Mathematical Journal of Okayama University

Volume 25, *Issue* 2 1983 *Article* 13

DECEMBER 1983

On arithmetical functions associated with higher order divisors

J. Chidambaraswamy* R. Sitaramachandrarao†

*University Of Toledo
†University Of Toledo

Math. J. Okayama Univ. 25 (1983), 185—194

ON ARITHMETICAL FUNCTIONS ASSOCIATED WITH HIGHER ORDER DIVISORS

J. CHIDAMBARASWAMY and R. SITARAMACHANDRARAO*)

1. Introduction. A positive integer d is said to be a divisor of the first order of the integer n if d is a divisor of n. We write $d|_1 n$ to denote that d is a first order divisor of n. By $(a, b)_1$ we mean the largest divisor of a which is a first order divisor of b. Clearly, this is simply the G.C.D. of a and b. We say d is a second order divisor of n, written $d|_2 n$, if $\left(\frac{n}{d}, d\right)_1 = 1$. Clearly, the second order divisors of n are simply the unitary divisors of n.

Similarly, for $r \geq 2$, we say d is an rth order divisor of n, written $d|_r n$, if $\left(\frac{n}{d}, d\right)_{r-1} = 1$. By $(a, b)_r$ we mean the largest divisor of a which is an rth order divisor of b. Clearly,

$$(a, b)_r = \max \{c \mid c|_1 a,\ c|_r b\}.$$

Motivated by the concept of semi-unitary divisors introduced by the first author [3] and that of bi-unitary divisor introduced by D. Suryanarayana [9], Alladi [1] recently introduced the notion of the rth order divisors. Among other things, he obtained asymptotic formulae for the summatory functions of $\tau_r(n)$, $\phi_r(n)$, $\sigma_{r,k}(n)$ and $\sigma^*_{r,k}(n)$ defined respectively by

$$\tau_r(n) = \sum_{d|_r n} 1 \quad , \quad \phi_r(n) = \sum_{\substack{1 \leq a \leq n \\ (a,n)_r = 1}} 1$$

$$\sigma_{r,k}(n) = \sum_{d|_r n} d^k, \quad \sigma^*_{r,k}(n) = \sum_{d|_r n} \left(\frac{n}{d}\right)^k.$$

He proved that

(1.1) $$\sum_{n \leq x} \tau_r(n) = A_r x \log x + O(x),$$

(1.2) $$\sum_{n \leq x} \phi_r(n) = B_r x^2 + O(x^{3/2+\varepsilon}), \text{ for each } \varepsilon > 0,$$

(1.3) $$\sum_{n \leq x} \sigma_{r,k}(n) = C_{r,k} x^{k+1} + O(x^{k+1/2})$$

and

(1.4) $$\sum_{n \leq x} \sigma^*_{r,k}(n) = D_{r,k} x^{k+1} + O(x^{k+1/2}),$$

*) On leave from Andhra University, Waltair, India.

where $k \geq 1$ is an integer and A_r, B_r, $C_{r,k}$ and $D_{r,k}$ are certain positive constants.

The purpose of this paper is to obtain the above asymptotic formulae with substantially better estimates for the error terms. In fact, we prove

(1.1)* $$\sum_{n \leq x} \tau_r(n) = A_r x \log x + A'_r x + O(x^{1/2})$$

(1.2)* $$\sum_{n \leq x} \phi_r(n) = B_r x^2 + O(x \log^2 x),$$

(1.3)* (1) $$\sum_{n \leq x} \sigma_{r,k}(n) = \frac{C_{r,k}}{k+1} x^{k+1} + E_k(x)$$

where $$E_k(x) = \begin{cases} O(x^{\max(1,k)}), & \text{if } 0 < k \neq 1 \\ O(x^{1+\varepsilon}), & \text{for each } \varepsilon > 0, \text{ if } k=1, \end{cases}$$

(2) $$\sum_{n \leq x} \sigma_{r,k}(n) = \begin{cases} (1-k)C_{r,-k}x + O(x^{\max(0,1+k)}), & \text{for } -1 \neq k < 0 \\ (1-k)C_{r,-k}x + O(x^{\varepsilon}), & \text{for each } \varepsilon > 0, \text{ for } k=-1 \end{cases}$$

(1.4)* (1) $$\sum_{n \leq x} \sigma^*_{r,k}(n) = \frac{C_{r,-k}}{k+1} x^{k+1} + F_k(x)$$

where $$F_k(x) = \begin{cases} O(x^{\max(1,k)}), & \text{if } k > 0,\ k \neq 1; \\ O(x^{1+\varepsilon}), & \text{for each } \varepsilon > 0, \text{ if } k=1; \end{cases}$$

(2) $$\sum_{n \leq x} \sigma^*_{r,k}(n) = (1-k)C_{r,-k}x + G_k(x)$$

where $$G_k(x) = \begin{cases} O(\log x), & \text{if } k < -1; \\ O(x^{1+k}), & \text{if } 0 > k > -1; \\ O(x^{\varepsilon}), & \text{for each } \varepsilon > 0, \text{ if } k=-1 \end{cases}$$

where A_r, A'_r, B_r, $C_{r,k}$ are given by (3.7), (3.8), (4.7) and (5.4) of this paper.

We remark that our methods are entirely different from those of Alladi and are simpler and direct. Also our expressions for the constants A_r, B_r etc. are more explicit.

2. Preliminaries. Let $\{F_n\}$, $n=0, 1, 2, \cdots$ be the Fibonacci sequence given by

(2.1) $$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2;$$

and $l(y)$ and $l^*(y)$ be respectively the smallest integer $> y$ and $\geq y$. Further, let

(2.2) $$f_r(x) = l\left(\frac{F_{r-1}}{F_r} x\right) \quad \text{for } r \equiv 1 \pmod 2,$$

(2.3) $$f_r(x) = l^*\left(\frac{F_{r-1}}{F_r} x\right) \quad \text{for } r \equiv 0 \pmod 2.$$

Alladi proved that (cf. [1], Theorem 1), if $n=\prod_{i=1}^{s} p_i^{a_i}$ is the canonical representation of n as a product of prime powers, and if $d=\prod_{i=1}^{s} p_i^{\beta_i}$ divides n, then $d|_r n$ if and only if

$$\beta_i=0 \text{ or } f_r(a_i) \leq \beta_i \leq a_i,\ 1 \leq i \leq s;$$

further, he proved (cf. [1], (2.7), Theorem 3, and (2.8)) that

$$\tau_r(n)=\prod_{i=1}^{s} (a_i-f_r(a_i)+2); \tag{2.4}$$

$$\phi_r(n)=n\prod_{i=1}^{s} \left(1-\frac{1}{p_i^{f_r(a_i)}}\right); \tag{2.5}$$

and for real k

$$\sigma_{r,k}(n)=\prod_{i=1}^{s} (1+p_i^{kf_r(a_i)}+p_i^{k(f_r(a_i)+1)}+\cdots+p_i^{ka_i}). \tag{2.6}$$

We shall also need the Möbius function $\mu(n)$, Euler totient function $\phi(n)$, and $\sigma_k(n)=\sum_{d|n} d^k$. For $s>1$, let $\zeta(s)=\sum_{n=1}^{\infty} \frac{1}{n^s}$. It is well known that (cf. [6], Theorems 280, 287, 288, 289 and 291)

$$\zeta(s)=\prod_p \left(\frac{1}{1-\frac{1}{p^s}}\right),\quad s>1; \tag{2.7}$$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}=\frac{1}{\zeta(s)},\quad s>1; \tag{2.8}$$

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}=\frac{\zeta(s-1)}{\zeta(s)},\quad s>2; \tag{2.9}$$

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^s}=\zeta^2(s),\quad s>1; \tag{2.10}$$

$$\sum_{n=1}^{\infty} \frac{\sigma_k(n)}{n^s}=\zeta(s)\zeta(s-k),\quad s>\max(1, k+1). \tag{2.11}$$

Lemma 2.1 (cf. [6], Theorem 320, and p. 272).

$$\sum_{n\leq x} \tau(n)=x\log x+(2\gamma-1)x+O(x^{\theta})$$

where γ is the Euler's constant and $1/4<\theta<1/2$.

Remark 2.1. We shall need the above only with $\theta<1/2$.

Mathematical Journal of Okayama University, Vol. 25 [1983], Iss. 2, Art. 13

Lemma 2.2 (cf. [6], Theorem 330).

$$\sum_{n\le x}\phi(n)=\frac{3x^2}{\pi^2}+O(x\log x).$$

Lemma 2.3 (cf. [2], Section 3.6).

$$\sum_{n\le x}\sigma_k(n)=\frac{\zeta(k+1)}{k+1}x^{k+1}+\begin{cases}O(x\log x), & \text{if } k=1,\\ O(x^{\max(1,k)}), & \text{if } 0<k\ne 1.\end{cases} \tag{2.12}$$

$$\sum_{n\le x}\sigma_k(n)=\zeta(1-k)x+\begin{cases}O(\log x), & \text{if } k=-1,\\ O(x^{\max(0,1+k)}), & \text{if } -1\ne k<0.\end{cases} \tag{2.13}$$

Lemma 2.4 (cf. [7], Section 157; q=2).

$$\sum_{n\le x}\mu(n)=O\left(\frac{x}{(\log x)^2}\right).$$

Lemma 2.5. *If $g(n)$ is multiplicative and $\prod_p\{\sum_{m=0}^{\infty}|g(p^m)|\}$ converges, then $\sum_{n=1}^{\infty}g(n)$ converges absolutely and*

$$\sum_{n=1}^{\infty}g(n)=\prod_p\{\sum_{m=0}^{\infty}g(p^m)\}.$$

This is well known.

Lemma 2.6.

$$f_r(1)=1 \quad \text{for all } r;$$

$$f_r(2)=\begin{cases}1 & \text{for } r=1,\\ 2 & \text{for } r\ge 2;\end{cases}$$

$$f_r(3)=\begin{cases}1 & \text{for } r=1,\\ 3 & \text{for } r=2,\\ 2 & \text{for } r\ge 3.\end{cases}$$

Proof. Follows easily from (2.1), (2.2) and (2.3).

Lemma 2.7. *For each prime p,*

$$\tau_r(p)=2 \quad \text{for all } r;$$

$$\tau_r(p^2)=\begin{cases}3, & \text{for } r=1,\\ 2, & \text{for } r\ge 2;\end{cases}$$

$$\tau_r(p^3)=\begin{cases}4, & \text{for } r=1\\ 2, & \text{for } r=2,\\ 3, & \text{for } r\ge 3.\end{cases}$$

Proof. Follows easily from (2.4) and Lemma 2.6.

Finally, we shall be using Vinogradov's notation. We recall $f \ll g$ means the same as $f(x)=O(g(x))$.

3. **Proof of (1.1)*.** Clearly Lemma 2.1 gives (1.1)* in case $r=1$ and we refer to Gioia and Vaidya [5] for the case $r=2$. So let $r \geq 3$. Clearly $\tau_r(n)$ is multiplicative and $\tau_r(n) \leq \tau(n)=O(n^\varepsilon)$ for all $\varepsilon > 0$. Hence the series $\sum_{n=1}^{\infty} \tau_r(n)n^{-s}$ converges absolutely for each $s > 1$ and so can be expanded into an infinite product of Euler type (cf. [6], Theorem 286). Thus by Lemma 2.8 and (2.10)

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\tau_r(n)}{n^s} &= \prod_p \left\{1+\frac{2}{p^s}+\frac{2}{p^{2s}}+\frac{3}{p^{3s}}+\cdots\right\}\\ &= \zeta^2(s)\prod_p \left\{\left(1+\frac{2}{p^s}+\frac{2}{p^{2s}}+\frac{3}{p^{3s}}+\cdots\right)\left(1-\frac{2}{p^s}+\frac{1}{p^{2s}}\right)\right\}\\ &= \zeta^2(s)\prod_p \left\{1-\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\cdots\right\}\\ &= \left(\sum_{n=1}^{\infty}\frac{\tau(n)}{n^s}\right)\left(\sum_{n=1}^{\infty}\frac{S(n)}{n^s}\right), \qquad (3.1)\end{aligned}$$

say. Also

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{S(n)}{n^s} &= \prod_p \left\{1-\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\cdots\right\}\\ &= \frac{1}{\zeta(2s)}\prod_p \left\{\left(1-\frac{1}{p^{2s}}+\frac{1}{p^{3s}}+\cdots\right)\left(1-\frac{1}{p^{2s}}\right)^{-1}\right\}\\ &= \frac{1}{\zeta(2s)}\prod_p \left\{1+\frac{1}{p^{3s}}+\cdots\right\}\\ &= \left(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{2s}}\right)\left(\sum_{n=1}^{\infty}\frac{T(n)}{n^s}\right) \qquad (3.2)\end{aligned}$$

say. Now by Lemma 2.5, we infer that $\sum_{n=1}^{\infty} T(n)n^{-s}$ converges absolutely for $s > 1/3$ and as such for each $\varepsilon > 0$, and as $n \to \infty$

$$\sum_{n \leq x}\frac{|T(n)|}{n^{1/3+\varepsilon}} = O(1).$$

Hence by the Theorem of partial summation (cf. [6], Theorem 421)

$$\sum_{n \leq x}|T(n)| = O(x^{1/3+\varepsilon}).$$

Consequently by (3.2) and Lemma 2.4

$$\sum_{n\le x} S(n)=\sum_{n\le x}\sum_{d^2\delta=n}\mu(d)T(\delta)$$
$$=\sum_{\delta\le x} T(\delta)\sum_{d\le\sqrt{x/\delta}}\mu(d)\ll\sum_{\delta\le x}\frac{|T(\delta)|(x/\delta)^{1/2}}{(\log 2x/\delta)^2}$$
$$=\sum_{n\le x}|T(n)|(x/n)^{5/12}\frac{(x/n)^{1/12}}{(\log 2x/n)^2}$$
$$\ll\frac{x^{1/12}}{(\log 2x)^2}x^{5/12}\sum_{n\le x}\frac{|T(n)|}{n^{5/12}}\ll\frac{x^{1/2}}{(\log 2x)^2} \tag{3.3}$$

since $\frac{x^{1/12}}{(\log 2x)^2}$ is increasing and the series $\sum_{n=1}^{\infty}\frac{T(n)}{n^s}$ converges absolutely for $s>1/3$.

Again, by partial summation, it is easily seen that

$$\sum_{n>x}\frac{S(n)}{n}=O\left(\frac{1}{x^{1/2}\log^2 x}\right), \tag{3.4}$$

and

$$\sum_{n>x}\frac{S(n)\log n}{n}=O\left(\frac{1}{x^{1/2}\log x}\right). \tag{3.5}$$

Also by (3.2),

$$\sum_{n\le x}|S(n)|\le\sum_{n\le x}\sum_{d^2\delta=n}|\mu(d)|\,|T(\delta)|$$
$$\le\sum_{\delta\le x}|T(\delta)|\sum_{d\le\sqrt{x/\delta}}1\le x^{1/2}\sum_{\delta\le x}\frac{|T(\delta)|}{\delta^{1/2}}$$
$$=O(x^{1/2}),$$

since the series $\sum_{n=1}^{\infty}\frac{|T(n)|}{n^{1/2}}$ converges. Again by partial summation, we have for any $\theta<1/2$

$$\sum_{n\le x}\frac{|S(n)|}{n^\theta}=O(x^{1/2-\theta}). \tag{3.6}$$

Now by (3.1), Lemma 2.1, Remark 2.1, (3.4), (3.5) and (3.6)

$$\sum_{n\le x}\tau_r(n)=\sum_{d\delta\le x}\tau(d)S(\delta)=\sum_{\delta\le x}S(\delta)\sum_{d\le x/\delta}\tau(d)$$
$$=\sum_{\delta\le x}S(\delta)\left\{\left(\frac{x}{\delta}\right)\log\frac{x}{\delta}+(2\gamma-1)\left(\frac{x}{\delta}\right)+O\left(\left(\frac{x}{\delta}\right)^\theta\right)\right\}$$
$$=x\{\log x+(2\gamma-1)\}\left\{\sum_{n=1}^{\infty}\frac{S(n)}{n}-\sum_{n>x}\frac{S(n)}{n}\right\}$$
$$-x\left\{\sum_{n=1}^{\infty}\frac{S(n)\log n}{n}-\sum_{n>x}\frac{S(n)\log n}{n}\right\}+O\left(x^\theta\sum_{n\le x}\frac{|S(n)|}{n^\theta}\right)$$
$$=A_r x\log x+A_r' x+O(x^{1/2})$$

where

$$A_r=\sum_{n=1}^{\infty}\frac{S(n)}{n}=\prod_p\left\{\left(1-\frac{1}{p}\right)^2\left(\sum_{m=0}^{\infty}\frac{\tau_r(p^m)}{p^m}\right)\right\} \tag{3.7}$$

and

$$A'_r=(2\gamma-1)\sum_{n=1}^{\infty}\frac{S(n)}{n}-\sum_{n=1}^{\infty}\frac{S(n)\log n}{n} \tag{3.8}$$

and this proves (1.1)* in case $r\geq 3$.

4. **Proof of (1.2)*.** Lemma 2.1 gives (1.2)* for $r=1$ and we refer to E. Cohen [4] for the case $r=2$. Let then $r\geq 3$. Since $\phi_r(n)\leq n$ for all n, the series $\sum_{n=1}^{\infty}\frac{\phi_r(n)}{n^s}$ converges absolutely for all $s>2$. Also since $\phi_r(n)$ is multiplicative, we have by Euler's infinite product factorization theorem and Lemma 2.5,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\phi_r(n)}{n^s}&=\prod_p\left\{1+\sum_{m=1}^{\infty}\frac{\phi_r(p^m)}{p^{ms}}\right\}\\&=\frac{\zeta(s-1)}{\zeta(s)}\prod_p\left[\left\{1+\sum_{m=1}^{\infty}\frac{\phi_r(p^m)}{p^{ms}}\right\}\left(1-\frac{1}{p^{s-1}}\right)\left(1-\frac{1}{p^s}\right)^{-1}\right]\\&=\frac{\zeta(s-1)}{\zeta(s)}\prod_p\left\{1+\sum_{m=1}^{\infty}\frac{\phi_r(p^m)}{p^{ms}}\right\}\prod_p\left\{1-(p-1)\sum_{m=1}^{\infty}\frac{1}{p^{ms}}\right\}\\&=\frac{\zeta(s-1)}{\zeta(s)}\prod_p\left[1+\sum_{m=1}^{\infty}\frac{\{\phi_r(p^m)-(p-1)(\phi_r(1)+\cdots+\phi_r(p^{m-1}))\}}{p^{ms}}\right]\\&=\frac{\zeta(s-1)}{\zeta(s)}\sum_{n=1}^{\infty}\frac{U(n)}{n^s},\end{aligned} \tag{4.1}$$

say. Now, using (2.5) we obtain, after some calculation

$$\sum_{n=1}^{\infty}\frac{U(n)}{n^s}=\prod_p\left(1+\sum_{m=1}^{\infty}\frac{a_m}{p^{ms}}\right),\text{ where}$$

$$a_m=1+(p-1)(p^{m-1-f_r(m-1)}+p^{m-2-f_r(m-2)}+\cdots+p^{1-f_r(1)})-p^{m-f_r(m)}. \tag{4.2}$$

Now, since $a_1=0$, $a_2=a_3=p-1$, we have

$$\begin{aligned}\frac{1}{\zeta(2s-1)}\sum_{n=1}^{\infty}\frac{U(n)}{n^s}&=\prod_p\left(1+\sum_{m=1}^{\infty}\frac{a_m}{p^{ms}}\right)\prod_p\left(1-\frac{p}{p^{2s}}\right)\\&=\prod_p\left\{1-\frac{1}{p^{2s}}+\frac{p-1}{p^{3s}}+\sum_{m=4}^{\infty}\frac{a_m-pa_{m-2}}{p^{ms}}\right\}\\&=\sum_{n=1}^{\infty}\frac{V(n)}{n^s},\end{aligned} \tag{4.3}$$

say. Using the fact that $f_r(a)\geq a/2$, we have from (4.2) that for $m\geq 4$

$$\begin{aligned}|a_m - pa_{m-2}| &\le (p-1)\sum_{k=2}^{m-1} p^{k-f(k)} + p(p-1)\sum_{k=1}^{m-3} p^{k-f(k)} \\ &\le p\sum_{k=0}^{m-1} p^{k/2} + p^2 \sum_{k=0}^{m-3} p^{k/2} \\ &\le \frac{2p^{(m+2)/2}}{p^{1/2}-1} \le Cp^{(m+1)/2},\end{aligned}$$

C being a positive absolute constant. Hence the infinite product defining the series in (4.3) converges absolutely at least for $s > 7/8$. Thus

$$\sum_{n\le x} \frac{|V(n)|}{n^{7/8+\varepsilon}} = O(1) \tag{4.4}$$

for each fixed $\varepsilon > 0$. Now by (4.3),

$$\sum_{n=1}^{\infty} \frac{U(n)}{n^s} = \sum_{n=1}^{\infty} \frac{n}{n^{2s}} \sum_{n=1}^{\infty} \frac{V(n)}{n^s}$$

so that

$$U(n) = \sum_{d^2\delta=n} dV(\delta).$$

Consequently, from (4.4),

$$\sum_{n\le x} |U(n)| \le \sum_{\delta\le x} |V(\delta)| \sum_{d\le\sqrt{x/\delta}} d = O\left(x \sum_{\delta\le x} \frac{|V(\delta)|}{\delta}\right) = O(x)$$

and hence by partial summation

$$\sum_{n\le x} \frac{|U(n)|}{n} = O(\log x), \tag{4.5}$$

$$\sum_{n>x} \frac{|U(n)|}{n^2} = O\left(\frac{1}{x}\right). \tag{4.6}$$

Now, by (4.1) and (2.9),

$$\sum_{n=1}^{\infty} \frac{\phi_r(n)}{n^s} = \sum_{n=1}^{\infty} \frac{U(n)}{n^s} \sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}$$

and so

$$\phi_r(n) = \sum_{d\delta=n} U(d)\phi(\delta).$$

Hence, by Lemma 2.2, (4.5) and (4.6), we have

$$\begin{aligned}\sum_{n\le x} \phi_r(n) &= \sum_{d\delta\le x} U(d)\phi(\delta) = \sum_{d\le x} U(d) \sum_{\delta\le x/d} \phi(\delta) \\ &= \sum_{d\le x} U(d)\left\{\frac{3x^2}{\pi^2}\frac{1}{d^2} + O\left(\frac{x}{d}\log\frac{x}{d}\right)\right\} \\ &= \frac{3x^2}{\pi^2}\sum_{n=1}^{\infty}\frac{U(n)}{n^2} + O\left(x^2 \sum_{d>x}\frac{|U(d)|}{d^2}\right) + O\left(x\log x \sum_{d\le x}\frac{|U(d)|}{d}\right)\end{aligned}$$

$$=B_r x^2+O(x\log^2 x)$$

where

$$B_r=\frac{3}{\pi^2}\sum_{n=1}^{\infty}\frac{U(n)}{n^2}=\frac{1}{2}\prod_p\left\{\left(1-\frac{1}{p}\right)\left(\sum_{m=0}^{\infty}\phi_r(p^m)p^{-2m}\right)\right\}. \tag{4.7}$$

This completes the proof of (1.2)*.

5. Proofs of (1.3)* and (1.4)*. Clearly $\sigma_{r,k}(n)$ is a multiplicative function of n and $\sigma_{r,k}(n)=O(n^{\max(k,0)+\varepsilon})$ for each $\varepsilon>0$. Hence by Euler's infinite product factorization theorem, (2.6) and (2.11) we have for $s>\max(k,0)+1$,

$$\sum_{n=1}^{\infty}\frac{\sigma_{r,k}(n)}{n^s}=\prod_p\left\{\sum_{m=0}^{\infty}\frac{\sigma_{r,k}(p^m)}{p^{ms}}\right\}$$
$$=\zeta(s)\zeta(s-k)\prod_p\left[\left\{\sum_{m=0}^{\infty}\frac{\sigma_{r,k}(p^m)}{p^{ms}}\right\}\left(1-\frac{1}{p^s}\right)\left(1-\frac{p^k}{p^s}\right)\right]$$

and this after a simple calculation is

$$=\zeta(s)\zeta(s-k)\prod_p\left\{1-\frac{p^k}{p^{2s}}+\cdots\right\}$$
$$=\sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^s}\sum_{n=1}^{\infty}\frac{W(n)}{n^s} \tag{5.1}$$

say, where the series on the extreme right of (5.1) converges absolutely for $s>(k+1)/2$. Thus, for each $\varepsilon>0$

$$\sum_{n\le x}\frac{|W(n)|}{n^{(k+1)/2+\varepsilon}}=O(1), \tag{5.2}$$

and by partial summation,

$$\sum_{n\le x}|W(n)|=O(x^{(k+1)/2+\varepsilon}). \tag{5.3}$$

Now (5.1) yields the identity

$$\sigma_{r,k}(n)=\sum_{d\delta=n}\sigma_k(d)W(\delta)$$

and the proof of (1.3)* follows on the same lines as in Section 4, except that now we make use of Lemma 2.3 and the needed results obtainable by partial summation from (5.3).

Also (1.4)* follows from (1.3)* by partial summation and the observation

$$\sigma^*_{r,k}(n)=n^k\sigma_{r,-k}(n).$$

The constant $C_{r,k}$ for $k>0$ is given by

$$C_{r,k}=\prod_p \left\{\left(1-\frac{1}{p}\right)\left(\sum_{m=0}^{\infty} \frac{\sigma_{r,k}(p^m)}{p^{m(k+1)}}\right)\right\}.$$

We omit the details.

REFERENCES

[1] K. ALLADI: On arithmetic functions and divisors of higher order, J. Austral. Math. Soc. **23A** (1977), 9—27.

[2] T.M. APOSTOL: Introduction to Analytic Number Theory, Springer, New York, 1976.

[3] J. CHIDAMBARASWAMY: Sum functions of unitary and semi-unitary divisors, J. Indian Math. Soc. **31** (1967), 117—126.

[4] E. COHEN: Arithmetic functions associated with the unitary divisors of an integer, Math. Z. **74** (1960), 66—80.

[5] A.A. GIOIA and A.M. VAIDYA: The number of square free divisors of an integer, Duke Math. J. **33** (1966), 797—799.

[6] G.H. HARDY and E.M. WRIGHT: An Introduction to the Theory of Numbers, 4th ed., Oxford, 1973.

[7] E. LANDAU: Primzahlen, Chelsea Reprint, New York, 1953.

[8] R. SITARAMACHANDRARAO and D. SURYANARAYANA: On $\sum_{n\le x}\sigma^*(n)$ and $\sum_{n\le x}\phi^*(n)$, Proc. Amer. Math. Soc. **41** (1973), 61—66.

[9] D. SURYANARAYANA: The number of biunitary divisors of an integer, The Theory of Arithmetic Functions, Lecture Notes in Math. **251**, Springer, Berlin, 1972.

[10] D. SURYANARAYANA and R. SITARAMACHANDRARAO: The number of biunitary divisors of an integer-II, J. Indian Math. Soc. **39** (1975), 261—280.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TOLEDO
TOLEDO, OHIO 43606, U.S.A.

(Received January 28, 1983)