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# A note on separable polynomials in skew polynomial rings of derivation type

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# A NOTE ON SEPARABLE POLYNOMIALS IN SKEW POLYNOMIAL RINGS OF DERIVATION TYPE

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Throughout the present note, $K$ will represent a ring with 1, $Z$ the center of $K$, and $D$ a derivation of $K$. Let $R = K[X; D]$ be the skew polynomial ring, where the multiplication is given by $cX = Xc + D(c)$ $(c \in K)$. Then the inner derivation $\widetilde{D}$ of $R$ effected by $X$ is an extension of $D$ and $\widetilde{D}(\sum_i X^i a_i) = [\sum_i X^i a_i, X] = \sum_i X^i D(a_i)$. A monic polynomial $f$ in $R$ with $Rf = fR$ is called a *separable polynomial*, if $R/Rf$ is a separable extension of $K$. The purpose of this note is to prove the following theorem which partially generalizes a theorem of T. Nagahara [2, Theorem 3. 4].

**Theorem.** *Let $p$ be a prime number, and assume that $K$ is of characteristic $p$ and $D(Z) = 0$. Then a monic polynomial $f = X^p - Xa - b$ in $R$ with $Rf = fR$ is separable if and only if $a$ is invertible in $K$.*

*Proof.* As is easily verified, the hypothesis $Rf = fR$ is equivalent to the following:

$$D(a) = D(b) = 0, \quad a \in Z \text{ and } D^p(c) - D(c)a = [c, b] \quad (c \in K).$$

Assume that $f$ is separable. Then, by [1, Theorem 3. 2], there exists a polynomial $y = X^{p-1}d_{p-1} + \cdots + Xd_1 + d_0$ in $R$ such that $\widetilde{D}^{p-1}(y) - ya = 1$ and $[c, y] = 0$ for all $c \in K$. Obviously, $[c, y] = 0$ $(c \in K)$ implies $d_{p-1} \in Z$, and therefore $D(d_{p-1}) = 0$. By induction, we shall prove that

$$(1) \qquad [c, \widetilde{D}^{i-1}(y)] = 0 \text{ for all } c \in K$$

and

$$(2) \qquad D^i(d_{p-i}) = 0 \qquad (i = 1, \cdots, p-1).$$

Assume that (1) and (2) have been proved for $i = 1, \cdots, k$ $(< p-1)$. Since $0 = D([c, \widetilde{D}^{k-1}(y)]) = [D(c), \widetilde{D}^{k-1}(y)] + [c, \widetilde{D}^k(y)] = [c, \widetilde{D}^k(y)] = [c, X^{p-k-1}D^k(d_{p-k-1}) + \cdots + XD^k(d_1) + D^k(d_0)]$, it follows $D^k(d_{p-k-1}) \in Z$, and therefore $D^{k+1}(d_{p-k-1}) = 0$, completing the induction. In view of (1) and (2), we readily see that

$$0 = [c, \widetilde{D}^{p-2}(y)] = [c, XD^{p-2}(d_1) + D^{p-2}(d_0)] \quad (c \in K),$$



 

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$$1=\widetilde{D}^{p-1}(y)-ya=D^{p-1}(d_0)-X^{p-1}d_{p-1}a-\cdots-Xd_1a-d_0a.$$

Hence, comparing the coefficients in both sides, we obtain

(3) $\quad D(c)D^{p-2}(d_1)+[c, D^{p-2}(d_0)]=0 \qquad (c\in K).$

(4) $\quad ad_1\ (=d_1\ a)=0.$

(5) $\quad D^{p-1}(d_0)-d_0a=1.$

Specializing $c=D^{p-2}(d_0)$ in (3), we have

(3)′ $\quad D^{p-1}(d_0)\,D^{p-2}(d_1)=0.$

Now, by (3), (5), (3)′ and (4), it follows that

$$\begin{aligned}[c, D^{p-2}(d_0)] &= -D(c)D^{p-2}(d_1)\\ &= -D(c)\{D^{p-1}(d_0)-d_0a\}D^{p-2}(d_1)\\ &= -D(c)\{D^{p-1}(d_0)D^{p-2}(d_1)-d_0D^{p-2}(ad_1)\}\\ &=0.\end{aligned}$$

Hence, $D^{p-2}(d_0)$ is in $Z$, and therefore by (5), $1=D^{p-1}(d_0)-d_0a=(-d_0)\,a.$

Conversely, assume that $a$ is invertible in $K$. Since $a$ is invertible in $Z$, there holds $D(a^{-1})=0$. Putting $y=-a^{-1}$, we heve $\widetilde{D}^{p-1}(y)-ya=1$ and $[c, y]=0$ for all $c\in K$. Thus, $f$ is separable by [1, Theorem 3.2].

**Remark.** Under the hypothesis of our theorem, $f=X^p-X-b$ is seen to be Galois provided $fR=Rf$. In fact, $\bar{R}=R/fR$ has an automorphism $\bar{\sigma}$ of order $p$ such that $\bar{\sigma}(\bar{X})=\bar{X}+1$ and $\bar{R}^{\langle\bar{\sigma}\rangle}=K$. Then the expansions of $\Pi_{j=1}^{p-1}\{j^{-1}(\bar{X}+j)-j^{-1}(\bar{X})\}=1$ and $\Pi_{j=1}^{p-1}\{j^{-1}(\bar{X}+j)-j^{-1}(\bar{X}+k)\}=0$ $(k=1, \cdots, p-1)$ enable us to see the existence of a $\langle\bar{\sigma}\rangle$-Galois coordinate system of $\bar{R}/K$.

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