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Equivalence of module categories

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EQUIVALENCE OF MODULE CATEGORIES

To the memory of Professor Tadao Tannaka

SHOJI KYUNO

Morita contexts and gamma rings are equivalent concepts ([2]). Therefore, the duality theory obtained in Morita contexts is interpreted in the terms of gamma rings and vice versa. Nobusawa [2] proved directly that when R and L contain the unities the categories of all R -modules and L -modules are equivalent, where R and L are the right operator ring and the left operator ring of a gamma ring of homomorphisms respectively. Furthermore, in [3] he obtained a generalization of one of Morita duality theorems, that is, if $R^2 = R$ and $L^2 = L$, then the categories of properly generated R -modules and L -modules are equivalent.

In this note, without the assumption $R^2 = R$ and $L^2 = L$, we shall prove the following theorem :

Theorem. *Let $(R, L, M, \Gamma, \tau, \mu)$ be a Morita context, in which τ and μ are surjective. It is not assumed that the rings R, L have unities nor that the modules are unitary. The categories of properly generated R -modules and L -modules are equivalent.*

We refer to Jacobson [1, p. 166] for the definition of a Morita context $(R, L, M, \Gamma, \tau, \mu)$, where R, L are rings, $M = {}_L M_R$ is an L - R -bimodule, $\Gamma = {}_R \Gamma_L$ is an R - L -bimodule. We shall use his notations, except that the products $\Gamma \times M$ to R and $M \times \Gamma$ to L will be denoted by γx and $x\gamma$ ($x \in M$ and $\gamma \in \Gamma$), since all relevant associative laws hold. It is not assumed that the rings have unities nor that the modules are unitary. But, we assume that τ and μ are surjective.

Let R be a ring and M be a right R -module. If it satisfies (1) $MR = M$, (2) $\{x \in M \mid xR = 0\} = \{0\}$, then according to Nobusawa [3] we say M is properly generated over R .

Let $\text{PGM}(R)$ be a category of properly generated right modules over R where the morphisms are R -module homomorphisms. Similarly, $\text{PGM}(L)$ denotes a category of properly generated right modules over L where the morphisms are L -module homomorphisms.

Proof of the theorem. Let $G \in \text{ob } \text{PGM}(R)$. Let A be a free additive

abelian group generated by the set of ordered pairs (g, γ) , where $g \in G$, $\gamma \in \Gamma$, and let B be the subgroup of elements $\sum_i m_i(g_i, \gamma_i) \in A$, where m_i are integers such that $\sum_i m_i g_i(\gamma_i x) = 0$ for all $x \in M$. Denote by $[G, \Gamma]$ the factor group A/B and, without causing any ambiguity, by $[g, \gamma]$ the coset $(g, \gamma) + B$. Every element in $[G, \Gamma]$ therefore can be expressed as a finite sum $\sum_i [g_i, \gamma_i]$. $[G, \Gamma]$ forms a right L -module with definition

$$\sum_i [g_i, \gamma_i] \sum_j x_j \beta_j = \sum_{i,j} [g_i(\gamma_i x_j), \beta_j]$$

for $\sum_i [g_i, \gamma_i] \in [G, \Gamma]$ and $\sum_j x_j \beta_j \in L$. It is well-defined, because $\sum_i [g_i, \gamma_i] = \sum_j [g'_j, \gamma'_j]$ means $\sum_i g_i(\gamma_i x) = \sum_j g'_j(\gamma'_j x)$ for any $x \in M$.

To see $[G, \Gamma] \in \text{ob PGM}(L)$, let $\sum_i [g_i, \gamma_i]$ be an element in $[G, \Gamma]$ such that $(\sum_i [g_i, \gamma_i])L = 0$, that is, $(\sum_i [g_i, \gamma_i])M\Gamma = 0$. By the definition, $\sum_i [g_i(\gamma_i M), \Gamma] = 0$, which implies $(\sum_i g_i(\gamma_i M))\Gamma M = 0$, that is, $(\sum_i g_i(\gamma_i M))R = 0$. Since $G \in \text{ob PGM}(R)$, $\sum_i g_i(\gamma_i M) = 0$. Hence, $\sum_i [g_i, \gamma_i] = 0$.

In addition, $[G, \Gamma]L = [G, \Gamma]M\Gamma = [G\Gamma M, \Gamma] = [G\Gamma, \Gamma] = [G, \Gamma]$. Therefore, $[G, \Gamma] \in \text{ob PGM}(L)$. Similarly, for $U \in \text{ob PGM}(L)$ we can define a right R -module $[U, M]$ and show that $[U, M] \in \text{ob PGM}(R)$.

An R -module homomorphism $f: A_R \rightarrow B_R$ determines an L -module homomorphism $g: [A, \Gamma]_L \rightarrow [B, \Gamma]_L$ by

$$g(\sum_i [a_i, \gamma_i]) = \sum_i [f(a_i), \gamma_i].$$

To see that g is well-defined, let $\sum_i [a_i, \gamma_i] = 0$. Then for any $\sum_j x_j \omega_j \in L$, $\sum_i [f(a_i), \gamma_i] \sum_j x_j \omega_j = \sum_{i,j} [f(a_i)(\gamma_i x_j), \omega_j] = \sum_{i,j} [f(a_i(\gamma_i x_j)), \omega_j] = \sum_j [f(\sum_i a_i(\gamma_i x_j)), \omega_j] = 0$. Hence, $\sum_i [f(a_i), \gamma_i] = 0$.

It is easy to see that g is an L -module homomorphism.

Similarly, an L -module homomorphism $h: U_L \rightarrow V_L$ determines an R -module homomorphism $k: [U, M]_R \rightarrow [V, M]_R$ by $k(\sum_j [u_j, x_j]) = \sum_j [h(u_j), x_j]$.

Let f_1 and f_2 be R -module homomorphisms such that $f_1: A \rightarrow B$ and $f_2: B \rightarrow C$. Let g_1 and g_2 be L -module homomorphisms determined by f_1 and f_2 respectively. Then, $f_2 f_1: A \rightarrow C$ determines an L -module homomorphism $p: [A, \Gamma] \rightarrow [C, \Gamma]$ such that $p = g_2 g_1$. Indeed, for any $\sum_i [a_i, \gamma_i] \in [A, \Gamma]$ we have $p(\sum_i [a_i, \gamma_i]) = \sum_i [f_2 f_1(a_i), \gamma_i] = \sum_i [f_2(f_1(a_i)), \gamma_i] = g_2(\sum_i [f_1(a_i), \gamma_i]) = g_2 g_1(\sum_i [a_i, \gamma_i])$.

Clearly, $1_A: A \rightarrow A$ determines $1_{[A, \Gamma]}: [A, \Gamma] \rightarrow [A, \Gamma]$. Thus, we have functors $F: \text{PGM}(R) \rightarrow \text{PGM}(L)$ and $H: \text{PGM}(L) \rightarrow \text{PGM}(R)$, where for $A \in \text{ob PGM}(R)$ $F(A) = [A, \Gamma]$ and for $U \in \text{ob PGM}(L)$ $H(U)$

$$= [U, M].$$

For any $A \in \text{ob PGM}(R)$ and $U \in \text{ob PGM}(L)$,

$$\begin{aligned} HF(A) &= H([A, \Gamma]) = [[A, \Gamma], M] \\ \text{and } FH(U) &= F([U, M]) = [[U, M], \Gamma]. \end{aligned}$$

Define the mapping $\eta_A : A = A(\Gamma M) \rightarrow [[A, \Gamma], M]$ by

$$a = \sum_i a_i(\gamma_i x_i) \mapsto \sum_i [[a_i, \gamma_i], x_i].$$

We show that η_A is an isomorphism.

$$\begin{aligned} a = \sum_i a_i(\gamma_i x_i) = 0 &\Leftrightarrow (\sum_i a_i(\gamma_i x_i))R = 0 \\ &\quad \text{(By } A \in \text{ob PGM}(R).) \\ &\Leftrightarrow (\sum_i a_i(\gamma_i x_i))\Gamma M = 0 \\ &\quad \text{(By } R = \Gamma M.) \\ &\Leftrightarrow [\sum_i a_i(\gamma_i x_i), \Gamma] = 0 \\ &\quad \text{(By the definition that a coset is 0.)} \\ &\Leftrightarrow \sum_i [a_i, \gamma_i](x_i \Gamma) = 0 \\ &\quad \text{(By the definition of an } L\text{-module} \\ &\quad \text{imposed on } [A, \Gamma].) \\ &\Leftrightarrow \sum_i [[a_i, \gamma_i], x_i] = 0 \\ &\quad \text{(By the definition that a coset is 0.)} \end{aligned}$$

Hence, η_A is a bijection of A onto $HF(A)$. It is easy to see that η_A is an R -module homomorphism.

For an R -module homomorphism $f : A_R \rightarrow B_R$ and for $a = \sum_i a_i(\gamma_i x_i) \in A$, we have

$$\begin{aligned} HF(f)\eta_A(a) &= HF(f)\eta_A(\sum_i a_i(\gamma_i x_i)) = HF(f)\sum_i [[a_i, \gamma_i], x_i] \\ &= \sum_i [F(f)([a_i, \gamma_i]), x_i] = \sum_i [[f(a_i), \gamma_i], x_i] \\ &= \eta_B f(a). \end{aligned}$$

Therefore, we have the following commutative diagram :

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & HF(A) \\ f \downarrow & & \downarrow HF(f) \\ B & \xrightarrow{\eta_B} & HF(B) \end{array}$$

Thus, $HF \cong 1_{\text{PGM}(R)}$.

Similarly, we have $FH \cong 1_{\text{PGM}(L)}$.

This completes the proof.

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