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On Commutative Rings over which all Semiprime Finitely Generated Algebras are Separable

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ON COMMUTATIVE RINGS OVER WHICH ALL SEMIPRIME FINITELY GENERATED ALGEBRAS ARE SEPARABLE

Dedicated to Professor Takasi Nagahara on his 60th birthday

YASUYUKI HIRANO and JAE KEOL PARK

In [1], Armendariz proved that every semiprime finitely generated algebra over a commutative von Neumann regular ring is an Azumaya algebra over its center. In this paper, we shall first prove the converse, that is, if every semiprime finitely generated algebra over a commutative semiprime ring R is Azumaya (over its center), then R is von Neumann regular. Using this, we shall characterize a commutative quasi-regular ring in terms of Azumaya algebra. Finally, we shall describe the structure of a commutative ring over which every semiprime finitely generated algebra is separable.

Throughout we will assume that rings have unit. Let R be a commutative ring. We describe an R-algebra A as finitely generated or faithful if A is finitely generated or faithful when considered as an R-module. The Jacobson radical of a ring A will be denoted by $J(A)$, the prime radical by $P(A)$, and the center by $Z(A)$.

An element a of a ring R is called *von Neumann regular* if there exists an $x \in R$ such that $a = axa$. We start with the following

Lemma 1. *Let A be a semiprime ring with center R and let a be an element of R. Then $B = \begin{pmatrix} A & aA \\ aA & A \end{pmatrix}$ is Azumaya if and only if A is Azumaya and a is von Neumann regular in R.*

Proof. Suppose that B is Azumaya. It is easily checked that the center of B is $Z(B) = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| x, y \in R, (x-y)a = 0 \right\}$. Clearly, $I = \begin{pmatrix} aA & aA \\ aA & aA \end{pmatrix}$ is an ideal of B. Since R is semiprime, we obtain $I \cap Z(B) = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \middle| x \in aR \right\}$. Hence, by [3, Corollary 2. 3. 7], we have $\begin{pmatrix} aA & aA \\ aA & aA \end{pmatrix} = \begin{pmatrix} aA & a^2A \\ a^2A & aA \end{pmatrix}$. Hence there exists $x \in A$ such that $a^2x = a$. If we set $z = ax^2$, then we can easily check that $z \in R$ and $a^2z = a$. Therefore a is von Neumann regular in R. Clearly, $e = az$ is a central idempotent of A, and so we have $B \cong M_2(eA) \oplus (1-e)A \oplus (1-e)A$. Hence A is Azumaya (cf. [3, Proposition 2.1.13]).

Now the reverse implication is clear.

We come to our first theorem.

Theorem 1. *Let R be a commutative semiprime ring. Then the following statements are equivalent:*

(1) *R is von Neumann regular.*

(2) *Every semiprime finitely generated R-algebra is Azumaya (over its center).*

(3) *For every finitely generated R-algebra A, $J(A)$ is nilpotent and $A/J(A)$ is Azumaya.*

Proof. (1) ⇒ (2). This follows from [1, Theorem 2].

(2) ⇒ (1). Take an arbirary $a \in R$, and consider the R-algebra $A = \begin{pmatrix} R & aR \\ aR & R \end{pmatrix}$. Then A is a semiprime finitely generated R-algebra. Then A is Azumaya, and hence a is von Neumann regular in R by Lemma 1.

(2) ⇒ (3). By the equivalence of (1) and (2), R is von Neumann regular. Hence this follows from [6, Proposition 2.2].

(3) ⇒ (2). Let A be a semiprime finitely generated R-algebra. Then, since $J(A)$ is nilpotent, the semiprimeness implies $J(A) = 0$. Hence $A(= A/J(A))$ is Azumaya.

Let A and B be rings with the same identity such that $A \subseteq B$. Then B is called a *finite liberal extension* of A if it contains a finite set of A-centralizing elements, $\{a_1, \dots, a_n\}$ say, such that $B = Aa_1 + \cdots + Aa_n$. Now we deal with the noncommutative version of Theorem 1.

Theorem 2. *Let A be a semiprime ring with center R. Then the following statements are equivalent:*

(1) *Every semiprime finite liberal extension of A is Azumaya.*

(2) *A is a finitely generated R-algebra and R is von Neumann regular.*

Proof. (1) ⇒ (2). Since A is a semiprime finite liberal extension of itself, A is Azumaya. Hence A is finitely generated over its center R. Take an arbitrary $a \in R$, and consider the ring $B = \begin{pmatrix} A & aA \\ aA & A \end{pmatrix}$. It is easily checked that B is a semiprime finite liberal extension of A. Hence B is Azumaya, and so a is von Neumann regular in R by Lemma 1.

(2) $\Rightarrow$ (1). If B is a semiprime finite liberal extension of A, then B is a finitely generated R-algebra, and so B is Azumaya by Theorem 1.

Let R be a commutative ring and let S be the set of non-zero-divisors of R. Then R is said to be *quasi-regular* provided its total quotient ring $S^{-1}R$ is von Neumann regular ([4]). For example, p.p. rings are quasi-regular (see e.g. [5]).

Theorem 3. *Let R be a commutative semiprime ring with total quotient ring Q. Then the following statements are equivalent :*

(1) *R is quasi-regular.*

(2) *For every semiprime finitely generated faithful R-algebra A, $A\otimes_R Q$ is Azumaya.*

(3) *For every semiprime finitely generated faithful R-algebra A, there exists a non-zero-divisor $d \in R$ such that $A_d = A\otimes_R R[d^{-1}]$ is Azumaya.*

Proof. (1) $\Rightarrow$ (2). Let A be a semiprime finitely generated faithful R-algebra. Then $A\otimes_R Q$ is a semiprime finitely generated algebra over the regular ring Q, and hence $A\otimes_R Q$ is Azumaya by Theorem 1.

(2) $\Rightarrow$ (3). Let A be a semiprime finitely generated faithful R-algebra and let S denote the set of non-zero-divisors of R. Then $S^{-1}A = A\otimes_R Q$ is Azumaya, and $Z(S^{-1}A) = S^{-1}Z(A)$. Since $S^{-1}A\otimes_{Z(S^{-1}A)}(S^{-1}A)^{\mathrm{op}} = (A\otimes_{Z(A)}A^{\mathrm{op}})\otimes_{Z(A)}S^{-1}Z(A)$, a separability idempotent e for $S^{-1}A$ can be written as fd^{-1} where $f\in A\otimes_{Z(A)}A^{\mathrm{op}}$ and $d\in S$. Then $e=fd^{-1}$ is in $A_d\otimes_{Z(A_d)}(A_d)^{\mathrm{op}}$, and so e is a separability idempotent for the $Z(A_d)$-algebra A_d. This implies that A_d is Azumaya.

(3) $\Rightarrow$ (1). Let a be an arbitrary element of R. Then $A=\begin{pmatrix} R & aR \\ aR & R\end{pmatrix}$ is a semiprime finitely generated faithful R-algebra. By hypothesis, there exists a non-zero-divisor $d\in R$ such that $A_d=\begin{pmatrix} R[d^{-1}] & aR[d^{-1}] \\ aR[d^{-1}] & R[d^{-1}]\end{pmatrix}$ is Azumaya. By Lemma 1, a is von Neumann regular in $R[d^{-1}]$ and hence in Q. Hence every element of R is von Neumann regular in Q. Let as^{-1} be an arbitrary element of Q, where a, $s\in R$ and s is a non-zero-divisor. Then there exists $x\in Q$ such that $a^2x=a$, and so we have $(as^{-1})^2xs = as^{-1}$. Therefore Q is von Neumann regular.

Finally, we describe the structure of a commutative semiprime ring R

such that every semiprime finitely generated R-algebra is separable. To do it, we introduce the following

Definition. A commutative ring R is called *perfect von Neumann regular* if R is von Neumann regular and every prime factor ring of R is a perfect field.

Examples. (a) Every commutative von Neumann regular $\mathbf{Q}$-algebra R is perfect von Neumann regular.

(b) A ring R is called a *J-ring* if, for each $x \in R$, there exists an integer $n = n(x) > 1$ such that $x = x^n$. Clearly, J-rings are perfect von Neumann regular.

We conclude this paper with the following

Theorem 4. *Let R be a commutative semiprime ring. Then the following statements are equivalent :*

(1) *R is perfect von Neumann regular.*

(2) *Every semiprime finitely generated R-algebra is separable over R.*

(3) *Every finitely generated R-algebra contains a separable subalgebra S such that $A = S \oplus P(A)$ as R-modules.*

Proof. (1) ⇒ (2). Let A be a semiprime finitely generated R-algebra. By [1, Theorem 2], A is Azumaya and von Neumann regular. Hence the center $Z(A)$ of A is also von Neumann regular. Since $Z(A)$ is a $Z(A)$-direct summand of A ([3, Lemma 2.3.1]), $Z(A)$ is a finitely generated R-algebra. Let M be a maximal ideal of R. Since $Z(A)/MZ(A)$ is von Neumann regular and finitely generated over R, $Z(A)/MZ(A)$ is a finite direct sum of fields each of which is a finite extension of R/M. By hypothesis, R/M is a perfect field, and hence $Z(A)/MZ(A)$ is a separable R/M-algebra. Hence, by [3, Theorem 2.7.1], $Z(A)$ is a separable R-algebra. Therefore, by [3, Theorem 2.3.8], A is a separable R-algebra.

(2) ⇒ (3). By Theorem 1, R is von Neumann regular. Let A be a finitely generated R-algebra. Then, by [6, Proposition 2.2] we have $J(A) = P(A)$. Now the assertion follows from [2, Theorem 1].

(3) ⇒ (2). This is trivial.

(2) ⇒ (1). By Theorem 1, R is von Neumann regular. Hence every prime ideal of R is maximal. Let M be a prime ideal of R. Suppose that the field R/M is not perfect. Then we can find a monic polynomial $f(X) \in$

$R[X]$ such that the natural homomorphic image $\bar{f}(X) \in (R/M)[X]$ is irreducible and the field $F = (R/M)[X]/\bar{f}(X)(R/M)[X]$ is inseparable over R/M. Let A denote the R-algebra $R[X]/f(X)R[X]$. Then $B = A/P(A)$ is a semiprime finitely generated (faithful) R-algebra. However, since $B/MB(\cong F)$ is not separable over R/M, B is not a separable R-algebra by [3, Theorem 3.7.1].

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References

[1] E. P. Armendariz: On semiprime P.I.-algebras over commutative regular rings: Pacific J. Math. **66** (1976), 23–28.

[2] W. C. Brown: A splitting theorem for algebras over commutative von Neumann regular rings, Proc. Amer. Math. Soc. **36** (1972), 369–374.

[3] F. DeMeyer and E. C. Ingraham: Separable algebras over commutative rings, Lecture Notes in Math., vol. 181, Springer-Verlag, Berlin and New York, 1971.

[4] S. Endo: On semi-hereditary rings, J. Math. Soc. Japan **13** (1961), 109–119.

[5] M. W. Evans: On commutative p.p. rings, Pacific J. Math. **41** (1972), 687–697.

[6] J. A. Wehlen: Algebras over absolutely flat commutative rings, Trans. Amer. Math. Soc. **196** (1974), 149–160.

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