i$  construct the circle  $\Gamma_i$ , which is orthogonal to the boundary. This circle is unique. Let  $\zeta_i$  be the second intersection point of the circle  $\Gamma_i$  and the boundary, which is given by  $\zeta = \phi_{\xi_0}(-\phi_{\xi_0}(\xi_i))$ . Because circles orthogonal to the boundary are geodesics of the invariant metric, the construction is invariant.

Let then  $\xi_j$  be a point at the boundary such that  $\xi_j \neq \xi_i$  and  $\xi_j \neq \zeta_i$ , the function



$$(44) \quad \mathcal{Q}_{0ij} = [\xi_i, \xi_j; \xi_0, \zeta_i] = \frac{\|\xi_0 - \xi_j\| \|\xi_i - \zeta_i\|}{\|\xi_0 - \zeta_i\| \|\xi_i - \xi_j\|}$$

is invariant under the action of Möbius transformations on the closed unit ball. Using the invariance we find

$$(45) \quad \mathcal{Q}_{0ij} = \frac{2}{\|\phi_{\xi_0}(\xi_i) - \phi_{\xi_0}(\xi_j)\|} = \frac{1}{1 - \langle \phi_{\xi_0}(\xi_i), \phi_{\xi_0}(\xi_j) \rangle}.$$

Remark that  $\mathcal{Q}_{0ij}(\xi_0 = 0) = \frac{2}{\|\xi_i - \xi_j\|}$  is an invariant for the action of  $SO(n)$  and hence translating this function over  $B_0$  by means of  $\phi_{\xi_0}$  yields the invariant  $\mathcal{Q}_{0ij}$ .

The Möbius transformation (3) for  $\xi$  in the boundary  $S^{n-1}$  is

$$(46) \quad \phi_{\xi_0}(\xi) = \frac{2(1 - \frac{\langle \xi, \xi_0 \rangle}{\langle \xi_0, \xi_0 \rangle})\xi_0 - (\|\xi_0\|^2 - 1)\xi}{\|\xi_0 - \xi\|}.$$

Let  $u = \xi_0/\|\xi_0\|$  and set  $\|\xi\| = t$  we then find

$$(47) \quad \phi_{tu}(\xi) = \frac{2(t - \langle \xi, u \rangle)u - (t^2 - 1)\xi}{\|tu - \xi\|^2}$$

and

$$(48) \quad D_t \phi_{tu}(\xi)|_{t=0} = 2[(1 - 2\langle \xi, u \rangle)^2 u + \langle \xi, u \rangle \xi].$$

From the above expressions we find the total derivatives of the invariant functions at the points  $\xi_0 = 0$ .

**Proposition 4.15.**

$$(49) \quad T_0 \mathcal{Q}_{0ij}(\xi_0 = 0) = -4 \frac{\langle \xi_i + \xi_j, \dot{\xi}_0 \rangle}{\|\xi_i - \xi_j\|^2}$$

$$(50) \quad T_j \mathcal{Q}_{0ij}(\xi_0 = 0) = 2 \frac{\langle \xi_j - \xi_i, \dot{\xi}_j \rangle}{\|\xi_i - \xi_j\|^3}$$

The singular subsets in  $\mathcal{E}_n$  are given by  $\Sigma_0 : \xi_i = \xi_j, i < j, i, j = 1, \dots, n - 1$  and  $\Sigma_1 : -\phi_{\xi_0}(\xi_i) = \phi_{\xi_0}(\xi_j), i < j, i, j = 1, \dots, n - 1$ . The subset  $\mathcal{W}_0 = \mathcal{E}_n \setminus (\Sigma_0 \cup \Sigma_1)$  is a regular subset. It is sufficient to verify that the set of fundamental vector fields has maximal rank at points in  $\mathcal{W}_0$  with  $\xi_0 = 0$ .

For  $u, v$  orthonormal vectors in  $\mathbb{R}^n$  we have

$$(51) \quad X_u|_{\xi_0=0} = -\frac{1}{2}u\partial_{\xi_0} + \sum_{i=1}^{n-1} [-\langle \xi_i, u \rangle \xi_i + \frac{1}{2}(\langle \xi_i, \xi_i \rangle + 1)u]\partial_{\xi_i} \quad (1)$$

$$X_{(u,v)}|_{\xi_0=0} = \sum_{i=1}^{n-1} \langle \xi_i, u \rangle v\partial_{\xi_i} - \langle \xi_i, v \rangle u\partial_{\xi_i} \quad (2)$$

Set (1) has rank  $n$  and is independent of the set (2), which are the rotations on the  $S^{n-1}$  spheres. Outside the singular subsets the set (2) is linear independent and of rank  $n(n-1)/2$ .

**Theorem 4.16.** *The set of functions  $\mathcal{Q}_{0ij}$ , with  $i < j$  is a generating set of invariants on  $\mathcal{W}_0$ .*

*Proof.* The dimension of  $\mathcal{E}_n$  equals  $n^2 - n + 1$  while the dimension of the group  $GM(n-1)$  equals  $\frac{1}{2}n(n+1)$ . On the other hand the set  $\{\mathcal{Q}_{0ij}\}$  contains  $\frac{1}{2}(n-1)(n-2)$  invariant functions on  $\mathcal{W}_0$ . To prove that this set has maximal rank it is sufficient to look at the particular points with  $\xi_0 = 0$  because the group  $GM(n-1)$  acts transitively on  $B^n$ . The set of differentials with respect to the  $j^{th}$  factor, taken at  $\xi_0 = 0$ , which is given by the set (50) with  $i < j$ , has maximal rank for the points in  $\mathcal{W}_0$  with  $\xi_0 = 0$ . Hence the set has maximal rank on the whole of  $\mathcal{W}_0$ .  $\square$

4.2.2. *At first order.*

(1) **First order at interior points of  $B^n$  Let**

$$\mathcal{J}^{(1,0,\dots,0)}\mathcal{E}_n = J^1B_0 \times \underbrace{S^{n-1} \times \dots \times S^{n-1}}_{n-1 \text{ times}}$$

The invariant sheaf  $\mathcal{A}_0$  is in a natural way an invariant sheaf on  $\pi_0^{-1}(\mathcal{W}_0)$ , where

$$\pi_o : J^1B_0 \times \prod_{l=1}^{n-1} S_l^{n-1} \rightarrow B_0 \times \prod_{l=1}^{n-1} S_l^{n-1}$$

is the projection. We will use the same notation for both. The rank of  $\mathcal{A}_0$  equals  $\frac{1}{2}(n-1)(n-2)$ . The sheaf  $\mathcal{A}_0$  contains a subsheaf of invariant functions which are independent of  $\xi_0$  and which has dimension  $\frac{1}{2}(n-1)(n-4)$  [5]. More generally we have following lemma.

**Lemma 4.17.** *Consider the space  $\mathcal{J}^{(1,0,\dots,0)}\mathcal{E}_{n-1} = J^1B^n \times \underbrace{S^{n-1} \times \dots \times S^{n-1}}_{k \text{ times}}$ . Let  $n \geq 4$  and  $k \geq 2$ , then outside the singular subset the number of functional independent, in the variables  $\xi_0$ , invariant functions contained in the set  $\{\mathcal{Q}_{0ij}; i < j = 1, \dots, k\}$ , equals  $k - 1$ .*

*Proof.* Let  $2 \leq k$ . The differentials in the first layer of the set  $\mathcal{Q}_{0ij}$ ,  $i, j = 1, \dots, k$  at  $\xi_0 = 0$  are given by (49) and form a linear independent set of rank  $k - 1$  outside the singular subsets of  $\mathcal{E}_n$ . □

On the space  $\mathcal{J}^{(1,0,\dots,0)}\mathcal{E}_n$  we find  $n - 1$  functional independent, in  $\xi_0$ , invariants. Hence in order to have a generating set of invariants we need one invariant function more. The invariant metric (19) has maximal rank on  $B_0$  and hence is a good candidate. We have following theorem.

**Theorem 4.18.** *The invariant sheaf  $\mathcal{A}_1$  on  $\pi^{-1}*\mathcal{W}_0 \setminus \{\xi_0 = 0\}$  is generated by the first prolongation  $\mathcal{A}_0^{(1)}$  together with  $\tilde{g}_o$*

Applying theorem (2.7) we have :

**Theorem 4.19.** *Let  $\gamma(t)$  be a  $B_0$ -regular curve in  $\mathcal{E}_n$ . Then  $\gamma(t)$  is a motion in  $B^n$  parametrized by  $n - 1$  points at the boundary  $S^{n-1}$  of  $B^n$  iff the curve is lying in an orbit of  $GM(n - 1)$  and  $\tilde{g}_0$  is constant along the curve.*

Using theorem (4.18) one finds a set of invariant generators on  $\mathcal{J}^{(1,0,\dots,0)}\mathcal{E}_{n-1}$ .

**Theorem 4.20.** *Let  $\mathcal{W}_1$  be the regular subset of  $\mathcal{J}^{(1,0,\dots,0)}\mathcal{E}_{n-1}$ . On  $\mathcal{W}_1$  the following set is a set of invariant generators.*

$$(52) \quad \mathcal{I}_1 = \{\mathcal{Q}_{0ij}, T_0\mathcal{Q}_{0ij}, \tilde{g}_o : i, j = 1, \dots, n - 2\}.$$

The proof follows from the diagonal embedding

$$(53) \quad j : \mathcal{J}^{(1,0,\dots,0)}\mathcal{E}_{n-1} \rightarrow \mathcal{J}^{1,0,\dots,0}\mathcal{E}_n$$

where the image is in the diagonal  $\xi_n = \xi_{n-1}$ .

let  $\mathcal{O} \subset \mathcal{W}_0$  be an orbit and  $\pi_o$  the projection  $\mathcal{E}_n \rightarrow B_0$ . Then  $\pi_o$  is surjective on  $\mathcal{O}$ . From

$$(54) \quad d\mathcal{Q}_{0ij}(\xi_0 = 0) = \frac{-1}{\|\xi_i - \xi_j\|} \left( (\xi_i + \xi_j)d\xi_0 + \frac{\langle \xi_i - \xi_j, d\xi_i - d\xi_j \rangle}{\|\xi_i - \xi_j\|^2} \right),$$

we find as a consequence of the implicit function theorem that  $(\xi_0)$  are coordinate functions on the orbits and  $\pi_o^{-1}(p) \cap \mathcal{O}$ , with  $p \in B^n$ , is isomorphic to  $SO(n)$ .

**(2) First order at boundary points** Let  $\xi_1$  and  $\xi_2$  be two points in the boundary. Then  $\mathcal{Q}_{012}$  is invariant and the inverse function  $\Psi = 1/(\mathcal{Q}_{012})^2$  is zero at the diagonal  $\Delta : \xi_1 = \xi_2$  of  $\mathcal{E}_3 = B_0 \times S^{n-1} \times S^{n-1}$  with zero first order total derivatives with respect to  $\xi_1$  and  $\xi_2$ . We set  $\xi = \xi_1 = \xi_2$ .

**Theorem 4.21.**

$$(55) \quad T_{\xi_1} T_{\xi_2} \mathcal{Q}_{012}|_{\Delta} = \frac{(\|\xi_0\|^2 - 1)^2}{\|\xi_0 - \xi\|^4} \langle \dot{\xi}, \dot{\xi} \rangle$$

*Proof.* From  $\langle \phi_{\xi_0}(\xi_i), \phi_{\xi_0}(\xi_i) \rangle = 1$  we find

$$(56) \quad \langle T_{\xi_i} \phi_{\xi_0}(\xi_i), \phi_{\xi_0}(\xi_i) \rangle = 0$$

With the use of (46) we then find

$$(57) \quad T_{\xi} \phi_{\xi_0}(\xi) = -\frac{2 \langle \xi_0, \dot{\xi} \rangle \xi_0 - (\|\xi_0\|^2 - 1) \dot{\xi}}{\|\xi_0 - \xi\|^2} + 2 \frac{\langle \xi_0, \dot{\xi} \rangle \phi_{\xi_0}(\xi)}{\|\xi_0 - \xi\|^2}.$$

Then  $\langle T_{\xi} \phi_{\xi_0}(\xi), T_{\xi} \phi_{\xi_0}(\xi) \rangle$  yields relation (55).

□

Let

$$(58) \quad f(\xi_0, \xi) = \frac{(1 - \langle \xi_0, \xi_0 \rangle)}{\|\xi_0 - \xi\|^2}$$

then  $g_S(\xi_0, \xi) = f^2(\xi_0, \xi) d\sigma^2$ , with  $d\sigma^2$  the standard spherical metric on  $S^{n-1}$ , defines an invariant Riemannian metric on the boundary  $S^{n-1}$  depending parametrically on  $\xi_0$ . The conformal function  $f(\xi_0, \xi)$  is nowhere zero on  $B_0 \times S^{n-1}$ . Define

$$(59) \quad \eta(\xi_0, \xi) = f^{n-1}(\xi_0, \xi) \nu$$

with  $\nu$  the spherical volume form normalized by ( $\int_{S^{n-1}} \nu = 1$ ). Remark that  $\int_{S^{n-1}} \eta$  is constant, because the integral is an invariant function on  $B_0$ . Moreover the form  $\eta(\xi_0, \xi)$  has to be a constant multiple of the associated volume form of  $g_S$  because invariant functions on  $B_0 \times S^{n-1}$  are constant. The choice of the normalization constant is of no importance in future calculations.

It is now sufficient to verify that  $f(\xi_0 = 0) = 1$  which implies that  $\int_{S^{n-1}} \eta = 1$ . This was the reason for choosing the multiplicative constant of  $f$  as above. The value of the differential with respect to  $B_0$  of  $\log f$  at  $\xi_0 = 0$  equals

**Proposition 4.22.**

$$(60) \quad \frac{d_o f}{f}(\xi_0 = 0) = 2 \langle \xi, \dot{\xi}_0 \rangle.$$

let  $p = (\xi_0, \xi_1, \dots, \xi_{n-1}) \in \mathcal{E}_n$  such that  $\xi_0 = 0$  and  $(\xi_i), i = 1, \dots, n-1$ , is an orthonormal set. Let  $\mathcal{O}_p$  be the orbit through  $p$  and choose  $\xi_n$  a unit vector orthogonal to the set  $(\xi_i), i = 1, \dots, n-1$ , such that the set is positively oriented.

With each  $\xi_a, a = 1, \dots, n$ , corresponds an invariant one form

$$(61) \quad \mu_a = d_0 \log f_a = -2 \left[ \frac{\xi_0}{1 - \|\xi_0\|^2} + \frac{\xi_0 - \xi_a}{\|\xi_0 - \xi_a\|^2} \right] d\xi_0$$

which is well defined on  $\mathcal{E}_n$  and is a basic one form for the projection  $\pi_o : \mathcal{O}_p \rightarrow B_0$ . The functions  $g_o^{-1}(\mu_a, \mu_b)$  are invariant and hence are constant on the orbit  $\mathcal{O}_p$ . The vector fields

$$(62) \quad Z_a = g_o^{-1} \mu_a = -\frac{(1 - \|\xi_0\|^2)^2}{2} \left[ \frac{\xi_0}{1 - \|\xi_0\|^2} + \frac{\xi_0 - \xi_a}{\|\xi_0 - \xi_a\|^2} \right] \partial_{\xi_0}$$

are invariant vector fields and are tangent to the first layer  $B^n$  of  $\mathcal{E}_n$ .

We restrict the vectorfields  $Z_a$  to the orbit  $\mathcal{O}_p$ . Because they are invariant and all invariant functions are constant on  $\mathcal{O}_p$  it is sufficient to calculate their brackets at  $p$ . We use following expressions.

$$(63) \quad \begin{aligned} (a) \quad & Z_a|_p = \frac{1}{2} \xi_a \partial_{\xi_0} \\ (b) \quad & Z_a(Z_b)|_p = -\frac{1}{2} \xi_a \partial_{\xi_0} \\ (c) \quad & Z_1(g_o(Z_a, Z_b))|_p = -\frac{1}{2} (g_o(\xi_1, \frac{1}{2} \xi_b) + g_o(\frac{1}{2} \xi_a, \xi_1)) \\ & = -\langle \xi_1, \xi_a + \xi_b \rangle \end{aligned}$$

We then find on  $\mathcal{O}_p$

$$(64) \quad [Z_a, Z_b] = Z_b - Z_a.$$

**Lemma 4.23.** *The Lie algebra spanned by the vector fields  $Z_a$  is the nilpotent Lie subalgebra complementing the subalgebra  $K \equiv \mathfrak{o}(n)$ , isotropy algebra at  $\xi_0 = 0$ .*

*Proof.* It is sufficient to verify  $[Z_1, [Z_1, Z_a]] = [Z_1, Z_a]$  and  $[[Z_1, Z_b], [Z_1, Z_b]] = 0$ . □

It is important to notice that the invariant subbundle of tangent vectors to  $B_0$  over  $\mathcal{O}_p$  not a subbundle is of  $T\mathcal{O}_p$ . Hence the vector fields  $Z_a$

are not tangent to the orbits. Define their orthonormalization with respect to  $g_o$ . Let  $\bar{Z}_1 = Z_1/\|Z_1\|$  and

$$(65) \quad \bar{Z}_a = \frac{Z_a - g_o(\bar{Z}_{a-1}, Z_a)\bar{Z}_{a-1} - \cdots - g_o(\bar{Z}_1, Z_a)\bar{Z}_1}{\|Z_a - g_o(\bar{Z}_{a-1}, Z_a)\bar{Z}_{a-1} - \cdots - g_o(\bar{Z}_1, Z_a)\bar{Z}_1\|}$$

for  $a > 1$ . The norm is take with respect to  $g_o$ .

**Proposition 4.24.** *For any  $a \neq 1$  we have :*

$$(66) \quad [\bar{Z}_1, \bar{Z}_a] = \bar{Z}_a.$$

*Proof.* It is now sufficient to verify the bracket at  $p$ . Using (63) and  $Z_1(N_a)(p) = 0$ , with  $N_a$  the denominator in (65) we find

$$\begin{aligned} [\bar{Z}_1, \bar{Z}_a]|_p &= [Z_1, Z_a]|_p - Z_1(g_o(Z_a, Z_1))Z_1|_p \\ &= Z_a|_p, \end{aligned}$$

which proves the proposition. □

We clearly have that  $\mu_1$  is a closed one form which also follows from the brackets of the vector fields (66).

For arbitrary  $a$  and  $b = 1$  we have

$$\begin{aligned} d_{B^n}\mu_1(\bar{Z}_a, \bar{Z}_1) &= \bar{Z}_a(\mu_1(\bar{Z}_1)) - \bar{Z}_1(\mu_1(\bar{Z}_a)) - \mu_1([\bar{Z}_a, \bar{Z}_1]) \\ &= \bar{Z}_a(g_o(\bar{Z}_1, \bar{Z}_1)) - \bar{Z}_1(g_o(\bar{Z}_1, \bar{Z}_a)) - g_o(\bar{Z}_1, [\bar{Z}_a, \bar{Z}_1]) = 0. \end{aligned}$$

**Proposition 4.25.** *Let  $\lambda = \mu_2 \wedge \cdots \wedge \mu_n$  and  $\omega_o = \mu_1 \wedge \lambda$  the associated volume form of  $g_o$ . Then*

$$(67) \quad d\lambda = -(n-1)\omega_o$$

and

$$(68) \quad d\eta = (n-1)\mu_1 \wedge \eta$$

*Proof.* The differential of  $\lambda$  is given by

$$\begin{aligned} d\lambda(\bar{Z}_1, \dots, \bar{Z}_n) &= \frac{1}{n} \sum_{i=1, \dots, n} (-1)^{i-1} \bar{Z}_i(\lambda(\bar{Z}_1, \dots, \check{\bar{Z}}_i, \dots, \bar{Z}_n)) \\ &\quad - \frac{1}{n} \sum_{1 \leq i < j \leq n} (-1)^{i+j} \lambda([\bar{Z}_i, \bar{Z}_j], \bar{Z}_1, \dots, \check{\bar{Z}}_i, \dots, \check{\bar{Z}}_j, \dots, \bar{Z}_n) \end{aligned}$$

which gives with the use of the definition of  $\lambda$  and the brackets of the vector fields

$$(69) \quad d\lambda(\bar{Z}_1, \dots, \bar{Z}_n) = -\frac{n-1}{n} \frac{1}{(n-1)!}.$$

But then because  $\omega_o(\bar{Z}_1, \dots, \bar{Z}_n) = \frac{1}{n!}$ , we find (1).

Part (2) follows from

$$(70) \quad d\eta = (n-1)f^{n-1}df \wedge \nu = (n-1)\frac{d_{B^n}f}{f} \wedge \eta$$

together with (61). In the proof we have been using the brackets determined over the orbit  $\mathcal{O}_p$ , but the result clearly is independent of the chosen vector fields and hence is valid in general. □

**Proposition 4.26.** *The function  $f^{n-1}$  is harmonic with respect to the Laplace-Beltrami operator determined by the invariant metric  $g_o$ .*

*Proof.* From

$$(71) \quad d_o(f^k \lambda) = k f^{k-1} d_{B^n} f \wedge \lambda - \frac{n-1}{2} f^k \omega_o$$

follows

$$(72) \quad d_o(f^k \lambda) = \frac{1}{2} k f^k \mu_1 \wedge \lambda - \frac{n-1}{2} f^k \omega_o.$$

Then by taking  $k = n - 1$  we find

$$(73) \quad d_o(f^{n-1} \lambda) = 0.$$

Now  $\bar{Z}_1 = g_o^{-1} \mu_1$  and  $d_{B^n} f = \frac{1}{2} f \mu_1$ , we find  $grad_o(f^{n-1}) = \frac{1}{2} f^{n-1} Z_1$ .  $grad_o$  stands for the gradient on  $B_o$  with respect to  $g_o$ . Hence

$$(74) \quad L_{grad f^{n-1}} \omega_o = d(\iota_{(grad_o f^{n-1})} \omega_o) = \frac{1}{2} d(f^{n-1} \lambda) = 0.$$

This proves the proposition. □

**Proposition 4.27.** *The vector field  $\bar{Z}_1$  is geodesic for each value of  $\xi_1, \dots, \xi_{n-1}$ .*

*Proof.* The statement follows from

$$\begin{aligned} g_o(\nabla_{\bar{Z}_1} \bar{Z}_1, \bar{Z}_a) &= -g_o(\bar{Z}_1, \nabla_{\bar{Z}_1} \bar{Z}_a) \\ &= -2g_o(\bar{Z}_1, \bar{Z}_a) = 0 \end{aligned}$$

for all  $a \neq 1$ .  $g_o(\nabla_{\bar{Z}_1} \bar{Z}_1, \bar{Z}_1) = 0$  because  $\bar{Z}_1$  is of constant length for  $g_o$ .

**(3) Volume forms and metrics** Consider the space  $B_0 \times S^{n-1}$ . One has the invariant metric  $g_o$  and volume form  $\omega_o$  on the first factor  $B_0$ . On the second factor the invariant metric  $g_S$ , which depends parametrically on  $B_0$ . Indicate this metric as  $g_1$  and call the associated invariant volume form  $\eta_1$ . We now have  $g_o + g_1$  as invariant metric on  $B_0 \times S^{n-1}$  and  $\omega_o \wedge \eta_1$  as invariant volume form. This procedure extends to all factors in the  $n - 1$  product  $S^{n-1} \times \dots \times S^{n-1}$ . The metric then becomes

$$(75) \quad g = g_o + \sum_{i=1}^{n-1} g_i$$

with  $g_i = f_i^2 d\sigma_i$ ,  $f_i = (1 - \|\xi_0\|^2) / (\|\xi_0 - \xi_i\| |1 - \langle \xi_i, \xi_0 \rangle|)$  and  $d\sigma_i^2$  the spherical metric on  $S_i^{n-1}$ .

4.2.3. *Higher order invariants.* Let  $n \geq 4$  and  $1 \leq k \leq n - 3$ . Consider the space

$$(76) \quad J^k B^n \times \prod_{i=1}^{n-(k+1)} S_i^{n-1}.$$

The invariant sheaf is generated by the set

$$(77) \quad \mathcal{I}_k = \left\{ \begin{array}{cccc} \mathcal{Q}_{0ij} & & & \\ T_0 \mathcal{Q}_{0ij} & \tilde{g}_o & & \\ T_0^2 \mathcal{Q}_{0ij} & T_0 \tilde{g}_o & \tilde{K}_2|_0 & \\ \vdots & \vdots & \vdots & \\ T_0^k \mathcal{Q}_{0ij} & T_0^{k-1} \tilde{g}_o & T_0^{k-2} \tilde{K}_2|_0 & \vdots \tilde{K}_k|_0 \end{array} \right\}$$

The space  $J^k B^n \times \prod_{i=1}^{n-(k+1)} S_i^{n-1}$  has dimension  $n^2 - n + (k + 1)$ . The rank of the set  $\{\mathcal{Q}_{0ij}\}$  equals  $\frac{(n-(k+1))(n-(k+2))}{2}$ . The rank of the set  $\{T_0 \mathcal{Q}_{0ij}, \dots, T_0^k \mathcal{Q}_{0ij}\}$  equals  $k(n - (k + 1))$  while the remaining set  $\{\tilde{g}_o, \dots, \tilde{K}_k^o\}$  has rank equal to  $\frac{k(k+1)}{2}$ . Then  $\dim J^k B^n \times \prod_{i=1}^{n-(k+1)} S_i^{n-1} - \text{rank } \mathcal{I}_k = \frac{n(n+1)}{2}$ , which equals the dimension of the group. But the regular subset is the subset such that the orbits have maximal dimension, which proves the statement.

### 5. THE SPECIAL DIMENSIONS

In determination the first order invariants we have been using condition  $n \geq 4$  (4.2.2). For this reason we need to review dimensions three and two.



5.1. **Dimension three.** Consider  $\mathcal{E}_3 = B_0 \times S^2 \times S^2$ , with  $B_0 \equiv B^3$ . The group of Möbius transformations has dimension 6. On the regular subset  $\mathcal{W}_0$  there exists one invariant generator, namely  $\mathcal{Q}_{012}$ . We construct as before the invariant metric

$$(78) \quad g = g_o + g_1 + g_2$$

where  $g_o$  is the invariant metric on  $B^3$  and  $g_i, i = 1, 2$  the invariant metric on  $i^{th}$  boundary. With respect to this metric we construct the invariant vector field  $Y = g^{-1}d\mathcal{Q}_{012}$ . This field is transversal to the orbits which allows the construction of an invariant metric by taking (1) the Killing form in the tangent space to the orbits and (2) defining the vector field  $Y$  as an orthonormal vector field to the orbits. Denote this metric by  $K_1$ . Let  $K_1^o$  be the restriction of  $K_1$  to the first layer, then the lift of  $K_1^o$  is an invariant function  $\tilde{K}_1^o$  on  $J^1B_0 \times S^2 \times S^2$ .

Let  $\mathcal{W}_0^{(1)} = \pi_o^{-1}\mathcal{W}_0 \cap \{\dot{\xi}_0 \neq 0\}$  as subset of  $J^1B_3 \times S^2 \times S^2$ . We then have

**Theorem 5.1.** *The set*

$$(79) \quad \mathcal{I}_0^{(1)} = \{\mathcal{Q}_{012}, T_0\mathcal{Q}_{0ij}, \tilde{g}_o, \tilde{K}_1^o\}$$

*is a generating set of invariants on  $\mathcal{W}_0^{(1)}$*

The diagonal embedding

$$(80) \quad \Delta : J^1B_3 \times S^2 \rightarrow J^1B_3 \times S^2 \times S^2$$

pulls back the generating functions defining a generating set  $\mathcal{I}_1$ . One verifies that

$$(81) \quad \mathcal{I}_1 = \{\tilde{g}_o, \tilde{K}_1^o\}$$

is the resulting generating set of invariants.

The construction for higher order generating sets is exactly the same as in the general case. On  $J^2B_3 \times S^2$  one finds

$$(82) \quad \mathcal{I}_2 = \{\tilde{g}_o, T_0\tilde{g}_o, \tilde{K}_1^o, T_0\tilde{K}_1^o, \tilde{K}_2^o\}$$

We omit the details.

5.2. **Dimension two.**

5.2.1. *At zero order.* This is the classical case of the unit disc  $B = B_2$  in  $\mathbb{R}^2$ . Let  $(x, y)$  be the natural coordinates on  $\mathbb{R}^2$ . The action of the connected component of  $GM(1)$  is given in complex form by [2] [7]  $\Phi : \tilde{z} = (az + b)/(\bar{b}z + \bar{a})$  with  $|a|^2 - |b|^2 = 1$ , where  $z, a, b$  are complex variables. The group  $GM(1)$  is isomorphic with  $PGL(2)$  and has Lie algebra  $\mathfrak{sl}(2)$ . With  $a = 2(\alpha + i\beta)$ ,  $b = 2(\gamma + i\sigma)$  and  $(\sigma, \beta, \gamma)$  as local coordinates on the group, a set of fundamental vector fields of the Möbius action on the unit disc is given by

$$\begin{aligned}
 (83) \quad H &= \Phi_* \partial_\sigma (g = e) = -xy\partial_x + \frac{1}{2}(1 + x^2 - y^2)\partial_y \\
 K &= \Phi_* \partial_\beta (g = e) = (y\partial_x - x\partial_y) \\
 L &= \Phi_* \partial_\gamma (g = e) = \frac{1}{2}(1 - x^2 + y^2)\partial_x - xy\partial_y.
 \end{aligned}$$

Their commutators are

$$(84) \quad [K, H] = -L, [K, L] = H, [H, L] = K.$$

At first order the regular subset  $\mathcal{W}_1$  in  $J^1B_2$  is given by  $\|(\dot{x}, \dot{y})\| \neq 0$ .

**Proposition 5.2.** *The sheaf of invariant functions on  $\mathcal{W}_1$  is generated by  $\tilde{g}_o$ .*

Applying corollary (2.7) we find

**Proposition 5.3.** *A motion depending parametrically on an extra point in  $B$  is a curve in  $\mathcal{E}_2 = B_0 \times B_1$ , which is regular with respect to the first factor, such that  $\mathcal{Q}_{01}$  and  $g_o$  are constant along the curve.*

To exhibit the geometric structure of  $J^1B_2$  we recall the complete lift of the hyperbolic metric :

$$(85) \quad g_o^c = 8 \frac{x\dot{x} + y\dot{y}}{(1 - x^2 - y^2)^3} (dx^2 + dy^2) + 4 \frac{dx d\dot{x} + dy d\dot{y}}{(1 - x^2 - y^2)^2}.$$

Then with  $\Gamma = g_o^{c-1} d\tilde{g}_o$ , the spray of the connection, we find

$$(86) \quad \Gamma = \dot{x}\partial_x + \dot{y}\partial_y + 2 \frac{2y\dot{x}\dot{y} + x(\dot{x}^2 - \dot{y}^2)}{-1 + x^2 + y^2} \partial_{\dot{x}} + 2 \frac{2x\dot{x}\dot{y} + y(\dot{y}^2 - \dot{x}^2)}{-1 + x^2 + y^2} \partial_{\dot{y}}.$$

The level surfaces of  $\tilde{g}_o$  in  $\mathcal{W}_1 \subset J^1B_2$  are codimension one surfaces, which are  $PGL(2)$ -orbits. The vector field  $I = \dot{x}\partial_{\dot{x}} + \dot{y}\partial_{\dot{y}}$  commutes with the vector fields  $H, K, L$  on  $\mathcal{W}_1$  because they are linear homogeneous in the fibre coordinates  $(\dot{x}, \dot{y})$ . Consider the set  $Z = (H^{(1)}, K^{(1)}, L^{(1)}, I)$  and let

$A$  be the matrix of this set. Then  $\Delta_1 = \det A = \frac{1}{4}(1 - x^2 - y^2)^2(\dot{x}^2 + \dot{y}^2)^2$ , which is different from zero on the regular subset  $\mathcal{W}_1 = J^1 B_2 \setminus \{\|(\dot{x}, \dot{y})\| = 0\}$ . Hence  $Z$  is a linear independent set at each point of  $\mathcal{W}_1$ .

The canonical one form  $\omega$  with values in  $\mathfrak{sl}(2) \oplus \mathbb{R}$  is determined by

$$\begin{aligned}
 (87) \quad \omega_1 &= \frac{2}{1 - x^2 - y^2} dy + \frac{2x\dot{y}d\dot{x} - 2x\dot{x}d\dot{y}}{(1 - x^2 - y^2)(\dot{x}^2 + \dot{y}^2)} \\
 \omega_2 &= \frac{-2ydx + 2xdy}{1 - x^2 - y^2} + \frac{(1 + x^2 + y^2)(\dot{y}d\dot{x} - \dot{x}d\dot{y})}{(1 - x^2 - y^2)(\dot{x}^2 + \dot{y}^2)} \\
 \omega_3 &= \frac{2dx}{1 - x^2 - y^2} - \frac{2y\dot{y}d\dot{x} - 2x\dot{x}d\dot{y}}{(1 - x^2 - y^2)(\dot{x}^2 + \dot{y}^2)} \\
 \omega_4 &= \frac{2xdx + 2ydy}{1 - x^2 - y^2} + \frac{\dot{x}d\dot{x} + \dot{y}d\dot{y}}{\dot{x}^2 + \dot{y}^2}
 \end{aligned}$$

Remark that  $\omega_4 = \frac{1}{2}d \log \tilde{g}_o$ . Using the Killing metric on the orbits we determine the invariant metric

$$(88) \quad K_2 = (w^1)^2 - (w^2)^2 + (w^3)^2 + (w^4)^2$$

According to the general case, the level surfaces of  $\tilde{g}_o$  are null surfaces with signature  $(0, +, -)$  for the metric  $g_o^c$ .

**Proposition 5.4.**

$$(89) \quad (1) K_2(\Gamma) = \theta, (2) K_2(I) = \omega^4, (3) K_2(I, \Gamma) = 0$$

Recall that the canonical form associated with  $g_o$  in coordinates equals

$$(90) \quad \theta = \frac{4(\dot{x}dx + \dot{y}dy)}{(1 - x^2 - y^2)^2}.$$

Proof of the above proposition is then obtained by direct verification. Next proposition, which then is a consequence of the former, asserts that  $\Gamma$  is a constant vector field along the orbits of  $PGL(2)$  with positive norm.

**Proposition 5.5.**

$$(91) \quad K_2(\Gamma, \Gamma) = 4 \frac{\dot{x}^2 + \dot{y}^2}{(1 - x^2 - y^2)^2}$$

Remark that the operator  $P = Id - I \otimes \omega^4$  is an invariant operator projecting upon the tangent space to the orbits.

**Proposition 5.6.** *The complete lift of a regular curve  $\gamma(t)$  which is lying in a level surface of  $\tilde{g}_o$  has constant angle with the vector field  $\Gamma$ , with respect to the metric  $K_2$ .*

*Proof.* It suffices to calculate  $K_2(\dot{\gamma}^c(t), \Gamma)$ . This equals

$$\theta(\dot{\gamma}^c(t)) = 4 \frac{\dot{x}^2 + \dot{y}^2}{(1 - x^2 - y^2)^2}(\gamma(t))$$

which is a constant if  $\gamma^c(t)$  is lying in a level surface of  $\mathcal{Q}_1$ . □

Let  $\mathcal{W}_2 = \pi^{-1}\mathcal{W}_1$ . The tensor field  $g_o$  itself is also a tensor field on  $J^1B_2$  ; we will use the same symbol for it. Any symmetric covariant tensor field  $T$  on a manifold is a function, denoted by  $\widehat{T}$ , on the tangent space. Let  $j : J^2B_2 \rightarrow TJ^1B_2$  be the canonical embedding, then  $\widetilde{g}_o = j^*g_o$ ,  $\widetilde{g}_o^c = j^*\widehat{g}_o^c$  and  $\widetilde{K}_2 = j^*\widehat{K}_2$  are invariant functions on  $J^2B_2$ .

**Theorem 5.7.** *The invariant sheaf on the regular subset  $\mathcal{W}_2$  is generated by the germs of the set*

$$(92) \quad \mathcal{I}_{\mathcal{W}_1} = \{\widetilde{g}, \widetilde{g}^c, \widetilde{K}_2\}.$$

**Theorem 5.8.** *Let  $\gamma(t)$  be a regular curve in  $B_0$ . Then  $\gamma(t)$  is the trajectory of a one parameter subgroup of  $PGL(2)$  iff  $\widetilde{g}_o$  and  $\widetilde{K}_2$  are constant along  $\gamma(t)$ .*

*Proof.* From  $g_o(\dot{\gamma}, \dot{\gamma}) = a$ ,  $a \in \mathbb{R}$  we find  $g_o^c(\dot{\gamma}^c, \dot{\gamma}^c) = 0$ , which implies that the tangent vector to the curve is a null vector in  $g_o^c$ . But constant  $g_o(\dot{\gamma}, \dot{\gamma})$  also implies, using proposition (5.6) that  $\dot{\gamma}^c(t)$  has constant angle (with respect to  $K_2$ ) with  $\Gamma$  and is lying in a null two plane formed by a null vector in  $g_o^c$  transversal to  $\Gamma$  and  $\Gamma$  itself. But requirement  $K_2(\dot{\gamma}^c(t), \dot{\gamma}^c(t)) = c$ ,  $c \in \mathbb{R}$  implies that  $\dot{\gamma}^c(t)$  is constant with respect to a complete set of right invariant vector fields in the orbit. This implies the theorem. □

5.2.2. *Invariant functions on  $J^1B_2 \times S^1$ .* We now consider sets of two points  $(\xi_0, \xi_1)$  with  $\xi_0 \in B_2$  and  $\xi_1 \in S^1$ , the boundary. Call this the space  $\mathcal{E}_2 = B_2 \times S^1$ . For this calculations we prefer the polar coordinates on  $B_2$  and hence use  $(\rho, \theta, \varphi)$  as coordinates on  $\mathcal{E}_2$ . The fundamental vector fields on  $\mathcal{E}_1$  then are

$$(93) \quad \begin{aligned} H &= \frac{1}{2}(1 - r^2) \sin \theta \partial_r + \frac{1 + r^2}{2r} \cos \theta \partial_\theta + \cos \varphi \partial_\varphi \\ K &= -\partial_\theta - \partial_\varphi \\ L &= \frac{1}{2}(1 - r^2) \cos \theta \partial_r - \frac{1 + r^2}{2r} \sin \theta \partial_\theta - \sin \varphi \partial_\varphi \end{aligned}$$

Let  $\Delta = 1 + r^2 - 2r \cos(\theta - \varphi)$ , the basic one forms are

$$\begin{aligned}
 \omega_1 &= 2 \frac{1}{\Delta} \left[ -\frac{(1+r^2) \sin \theta - r \sin \varphi}{(-1+r^2)} dr + r \cos \theta d\theta - r \cos \theta d\varphi \right] \\
 (94) \quad \omega_2 &= \frac{1}{\Delta} \left[ -2 \frac{(1+r^2) \sin(\theta - \varphi)}{(-1+r^2)} dr + 2r \cos(\theta - \varphi) d\theta - (1+r^2) d\varphi \right] \\
 \omega_3 &= 2 \frac{1}{\Delta} \left[ -\frac{(1+r^2) \cos \theta - r \cos \varphi}{(-1+r^2)} dr - r \sin \theta d\theta + r \sin \theta d\varphi \right]
 \end{aligned}$$

The Killing metric  $K = \omega_1^2 - \omega_2^2 + \omega_3^2$  restricted to the first factor of  $\mathcal{E}_2$  equals  $K|^\circ = \mu.\mu$ , with

$$(95) \quad \mu = 2 \frac{-2r + (1+r^2) \cos(\theta - \varphi)}{(-1+r^2)(1+r^2 - 2r \cos(\theta - \varphi))} dr + 2 \frac{r \sin(\theta - \varphi)}{1+r^2 - 2r \cos(\theta - \varphi)} d\theta$$

Because  $K|^\circ$  is invariant the one form  $\mu$  is an invariant one form. As a consequence of the invariance of the metric  $g_o$  the norm  $g_o^{-1}(\mu, \mu)$  has to be constant. We find  $g_o^{-1}(\mu, \mu) = 1$ . We are now able to construct a one parameter family of invariant metrics. Let

$$(96) \quad g(k) = g_o + k \mu.\mu$$

The determinant of this metric equals  $(1+k)r^2/(1-r^2)^4$  from which it follows that the metrics  $g(k)$  are Riemannian for  $-1 < k$  and singular for  $k = -1$ . For  $k = -1$  we find  $g(k = 0) = \nu.\nu$  with

$$(97) \quad \nu = -2 \frac{r(-2r + (1+r^2) \cos(\theta - \varphi))}{(-1+r^2)(1+r^2 - 2r \cos(\theta - \varphi))} d\theta + 2 \frac{\sin(\theta - \varphi)}{1+r^2 - 2r \cos(\theta - \varphi)} dr.$$

It follows that the one parameter family of invariant metrics can be written as

$$(98) \quad g(k) = \nu^2 + (k+1)\mu^2.$$

The one forms  $\mu$  and  $\nu$  are defined on  $B_2$  and  $d_{B_2}\mu = 0$  but  $d_{B_2}\nu \neq 0$ . Because  $\mu$  is  $d_{B_2}$ -closed it is  $d_{B_2}$ -exact and hence there exists a function  $\phi$  on  $\mathcal{E}_2$  such that  $\mu = d_{B_2}\phi$ , namely

$$(99) \quad \phi = \log \frac{1+r^2 - 2r \cos(\theta - \varphi)}{1-r^2}.$$

Remark that the function  $\phi$  is not an invariant function on  $\mathcal{E}_2$  because the only invariant functions are constants. We now are able to describe the invariant sheaf on  $J^1B_2 \times S^1$ .

**Theorem 5.9.** *Let  $\mathcal{W}_{(1,0)} = J^1B_2 \times S^1 \setminus \{\dot{x} = 0\}$ . The sheaf of invariant functions  $\mathcal{A}_{\mathcal{W}_{(1,0)}}$  is generated by the set*

$$(100) \quad \mathcal{I}_{\mathcal{W}_{(1,0)}} = \{g_o, K|^\circ\}.$$

The set  $\{\dot{x} = 0\}$  stands for the zero section of the bundle  $J^1B_2 \rightarrow B_2$ . As a consequence of the decomposition of the invariant metric we may reformulate this proposition.

**Theorem 5.10.** *The set  $\mathcal{J}_{\mathcal{W}_{1,0}} = \{\tilde{\mu}, \tilde{\nu}\}$  is a generating set of invariants on  $\mathcal{W}_{(1,0)}$ .*

The one forms are seen as functions on the 1-jets of curves in  $B_2$ . Again we are able to formulate next proposition.

**Properties 5.11.** *A regular curve in  $B_2 \times S$  is a motion of  $Pl(2)$  in  $B_2$  depending parametrically on  $S$  iff  $\tilde{\mu}, \tilde{\nu}$  are constant along the curve.*

We make this more explicit. Let  $(\xi_0^\circ, \xi_1^\circ)$  be a given point in  $\mathcal{E}_2$  and  $\mathcal{I}_{\xi_0^\circ}$  be the isotropy group at  $\xi_0^\circ \in B_2$ . A curve  $\gamma(t) = (\xi_0(t), \xi_1(t))$  is said to be a motion depending parametrically on a point in the boundary if  $\gamma(t)$  is of the form  $\gamma(t) = (g_t \cdot \xi_0^\circ, g_t \cdot f(t) \cdot \xi_1^\circ)$  with  $g_t$  a one parameter subgroup of  $Pl(2)$  and  $f(t)$  a curve in  $\mathcal{I}_{\xi_0^\circ}$ . The curve satisfies the above conditions if  $K|^\circ(\dot{\gamma}(t), \dot{\gamma}(t))$  and  $g_o(\dot{\gamma}(t), \dot{\gamma}(t))$  are constant.

5.2.3. *Invariant functions on  $B_2 \times J^1S^1$ .* Taking the restriction of the Killing form to the tangent space of the second factor in  $\mathcal{E}_2 = B_2 \times S^1$  gives  $K|^\circ = \eta \cdot \eta$  with

$$(101) \quad \eta = \frac{(1 - r^2)}{1 + r^2 - 2r \cos(\theta - \varphi)} d\varphi$$

The one form  $\eta$  defines an invariant measure on the boundary  $S^1$  depending parametrically on a point in  $B_2$ . One easily finds that  $\int_0^{2\pi} \eta = 2\pi$ . The integral has to be a constant because it is an invariant and the only invariants on  $B_2$  are constants. The invariant one form (101) is the Poisson kernel on the unit disk [1]. Accordingly to (4.26) the function  $(1 - r^2)/(1 + r^2 - 2r \cos(\theta - \phi))$  is harmonic in the  $B_2$ -variables and  $d\varphi$  is the rotation invariant Lebesgue measure on  $S^1$  when the integral is taken as a Lebesgue integral.

We now have the theorem

**Theorem 5.12.** *Let  $\mathcal{W}_{(0,1)}$  be the subset  $B_2 \times J^1S^1 \setminus \{\dot{\varphi} = 0\}$ . The sheaf of invariant functions  $\mathcal{A}_{\mathcal{W}_{(0,1)}}$  is generated by the function  $\tilde{\eta}$ .*

and the proposition

**Properties 5.13.** *Let  $\gamma(t)$  being a regular curve in the boundary  $B_2 \times S$ . Then  $\gamma(t)$  is a motion of  $Pl(2)$  in the boundary  $S$  depending parametrically on a point in  $B_2$  if and only if the one form  $\eta$  is constant along the curve.*

5.2.4. *Invariant functions on  $J^1B_2 \times J^1S^1$ .* We next define the invariant vector fields  $X_\mu = g_o^{-1}\mu$ ,  $X_\nu = g_o^{-1}\nu$  and  $X_\eta$ , where  $X_\eta$  is determined by the invariant condition  $\eta(X_\eta) = 1$  and  $\pi_o_*X_\eta = 0$ ,  $\pi_o$  being the projection  $\mathcal{E}_2 \rightarrow B_2$ . We then have

$$X_\mu = \frac{(1-r^2)(-2r+(1+r^2)\cos(\theta-\varphi))}{2(1+r^2-2r\cos(\theta-\varphi))} \partial_r + \frac{(-1+r^2)^2 \sin(\theta-\varphi)}{2r(1+r^2-2r\cos(\theta-\varphi))} \partial_\theta$$

$$X_\nu = \frac{(-1+r^2)^2 \sin(\theta-\varphi)}{2(1+r^2-2r\cos(\theta-\varphi))} \partial_r + \frac{r(-1+r^2) - (-1+r^4)\cos(\theta-\varphi)}{r(1+r^2-2r\cos(\theta-\varphi))} \partial_\theta$$

$$X_\eta = \frac{1+r^2-2r\cos(\theta-\varphi)}{1-r^2} \partial_\varphi.$$

Because the bracket of two invariant vector fields is an invariant vector field, the above vector fields constitute the dual Lie algebra up to automorphism. Their brackets are

$$(102) \quad [X_\mu, X_\nu] = -X_\nu, [X_\mu, X_\eta] = X_\nu + X_\eta, [X_\nu, X_\eta] = -X_\mu.$$

**Theorem 5.14.** *Let  $\mathcal{W}_{(1,1)} = J^1B_2 \times J^1S^1 \setminus (\{\dot{x} = 0\} \cup \{\dot{\varphi} = 0\})$ . The sheaf of invariant functions on  $\mathcal{W}_{(1,1)}$  is generated by the set  $\mathcal{I}_{\mathcal{W}_{(1,1)}} = \{\tilde{\mu}, \tilde{\nu}, \tilde{\eta}\}$ .*

**Proposition 5.15.** *Let  $\gamma(t)$  be a regular curve in  $\mathcal{E}_2$ . Then  $\gamma(t)$  is a motion of  $Pl(2)$  iff the forms  $\mu, \nu$  and  $\eta$  on  $\mathcal{W}_{(1,1)}$  are constant along the curve.*

We also have from the general case

**Proposition 5.16.** *For each value of  $\varphi$  the vector field  $X_\mu$  is geodesic.*

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*(Received June 29, 1999)*