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THE PRIME IDEAL FACTORIZATION OF 2 IN PURE QUARTIC FIELDS WITH INDEX 2

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Abstract. The prime ideal decomposition of 2 in a pure quartic field with field index 2 is determined explicitly.

1. Introduction

Let K be an algebraic number field and O_K its ring of integers. When determining generators of the ideals in the prime ideal factorization of a (rational) prime p in O_K, the most difficult case occurs when p divides the field index $i(K)$ of K. In this paper we examine the case when K is a pure quartic field. Here $i(K) = 1$ or 2, and we determine explicit generators of the prime ideals in the decomposition of 2 when $i(K) = 2$.

Let K be a pure quartic field. Then there exists a fourth power free integer m such that $K = \mathbb{Q}(m^{1/4})$. It follows from the work of Funakura [1, p. 36] that the field index $i(K)$ of K is given by

$$i(K) = \begin{cases} 2, & \text{if } m \equiv 1 \pmod{16}, \\ 1, & \text{if } m \not\equiv 1 \pmod{16}. \end{cases}$$

From now on we assume that $i(K) = 2$ so that $m \equiv 1 \pmod{16}$, say $m = 16k + 1$. In this case the prime ideal factorization of $< 2 >$ in O_K is

$$< 2 >= P_1^2 P_2 P_3,$$

where P_1, P_2, P_3 are distinct prime ideals, see [1, p. 36]. In this paper we determine explicit generators of P_1, P_2 and P_3.

Theorem. *Let m be a fourth power free integer such that $K = \mathbb{Q}(m^{1/4})$ is a pure quartic field with $i(K) = 2$. Then $< 2 >= P_1^2 P_2 P_3$, where the*

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distinct prime ideals P_1, P_2, P_3 of O_K are given by

$$P_1 = \; < 2, \frac{3}{2} + m^{1/4} + \frac{1}{2}m^{1/2} >,$$

$$P_2 = \begin{cases} < 2, \frac{5}{4} + \frac{1}{4}m^{1/4} + \frac{1}{4}m^{1/2} + \frac{1}{4}m^{3/4} >, & \text{if } m \equiv 1 \pmod{32}, \\ < 2, \frac{3}{4} + \frac{5}{4}m^{1/4} + \frac{3}{4}m^{1/2} + \frac{1}{4}m^{3/4} >, & \text{if } m \equiv 17 \pmod{32}, \end{cases}$$

$$P_3 = \begin{cases} < 2, \frac{5}{4} - \frac{1}{4}m^{1/4} + \frac{1}{4}m^{1/2} - \frac{1}{4}m^{3/4} >, & \text{if } m \equiv 1 \pmod{32}, \\ < 2, \frac{3}{4} - \frac{5}{4}m^{1/4} + \frac{3}{4}m^{1/2} - \frac{1}{4}m^{3/4} >, & \text{if } m \equiv 17 \pmod{32}. \end{cases}$$

2. Proof of Theorem

Let $L = \mathbb{Q}(m^{1/2})$ so that $\mathbb{Q} \subset L \subset K$ and $[L : \mathbb{Q}] = 2$. Set

$$Q_1 =< 2, \frac{1+m^{1/2}}{2} >, \quad Q_2 =< 2, \frac{1-m^{1/2}}{2} > .$$

Q_1 and Q_2 are distinct prime ideals of O_L such that $< 2 >= Q_1Q_2$. Let m_2 be the largest integer such that $m_2^2 \mid m$. Set $m_1 = m/m_2^2$ so that m_1 is a squarefree integer having the same sign as m. Clearly $m^{1/2} = m_2m_1^{1/2}$. Then

$$Q_1 = \begin{cases} < 2, \frac{1+m_1^{1/2}}{2} >, & \text{if } m_2 \equiv 1 \pmod 4, \\ < 2, \frac{1-m_1^{1/2}}{2} >, & \text{if } m_2 \equiv 3 \pmod 4. \end{cases}$$

Next, by [2, Table D, cases D1, D2, p. 92], we see that

$$Q_1 = P_1^2$$

for some prime ideal P_1 of O_K. We claim that

$$P_1 =< 2, \frac{3}{2} + m^{1/4} + \frac{1}{2}m^{1/2} > .$$

First we show that P_1 is a prime ideal of O_K. The minimal polynomial of $\theta = \frac{3}{2} + m^{1/4} + \frac{1}{2}m^{1/2}$ over $\mathbb{Q}$ is

$$g(x) = x^4 - 6x^3 + (13 - 8k)x^2 + (-14 - 8k)x + (6 + 16k + 16k^2).$$

Hence $N(\theta) = \pm(6 + 16k + 16k^2) \equiv 2 \pmod 4$. Let $< \theta >= S_1S_2 \cdots S_r$ be the prime ideal factorization of $< \theta >$ in O_K. Hence $N(< \theta >) =$

$N(S_1)N(S_2)\cdots N(S_r)$. As $2 \parallel N(<\theta>)$ there exists a unique $S = S_i$ such that $2 \parallel N(S)$, that is $N(S) = 2$. Thus $<\theta>$ has exactly one prime ideal to exponent 1 in its prime factorization lying above 2. As $P_1 =< 2, \theta >$ we deduce that $P_1 = S$ so that P_1 is a prime ideal of O_K. Next we show that $P_1 \mid Q_1$. We set $\phi = \frac{3}{2} - m^{1/4} + \frac{1}{2}m^{1/2}$. An easy calculation shows that

$$\frac{1+m^{1/2}}{2} = \theta\phi - (2k+1)2.$$

Hence, as $2 \in P_1$ and $\theta \in P_1$, we deduce that $\frac{1+m^{1/2}}{2} \in P_1$. Thus we have $Q_1 =< 2, \frac{1+m^{1/2}}{2} > \subseteq P_1$, and so $P_1 \mid Q_1$. As Q_1 is the square of a prime ideal in O_K, we deduce that $Q_1 = P_1^2$ as asserted.

Let

$$k = \begin{cases} 2g, & \text{if } m \equiv 1 \pmod{32}, \\ 2g+1, & \text{if } m \equiv 17 \pmod{32}. \end{cases}$$

For $\epsilon = \pm 1$, the minimal polynomial of

$$\alpha(\epsilon) = \begin{cases} \frac{5}{4} + \frac{\epsilon}{4}m^{1/4} + \frac{1}{4}m^{1/2} + \frac{\epsilon}{4}m^{3/4}, & \text{if } m \equiv 1 \pmod{32}, \\ \\ \frac{3}{4} + \frac{5\epsilon}{4}m^{1/4} + \frac{3}{4}m^{1/2} + \frac{\epsilon}{4}m^{3/4}, & \text{if } m \equiv 17 \pmod{32}, \end{cases}$$

is

$$x^4 - 5x^3 + (9-12g)x^2 + (-7+24g-64g^2)x + (2-12g+64g^2-128g^3),$$

if $m \equiv 1 \pmod{32}$, and

$$x^4 - 3x^3 + (-37-76g)x^2 + (-75-240g-192g^2)x$$
$$+(-38-172g-256g^2-128g^3),$$

if $m \equiv 17 \pmod{32}$. Clearly $N(\alpha(\epsilon)) \equiv 2 \pmod{4}$ in both cases, and similarly to the argument above, we deduce that $I_+ =< 2, \alpha(1) >$ and $I_- =< 2, \alpha(-1) >$ are conjugate prime ideals of O_K lying above 2. If $m \equiv 1$ (mod 32) we have

$$\frac{1-m^{1/2}}{2} = 2(1-g-gm^{1/2}) - \alpha(1)\alpha(-1) \in I_+ \cap I_-$$

and if $m \equiv 17$ (mod 32)

$$\frac{1-m^{1/2}}{2} = 2(-g-(1+g)m^{1/2}) - \alpha(1)\alpha(-1) \in I_+ \cap I_-.$$

Hence $\dfrac{1-m^{1/2}}{2} \in I_+ \cap I_-$. Thus I_+ and I_- are conjugate prime ideals of O_K lying above the prime ideal Q_2 of O_L. As $<2>= P_1^2 P_2 P_3 = Q_1 Q_2$ and $Q_1 = P_1^2$, we see that $Q_2 = P_2 P_3$ and that we can take

$$P_2 = I_+ =< 2, \alpha(1) >$$

and

$$P_3 = I_- =< 2, \alpha(-1) > .$$

This completes the proof.

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