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Supplements to the previous paper "Some commutativity theorems for rings"

Yasuyuki Hirano* Motoshi Hongan† Hisao Tominaga‡

*Okayama University
†Tsuyama College of Technology
‡Okayama University

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SUPPLEMENTS TO THE PREVIOUS PAPER "SOME COMMUTATIVITY THEOREMS FOR RINGS"

YASUYUKI HIRANO, MOTOSHI HONGAN and HISAO TOMINAGA

In the previous paper [1], we considered the following properties of a ring R:

$1)_n$ $[x^n, y^n]=0$ for all $x, y\in R$.
$2)_n$ $(xy)^n=x^ny^n$ and $(xy)^{n+1}=x^{n+1}y^{n+1}$ for all $x, y\in R$.
$3)_n$ $(xy)^n=(yx)^n$ for all $x, y\in R$.
$4)_n$ $[x, (xy)^n]=0$ for all $x, y\in R$.
$5)_n$ $[x^n, y]=0$ for all $x, y\in R$.
$6)_n$ $[x^n, y]=[x, y^n]$ for all $x, y\in R$.
$9)_n$ For each pair of elements x, y in R, $n[x, y]=0$ implies $[x, y]=0$.

The purpose of the present note is to add two results to the previous paper [1]. As for notations and terminologies used here, we follow [1].

First, we prove the following that includes essentially Theorem 5 of [1].

Theorem 1. *Let i, j be integers in the set $\{1, 2, 3, 4, 5, 6\}$, and $m, n>1$. Suppose an s-unital ring R has the properties $i)_m$ and $j)_n$. If $(m, n)=1$, then R is commutative.*

Proof. According to [1, Propositions 2 and 3], there exists a positive integer α such that R has the properties $1)_{m^\alpha}$ and $1)_{n^\alpha}$. Therefore, R is commutative by [2, Theorem 4].

Let $n>1$. A ring-property P will be called a $C(n)$-property if every ring with identity having the properties P and $9)_n$ is commutative. In view of [1, Theorem 2], the properties $2)_n-6)_n$ are $C(n)$-properties.

Theorem 2. *Let i, j be integers in the set $\{2, 3, 4, 5, 6\}$, and $m, n>1$. Suppose an s-unital ring R has the properties $i)_m$ and $j)_n$. If R has the property $9)_{(m, n)}$, then R is commutative.*

Proof. Let e be a pseudo-identity of $\{a, b\}\subseteq R$, and e' a pseudo-identity of $\{a, b, e\}$. Let $S=\langle a, b, e, e'\rangle$ be the subring of R generated by $\{a, b, e, e'\}$, and $A=l_S(e)$ $(=r_S(e))$. Then, $e'+A$ is the identity of

S/A. Since $\langle a, b\rangle \cap A = 0$, we may regard $\langle a, b\rangle$ as a subring of S/A. Obviously, S/A has the properties $i)_m$ and $j)_n$. Moreover, we can easily see that S/A has the property $9)_{(m,n)}$. Now, the rest of the proof is immediate by the proposition below.

Proposition 1. *Let P_i be a $C(n_i)$-property which is inherited by every finitely generated subring* ($i = 1, 2, \cdots, t$), and $d = (n_1, \cdots, n_t)$. *Suppose a ring R with identity has the properties $P_1, \cdots, P_t$. If R has the property $9)_d$ then R is commutative.*

Proof. It suffices to prove the case $t = 2$. We show that R has the property $9)_{n_1}$ (and therefore R is commutative). Suppose $n_1[a, b] = 0$ for some $a, b \in R$, and let R' be the subring of R generated by $\{1, a, b\}$. Then, we can easily see that $n_1[x, y] = 0$ for all $x, y \in R'$. Since R' has the property $9)_d$, the above implies that R' has the property $9)_{n_2}$. Hence, R' is commutative, namely $[a, b] = 0$.

References

[1] Y. Hirano and H. Tominaga: Some commutativity theorems for rings, Hiroshima Math. J. **11** (1981), 457—464.

[2] M. Hongan and H. Tominaga: A commutativity theorem for s-unital rings, Math. J. Okayama Univ. **21** (1979), 11—14.

Okayama University

Tsuyama College of Technology and Okayama University

Okayama University

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Added in proof. A ring-property P is called an *H-property* if P is inherited by every finitely generated subring and every canonical image modulo the annihilator of a central element, and is called an *F-property*, provided a ring has the property P if and only if all its finitely generated subrings have. Obviously, all the properties $1)_n$— $9)_n$ considered in [1] are H-properties, and the commutativity is an F-property. By making use of the argument employed in the proof of Theorem 2, we can easily see the following.

Proposition 2. *Let P be an H-property, and Q an F-property. Then the following are equivalent:*

i) *Every ring with identity having the property P has the property Q.*

ii) *Every s-unital ring having the property P has the property Q.*

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