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INFINITE MATRICES ASSOCIATED WITH POWER SERIES AND APPLICATION TO OPTIMIZATION AND MATRIX TRANSFORMATIONS

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Abstract

In this paper we first recall some properties of triangle Toeplitz matrices of the Banach algebra S_r aociated with power series. Then for boolean Toeplitz matrices M we explicitly calculate the product M^N that gives the number of ways with N arcs aociated with M. We compute the matrix B^N (i, j), where B (i, j) is an infinite matrix whose the nonzero entries are on the diagonals m − n = i or m − n = j. Next among other things we consider the infinite boolean matrix $B^+\infty$ that have infinitely many diagonals with nonzero entries and we explicitly calculate $(B^+\infty)^N$. Finally we give neceary and sufficient conditions for an infinite matrix M to map c (B^N (i, 0)) to c.

KEYWORDS: Matrix transformations, Banach algebra, boolean infinite matrix, optimization

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Abstract. In this paper we first recall some properties of triangle Toeplitz matrices of the Banach algebra S_r associated with power series. Then for boolean Toeplitz matrices $\mathcal{M}$ we explicitly calculate the product $\mathcal{M}^N$ that gives the number of ways with N arcs associated with $\mathcal{M}$. We compute the matrix $B^N(i,j)$, where $B(i,j)$ is an infinite matrix whose the nonzero entries are on the diagonals $m-n=i$ or $m-n=j$. Next among other things we consider the infinite boolean matrix B_∞^+ that have infinitely many diagonals with nonzero entries and we explicitly calculate $\left(B_\infty^+\right)^N$. Finally we give necessary and sufficient conditions for an infinite matrix $\mathcal{M}$ to map $c\left(B^N(i,0)\right)$ to c.

1. Introduction

In this paper among other things our aim is to give the number of ways with N arcs associated with a *boolean Toeplitz infinite matrix* $\mathcal{M}$. For this we need to compute the infinite boolean matrix $\mathcal{M}^N$. It is well-known that this number is equal to the entry $\left[\mathcal{M}^N\right]_{nm}$ lying in the n-th row and the m-th column of $\mathcal{M}^N$. Since we are led to make computations with infinite matrices it is natural to focus on Toeplitz triangular matrices. We then consider $\mathcal{M}$ as an operator from s_r to itself, where $s_r=(1/\alpha)^{-1}*l_\infty$with $\alpha_n=r^n$ for all n, (cf. [16, 18]). Then the isomorphism φ allows us to associate with a power series a triangle Toeplitz matrix mapping s_r to itself. Since S_r can be considered as a *Banach algebra of infinite matrices* and is also the set of all matrices mapping s_r to itself, we will make computations in this space. We will consider the boolean matrix $B(i,j)$ whose the nonzero entries are on the diagonals defined by $m-n=i$ or $m-n=j$ and compute the matrix $B^N(i,j)$, to obtain the number of ways with N arcs associated with $B(i,j)$. We will see that in each of the cases $i<j\le 0$ or $0\le i<j\le 0$ the matrix $B^N(i,j)$ is of Toeplitz and the problem is more complicated in the case when $i<0<j$, since $B^N(i,j)$ is not a triangular Toeplitz matrix. In Subsection 4.1.3 we deal with the case $i=-1$ and $j=1$ that was used in the study of the stability and the convergence of numerical schemes for

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finite difference method (cf. [1, 17]). Furthermore since an infinite boolean matrix can be considered as a matrix map between sequence spaces we focus on the characterization of the set $\left(c\left(B^N(i,0)\right),c\right)$. This result extends in a certain sense some of those given in [14] such as the characterization of $\left(c\left(\Delta^N\right),c\right)$ where Δ is the well-known operator of first differences.

This paper is organized as follows. In Section 2 we recall some definitions and properties of matrix transformations. In Section 3 we give some properties of the map φ which associate with a power series *an infinite Toeplitz matrix* and give an application to the infinite linear system $\varphi(e^{az})x=b$. In Section 4 we consider the *boolean matrix* $B(i,j)$ whose the nonzero entries are on the diagonals $m-n=i$ or $m-n=j$ and compute the number of ways with N arcs associated with $B(i,j)$. Next we consider infinite matrices which have *infinitely many diagonals with nonzero entries*, such as B_∞^+ which is usually denoted by Σ^T in the literature and we explicitly calculate $(B_\infty^+)^N$. Finally in Section 5 we study *matrix transformations mapping* $c\left(B^N(i,0)\right)$ to c.

2. Matrices considered as operators in s_r or s_α

We will denote by s, c_0, c, l_∞ the sets of all sequences, the set of sequences that converge to zero, that are convergent and that are bounded respectively. By cs we will denote the set of all the convergent series. Using Wilansky's notation we will write $s_r=(1/\alpha)^{-1}*l_\infty$with $\alpha_n=r^n$ for all n, (cf. [5, 18]), that is

$$s_r=\left\{x=(x_n)_{n\geq 1}:\|x\|_{s_r}=\sup_n\left(\frac{|x_n|}{r^n}\right)<\infty\right\}$$

where $r>0$. It is known that s_r with the norm $\|x\|_{s_r}$ is a Banach space. For a given infinite matrix $\mathcal{M}=(a_{nm})_{n,m\geq 1}$ we define the *operators* $\mathcal{M}_n$ for any integer $n\geq 1$, by $\mathcal{M}_n(x)=\sum_{m=1}^\infty a_{nm}x_m$ where $x=(x_m)_{m\geq 1}$, and the series are assumed convergent for all n. So we are led to the study of the operator $\mathcal{M}$ defined by $\mathcal{M}x=(\mathcal{M}_n(x))_{n\geq 1}$ mapping between sequence spaces.

The product $\mathcal{M}\mathcal{M}'$ of two infinite matrices $\mathcal{M}$ and $\mathcal{M}'=(a'_{nm})_{n,m\geq 1}$ is well defined if the series $\sum_{k=1}^\infty a_{nk}a'_{km}$ are convergent for all n, m.

By (E,F) where E, $F\subset s$, we will denote the *set of all matrices* $\mathcal{M}=(a_{nm})_{n,m\geq 1}$ *mapping from* E *to* F.

Now let r, $u>0$ and denote by $S_{r,u}$ the set of infinite matrices $\mathcal{M}$ such that

$$\|\mathcal{M}\|_{S_{r,u}}=\sup_{n\geq 1}\left(\frac{1}{u^n}\sum_{m=1}^\infty|a_{nm}|r^m\right)<\infty.$$

The set $S_{r,u}$ is a *Banach space with the norm* $\|\mathcal{M}\|_{S_{r,u}}$. Let E and F be any subsets of s. It was proved in [13] that $\mathcal{M} \in (s_r, s_u)$ if and only if $\mathcal{M} \in S_{r,u}$. So we can write that $(s_r, s_u) = S_{r,u}$.

When $r = u$ we obtain the *Banach algebra with identity* $S_{r,u} = S_r$, (see [5]) normed by $\|\mathcal{M}\|_{S_r} = \|\mathcal{M}\|_{S_{r,r}}$. We also have $\mathcal{M} \in (s_r, s_u)$ if and only if $\mathcal{M} \in S_r$.

When $r = 1$, we obtain $s_1 = l_\infty$. It is well known, see [2, 14] that $(s_1, s_1) = (c_0, s_1) = (c, s_1) = S_1$. We also have $\mathcal{M} \in (c_0, c_0)$ if and only if $\mathcal{M} \in S_1$ and $\lim_{n\to\infty} a_{nm} = 0$ for all $m \geq 1$.

By U^+ we denote the set of all sequences $\alpha = (\alpha_n)_{n\geq 1}$ with $\alpha_n > 0$ for all n. We obtain similar results considering the set S_α of all matrices $\mathcal{M}$ such that $\|\mathcal{M}\|_{S_\alpha} = \sup_{n\geq 1} \left(\alpha_n^{-1} \sum_{m=1}^{\infty} |a_{nm}|\, \alpha_m\right) < \infty$. The set S_α with the norm $\|\mathcal{M}\|_{S_\alpha}$ is a Banach space and we have $S_\alpha = (s_\alpha, s_\alpha)$, where $s_\alpha = (1/\alpha)^{-1} * l_\infty$, (cf. [1, 5 , 7, 8, 10, 11, 12, 13]).

For any subset E of s, we put

$$\mathcal{M}E = \{Y \in s \ : \ \text{there is } X \in E \quad Y = \mathcal{M}X\}.$$

If F is a subset of s, we shall denote $F(\mathcal{M}) = \{X \in s \ : \ Y = \mathcal{M}X \in F\}$.

To explicitly calculate $\mathcal{M}^N$ where $\mathcal{M}$ is an infinite Toeplitz boolean matrix, we need the following results.

3. Triangular Toeplitz matrices of S_r and power series

A *Toeplitz matrix* is an infinite matrix whose entries are of the form $[\mathcal{M}]_{nm} = a_{m-n}$ with $n, m \geq 1$. Here we focus on *triangular Toeplitz matrices* and consider $\mathcal{M}$ as an operator mapping s_r into itself, with $r > 0$. Let

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{3.1}$$

be a power series defined in the open disk $|z| < R$. We can associate with f the *upper infinite triangular Toeplitz matrix* $\mathcal{M} = \varphi(f) \in \bigcap_{0<r<R} S_r$ defined by

$$\varphi(f) = \begin{pmatrix} a_0 & a_1 & a_2 & . \\ & a_0 & a_1 & . \\ \mathbf{0} & & a_0 & . \\ & & & . \end{pmatrix}.$$

For pratical reasons, we will write $\varphi[f(z)]$ instead of $\varphi(f)$. So we can associate with 1 the matrix I and we can associate with z^k for k integer, the matrix whose the only nonzero entries are equal to 1 and are on the diagonal of equation $m = n + k$.

In the following we will use the notation $|f|^{\bullet}(z) = \sum_{k=0}^{\infty} |a_k| z^k$. It is obvious that $|fg|^{\bullet}(r)$ is not equal to $|f|^{\bullet}(r) \times |g|^{\bullet}(r)$ for $r < R$, when $f(z)$ and $g(z)$ are power series defined for $|z| < R$. On the other hand if we take $f(z) = 1 + z$ it can easily be seen that $1/|f|^{\bullet}(r)$ is not equal to $|1/f|^{\bullet}(r) = \sum_{k=0}^{\infty} \left|(-1)^k\right| r^k = 1/(1-r)$ for $r < 1$. From [16] we get the next result.

Lemma 3.1. *[16] The map $\varphi : f \mapsto \mathcal{M}$ is an isomorphism from the algebra of the power series defined in $|z| < R$ into the algebra $\overline{\mathcal{M}}$ of the corresponding matrices.*

More precisely we can state the following where we have $\varphi\left(f^N\right) = [\varphi(f)]^N$.

Lemma 3.2. *Let f be defined by (3.1) and let $0 < r < R$. Then*

(i) a) $\|\varphi(f)\|_{S_r} = \left\|[\varphi(f)]^T\right\|_{S_{1/r}} = |f|^{\bullet}(r)$.

b) $\|\varphi(-f)\|_{S_r} = \|\varphi(f)\|_{S_r}$.

(ii) a) For any integer N we have

$$\varphi\left(f^N\right) \in S_r \text{ and } \left[\varphi\left(f^N\right)\right]^T \in S_{1/r};$$

b) $\left|\varphi\left(f^N\right)\right|_{S_r} \leq \left[|f|^{\bullet}(r)\right]^N$ *and* $\left|\left[\varphi\left(f^N\right)\right]^T\right|_{S_{1/r}} \leq \left[|f|^{\bullet}(r)\right]^N$.

c) If $a_n \geq 0$ for all n then

$$\left|[\varphi(f)]^N\right|_{S_r} = |\varphi(f)|_{S_r}^N = f^N(r).$$

(iii) Assume that $a_0 \neq 0$ and that the series

$$\frac{1}{f(z)} = \sum_{k=0}^{\infty} a'_k z^k$$

has $R' > 0$ as its radius of convergence. Then for each $0 < r < R'$ we have

$$\varphi\left(\frac{1}{f}\right) = [\varphi(f)]^{-1} \in S_r$$

and

$$\left\|[\varphi(f)]^{-1}\right\|_{S_r} \geq \frac{1}{|f|^{\bullet}(r)}. \tag{3.2}$$

Proof. (i) a) Since the series (3.1) is convergent for $|z| < R$ we have

$$\|\varphi(f)\|_{S_r} = \sup_n \left(\frac{1}{r^n} \sum_{m=n}^{\infty} |a_{m-n}| r^m\right)$$

$$= \sum_{k=0}^{\infty} |a_k| r^k = |f|^{\bullet}(r) < \infty \text{ for all } r < R.$$

Concerning the transpose of $\varphi(f)$ we have

$$\left[[\varphi(f)]^T \right]_{nm} = \begin{cases} a_{n-m} & \text{for } m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

We deduce that

$$\begin{aligned} \left\| \varphi(f)^T \right\|_{S_{1/r}} &= \sup_n \left(r^n \sum_{m=1}^{n} |a_{n-m}| \frac{1}{r^m} \right) \\ &= \sum_{k=0}^{\infty} |a_k| r^k = |f|^{\bullet}(r) < \infty \text{ for all } r < R. \end{aligned}$$

b) is a direct consequence of a).

(ii) a) Since S_r is a Banach algebra and $\varphi(f) \in S_r$ we have $\varphi\left(f^N\right) \in S_r$. Similarly $[\varphi(f)]^T \in S_{1/r}$ implies $\left[\varphi\left(f^N\right)\right]^T \in S_{1/r}$ for each $0 < r < R$. (ii) b) comes from (i) a) and from the fact that in the Banach algebra S_r we have $\left\|\varphi\left(f^N\right)\right\|_{S_r} \leq \left(\|\varphi(f)\|_{S_r}\right)^N$.

(ii) c) Since the a_k are positive the power series $f^N(z)$ is of the form $\sum_{k=0}^{\infty} c_k z^k$ with $c_k \geq 0$ and by (i) a) we have

$$\left\|\varphi\left(f^N\right)\right\|_{S_r} = \left|f^N\right|^{\bullet}(r) = f^N(r) = \|\varphi(f)\|_{S_r}^N .$$

(iii) comes from [16] and inequality (3.2) comes from (i) a) and from the fact that S_r is a Banach algebra, so we have

$$\left\| \varphi\left(\frac{1}{f}\right) \right\|_{S_r} = \left\| [\varphi(f)]^{-1} \right\|_{S_r} \geq \frac{1}{\|\varphi(f)\|_{S_r}} = \frac{1}{|f|^{\bullet}(r)}.$$

□

Remark 1. From (ii) c) we deduce that the identity

$$\sum_{m=n}^{\infty} \left[[\varphi(f)]^N \right]_{nm} r^m = r^n f^N(r),$$

is satisfied for all integers n and for all r satisfying $0 < r < R$.

We now give a direct application of this lemma to the solvability of infinite linear systems.

Example 1. Let $a \in \mathbb{C}$ and put

$$\mathcal{M} = \varphi\left(e^{az}\right) = \begin{pmatrix} 1 & \frac{a}{1!} & \frac{a^2}{2!} & . & . \\ & 1 & \frac{a}{1!} & \frac{a^2}{2!} & . \\ & & 1 & \frac{a}{1!} & . \\ \mathbf{0} & & & . & . \\ & & & & . \end{pmatrix}.$$

Consider the infinite linear system represented by

$$\mathcal{M}x = b, \tag{3.3}$$

where $b \in s_r$. This system can be written as

$$\sum_{m=n}^{\infty} \frac{a^{m-n}}{(m-n)!} x_m = b_n \quad n = 1, 2, ...$$

Then $I - \mathcal{M} = \varphi(g)$ where

$$g(z) = 1 - e^{az} = -\sum_{k=1}^{\infty} \frac{a^k}{k!} z^k,$$

and by Lemma 2 (i) we have

$$\|I - \mathcal{M}\|_{S_r} = |g|^{\bullet}(r) = \sum_{k=1}^{\infty} \frac{|ar|^k}{k!} = e^{|a|r} - 1 < 1,$$

so $\|I - \mathcal{M}\|_{S_r} < 1$ for $r < (\ln 2) / |a|$. Since S_r is a Banach algebra $\mathcal{M}$ is invertible and $\mathcal{M}^{-1} \in S_r$. Then equation (3.3) is equivalent to $\mathcal{M}^{-1}(\mathcal{M}x) = x = \mathcal{M}^{-1}b$ for all $x \in s_r$, (cf. [1, 3, 4]). We conclude that for $r < (\ln 2) / |a|$ equation (3.3) where $b \in s_r$ has a unique solution in s_r given by

$$x = \mathcal{M}^{-1}b = \varphi\left(e^{-az}\right) b,$$

that is

$$x_n = \sum_{m=n}^{\infty} (-1)^{m-n} \frac{a^{m-n}}{(m-n)!} b_m \quad n = 1, 2, ... \tag{3.4}$$

4. Application to the boolean matrices $B(i,j)$, $B(0,1,2)$, B_∞^+ and $(B_\infty^+)^T$

In this section we say that an infinite matrix $\mathcal{M} = (a_{nm})_{n,m \geq 1}$ is boolean if a_{nm} is either equal to 0 or 1. Let A_1, A_2,..., A_n,... be a sequence of points in the plane. For any n, $m \in \mathbb{N}^*$ we define the relation $A_n \mathcal{R} A_m$ if there is an arc going from A_n to A_m. In this case we put $a_{nm} = 1$. If there is no arc going from A_n to A_m we then put $a_{nm} = 0$.

It is well-known that the number of ways with N arcs going from A_n to A_m, where n, $m = 1, 2, \ldots$ associated with $\mathcal{M}$ is equal to $\left[\mathcal{M}^N\right]_{nm}$. Note that for each integers n we have

$$\sum_{m=1}^{\infty} \left[\mathcal{M}^N\right]_{nm} \alpha_m \leq \left\|\mathcal{M}^N\right\|_{S_\alpha} \alpha_n \text{ for } \mathcal{M} \in S_\alpha$$

and similarly we have

$$\sum_{m=1}^{\infty} \left[\mathcal{M}^N\right]_{nm} r^m \leq \left\|\mathcal{M}^N\right\|_{S_r} r^n \text{ for } \mathcal{M} \in S_r.$$

4.1. **The boolean matrices** $B(i,j)$**.** Let i, $j \in \mathbb{Z}$ with $i < j$ and put $d = j - i$. Here we define the boolean matrix $B(i,j)$ by

$$[B(i,j)]_{nm} = \begin{cases} 1 \text{ for } m - n = i, \text{ or } m - n = j, \\ 0 \text{ otherwise.} \end{cases}$$

1. For instance for $i = -2$ and $j = -1$ we have

$$B(-2,-1) = \left[\varphi\left(z+z^2\right)\right]^T = \begin{array}{cc} & \begin{array}{cccccc} A_1 & A_2 & . & A_m & . & . \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ . \\ A_n \\ . \\ . \end{array} & \left[\begin{array}{cccccc} 0 & & & & & \\ 1 & 0 & & & \mathbf{0} & \\ 1 & 1 & 0 & & & \\ 0 & 1 & 1 & 0 & & \\ \mathbf{0} & & . & . & . & \\ & & & & & \end{array}\right] \end{array}.$$

We easily see that if $j \leq 0$ the matrix $B(i,j)$ is lower triangular, especially the matrix $B(i,0)$ is a triangle and so is invertible. For $i \geq 0$ the matrix $B(i,j)$ is upper triangular.

First we deal with the matrix $B^N(i,j)$ considered as operator in s_α and we explicitly give its expression in either of the cases $i < j \leq 0$, $0 \leq i < j$, and $i = -1$ and $j = 1$. We will see that the expression of $B^N(i,j)$ in the two previous cases is natural since this matrix is of Toeplitz. The problem is more complicated in the case $i < 0 < j$ as we will see in Subsection 4.1.3 where $i = -1$ and $j = 1$.

4.1.1. *The matrix* $B^N(i,j)$ *as operator in* s_α. Here we consider $B^N(i,j)$ as an operator in $S_\alpha = (s_\alpha, s_\alpha)$. We let

$$\kappa'_{ij}(\alpha) = \sup_{n \geq \max\{1, -i+1\}} \left(\frac{\alpha_{n+i} + \alpha_{n+j}}{\alpha_n}\right) < \infty. \tag{4.1}$$

Note that we obviously have

$$\kappa'_{ij}(\alpha) = \sup_{n \geq 1} \left(\frac{\alpha_{n+i} + \alpha_{n+j}}{\alpha_n}\right) \text{ for } i \geq 0.$$

We can state the next result.

Proposition 1. Let i, $j \in \mathbb{Z}$ and $N \geq 1$ be an integer. Then

(i) for $0 \leq i < j$ we have $B^N(i,j) \in S_\alpha$ for α satisfying condition (4.1) and

$$\left\| B^N(i,j) \right\|_{S_\alpha} \leq \left(\kappa'_{ij}(\alpha) \right)^N. \tag{4.2}$$

(ii) Let $i < j \leq 0$. Then $B^N(i,j) \in S_\alpha$ for α satisfying (4.1) and

$$\left\| B^N(i,j) \right\|_{S_\alpha} \leq \left[\max \left\{ \kappa'_{ij}(\alpha), \sup_{-j+1 \leq n \leq -i} \left(\frac{\alpha_{n+j}}{\alpha_n} \right) \right\} \right]^N < \infty. \tag{4.3}$$

(iii) Let $i < 0 < j$. Then $B^N(i,j) \in S_\alpha$ for α satisfying (4.1) and

$$\left\| B^N(i,j) \right\|_{S_\alpha} \leq \left[\max \left\{ \kappa'_{ij}(\alpha), \sup_{1 \leq n \leq -i} \left(\frac{\alpha_{n+j}}{\alpha_n} \right) \right\} \right]^N < \infty. \tag{4.4}$$

Proof. (i) For $0 \leq i < j$ we have $\|B(i,j)\|_{S_\alpha} = \kappa'_{ij}(\alpha)$. By (4.1) and since S_α is a Banach algebra we deduce $B^N(i,j) \in S_\alpha$ and (4.2) holds.

(ii) We have

$$\frac{1}{\alpha_n} \sum_{m=1}^{\infty} \left[B^N(i,j) \right]_{nm} \alpha_m = \begin{cases} \dfrac{\alpha_{n+i} + \alpha_{n+j}}{\alpha_n} & \text{for } n \geq -i+1, \\ \dfrac{\alpha_{n+j}}{\alpha_n} & \text{for } -j+1 \leq n \leq -i, \\ 0 & \text{for } n \leq -j. \end{cases}$$

Then

$$\|B(i,j)\|_{S_\alpha} = \max \left\{ \kappa'_{ij}(\alpha), \sup_{-j+1 \leq n \leq -i} \left(\frac{\alpha_{n+j}}{\alpha_n} \right) \right\}$$

and since in the Banach algebra S_α we have $\left\| B^N(i,j) \right\|_{S_\alpha} \leq \|B(i,j)\|_{S_\alpha}^N$ we conclude that (4.3) holds.

(iii) comes from the identity

$$\|B(i,j)\|_{S_\alpha} = \max \left\{ \kappa'_{ij}(\alpha), \sup_{1 \leq n \leq -i} \left(\frac{\alpha_{n+j}}{\alpha_n} \right) \right\}$$

and reasoning as above we deduce (4.4). □

We immediately deduce the next result.

Proposition 2. Let i, $j \in \mathbb{Z}$ and let $N \geq 1$ be an integer. Then

(i) a) $B^N(i,j) \in S_r$ for all $r > 0$,

$$\left\| B^N(i,j) \right\|_{S_r} \leq \left(r^i + r^j \right)^N \text{ and } \left[B^N(i,j) \right]_{nm} \leq \inf_{r>0} \left\{ \frac{\left(r^i + r^j \right)^N}{r^{m-n}} \right\} \text{ for all } n, m;$$

b) in the case when $i \geq 0$ we have $\left\| B^N(i,j) \right\|_{S_r} = \left(r^i + r^j \right)^N$.

(ii) Let ξ be a real with $0 < \xi < i$. There is $r_0 > 0$ such that for each $r < r_0$ we have

$$\frac{\left\| B^N(i,j) \right\|_{S_r}}{r^{\xi N}} \to 0 \quad (N \to \infty). \tag{4.5}$$

Proof. (i) a) It can easily be verified that

$$\left\| B(i,j) \right\|_{S_r} = \kappa'_{ij}(\alpha) = r^i + r^j$$

since $\alpha_n = r^n$ for all n. We conclude that

$$\left\| B^N(i,j) \right\|_{S_r} \leq \left\| B(i,j) \right\|_{S_r}^N = \left(r^i + r^j \right)^N.$$

Since

$$\left[B^N(i,j) \right]_{nm} r^{m-n} \leq \left\| B^N(i,j) \right\|_{S_r} \leq \left(r^i + r^j \right)^N \text{ for all } r > 0$$

we deduce

$$\left[B^N(i,j) \right]_{nm} \leq \inf_{r>0} \left\{ \left(r^i + r^j \right)^N r^{n-m} \right\} \text{ for all } n, m \geq 1.$$

This concludes the proof of (i) a).

b) Again by Lemma 3.2 (ii) c) and since $i \geq 0$ we have

$$\left\| B^N(i,j) \right\|_{S_r} = \left\| B(i,j) \right\|_{S_r}^N = \left(r^i + r^j \right)^N.$$

This concludes the proof of b).

(ii) Since $r^{i-\xi} + r^{j-\xi} \to 0$ as $r \to 0$, we deduce there is $r_0 > 0$ such that $r^{i-\xi} + r^{j-\xi} < 1$ for all $r < r_0$ and (4.5) holds. □

Remark 2. In the case (ii) of Proposition 2 we easily see that for each n we successively obtain

$$\sum_{m=n+Ni}^{n+Nj} \left[B^N(i,j) \right]_{nm} r^{m-n} = r^{\xi N} o(1) \quad (N \to \infty)$$

and

$$\left[B^N(i,j) \right]_{nm} = r^{\xi N + n - m} o(1) \quad (N \to \infty)$$

for $m \in [n + Ni, n + Nj]$ and for r small enough.

4.1.2. *Number of ways with N arcs starting from A_n to A_m associated with $B^N(i,j)$ in the cases $i < j \leq 0$, or $0 \leq i < j$.* To obtain the number of ways with N arcs we use the well known formula

$$C_N^k = N(N-1)\ldots(N-k+1)/k!$$

for $0 \leq k \leq N$, which gives the number of combinations of N things k at a time. We have the next result.

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Proposition 3. The number of ways with N arcs starting from A_n to A_m associated with $B^N(i,j)$ is given by the next formulas.

(i) Let $0 \leq i < j$.

(4.6)
$$\left[B^N(i,j)\right]_{nm} = \begin{cases} C_N^{\frac{m-n-Ni}{d}} & \text{for } m-n-Ni = 0, d, 2d..., Nd, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Let $i < j \leq 0$. Then we have

(4.7)
$$\left[B^N(i,j)\right]_{nm} = \begin{cases} C_N^{\frac{n-m+Nj}{d}} & \text{for } n-m+Nj = 0, d, 2d..., Nd, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (i) To obtain the matrix $B^N(i,j)$ we calculate

$$B^N(i,j) = \varphi\left[\left(z^i + z^j\right)^N\right] = \varphi\left[z^{iN}\left(1+z^d\right)^N\right] = \varphi\left(\sum_{k=0}^{N} C_N^k z^{iN+dk}\right).$$

Then if $m - n = iN + dk$, $k = 0, 1, .., N$ we have $\left[B^N(i,j)\right]_{nm} = C_N^k$. This shows (4.6).

(ii) For i and j integers with $i < j \leq 0$ we have

$$B(i,j) = [B(-j,-i)]^T. \tag{4.8}$$

For $0 \leq -j < -i$ and from (i) we obtain

$$\begin{aligned} &\left[\left(B^N(-j,-i)\right)\right]_{nm} \\ &= \begin{cases} C_N^k & \text{for } m = n - Nj + (-i+j)k,\ k = 0, 1, ..., N, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Then

$$\begin{aligned} \left[B^N(i,j)\right]_{nm} &= \left[\left(B^N(-j,-i)\right)^T\right]_{nm} \\ &= \begin{cases} C_N^k & \text{for } n = m - Nj + dk,\ k = 0, 1, ..., N, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

with $d = j - i$. We deduce (4.7). ☐

We immediately obtain the next corollary.

Corollary 1. *(i) For $j > 0$ we have*

(4.9)
$$\left[B^N(0,j)\right]_{nm} = \begin{cases} C_N^k & \textit{for } m = n + jk,\ k = 0, 1, ..., N, \\ 0 & \textit{otherwise.} \end{cases}$$

(ii) For $i<0$ we have
(4.10)
$$\left[B^{N}(i,0)\right]_{nm}=\begin{cases} C_N^k & \text{for } n=m-ik,\ k=0,1,...,N,\\ 0 & \text{otherwise.}\end{cases}$$

4.1.3. *The infinite boolean matrix $B(-1,1)$..* To compute $B^N(i,j)$ in the case when $i<0<j$ the previous formulas cannot be applied. We will see that $B^N(i,j)$ is not a Toeplitz matrix since the entries $\left[B^N(i,j)\right]_{nm}$ are not of the form a_{m-n}.

Here we consider the case when $i=-1$ and $j=1$, that is the infinite boolean matrix

$$B(-1,1)=\begin{array}{cc} & \begin{array}{ccccc} A_1 & A_2 & . & A_m & . \end{array}\\ \begin{array}{c} A_1\\ A_2\\ .\\ A_n\\ .\\ . \end{array} & \left[\begin{array}{ccccccc} 0 & 1 & & & & & \\ 1 & 0 & 1 & & \mathbf{0} & & \\ & . & . & . & & & \\ \mathbf{0} & & 1 & 0 & 1 & & \\ & & & . & . & . & \\ & & & & & & \end{array}\right] \end{array}.$$

We easily see that there is no way from A_n to A_m, in the cases $n=m$, or $|n-m|\geq 2$, where n, $m=1$, 2,.... and there is a unique way starting from A_n to A_m for $n=m-1$, $m\geq 2$, or $n=m+1$ with $m=1$, 2, 3,....

The previous formulas cannot be applied here since $B(-1,1)$ is not triangular. From Proposition 1 and from [1, Lemma 1, pp. 166, 167] the matrix $B^N(-1,1)$ is defined as follows.

Lemma 4.1. *Let $N\geq 1$ be an integer. Then*
(i) a) $B^N(-1,1)\in S_r$ for all $r>0$,
b) $B^N(-1,1)\in S_\alpha$ with $\kappa'_{2,1}(\alpha)<\infty$ and

$$\left\|B^N(-1,1)\right\|_{S_\alpha}\leq\left(\max\left\{\kappa'_{2,1}(\alpha),\frac{\alpha_2}{\alpha_1}\right\}\right)^N.$$

(ii) a) $\left[B^N(-1,1)\right]_{nm}=0$ in each of the next cases, N is even and $|m-n|$ is odd, N is odd and $|m-n|$ is even, or $|m-n|\geq N+1$ for all n, $m\geq 1$;
b) for $n\geq N-k+1$, with $k=0$,1,..., N we have

$$\left[B^N(-1,1)\right]_{n,n-N+2k}=C_N^k,$$

c) if N is odd and $n\leq N$, or N is even, $n\leq N-1$ and $k\geq 2n-N$, then

$$\left[B^N(-1,1)\right]_{n,n-k}=C_N^{\frac{N+k}{2}}-C_N^{\frac{N-2n+k}{2}}.$$

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As a direct consequence of Lemma 4.1 we immediately obtain the next reformulation of the previous result which gives the number $\left[B^N(-1,1)\right]_{nm}$ of ways with N arcs associated with the matrix $B(-1,1)$.

Theorem 4.2. *The number $\left[B^N(-1,1)\right]_{nm}$ of ways with N arcs associated with the matrix $B(-1,1)$ is given by the next formulas.*

(i) $\left[B^N(-1,1)\right]_{nm} = 0$ for $|m-n| \geq N+1$ with $n,\ m \geq 1$.

(ii) Let N be even.

a) If $|m-n|$ is odd then $\left[B^N(-1,1)\right]_{nm} = 0$;

b) if $|m-n|$ is even we have

$$\left[B^N(-1,1)\right]_{nm} = \begin{cases} C_N^{\frac{N-n+m}{2}} & \text{for } N-n+2 \leq m \leq n+N+1; \\ C_N^{\frac{N+n-m}{2}} - C_N^{\frac{N-n-m}{2}} & \text{for } n+m \leq N. \end{cases}$$

(iii) Let N be odd. Then

a) If $|m-n|$ is even then $\left[B^{(N)}(-1,1)\right]_{nm} = 0$;

b) if $|m-n|$ is odd we have

$$\left[B^N(-1,1)\right]_{nm} = \begin{cases} C_N^{\frac{N-n+m}{2}} & \text{for } N-n-2 \leq m \leq n+N; \\ C_N^{\frac{N+n-m}{2}} - C_N^{\frac{N-n-m}{2}} & \text{for } n+m \leq N. \end{cases}$$

For example we have

$$B^5(-1,1) = \begin{array}{cc} & \begin{array}{ccccccccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_9 & . & . \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ . \\ . \\ . \\ . \end{array} & \left[\begin{array}{ccccccccccc} 0 & 5 & 0 & 4 & 0 & 1 & & & & & \\ 5 & 0 & 9 & 0 & 5 & 0 & 1 & & & \mathbf{0} & \\ 0 & 9 & 0 & 10 & 0 & 5 & 0 & 1 & & & \\ 4 & 0 & 10 & 0 & 10 & 0 & 5 & 0 & 1 & & \\ 0 & 5 & 0 & 10 & 0 & 10 & 0 & 5 & 0 & 1 & \\ 1 & 0 & 5 & 0 & 10 & 0 & 10 & 0 & 5 & 0 & . \\ & 1 & 0 & 5 & 0 & 10 & 0 & 10 & 0 & 5 & . \\ \mathbf{0} & & & . & . & . & . & . & . & . & . \\ & & & & . & . & . & . & . & . & . \\ & & & & & . & . & . & . & . & . \\ & & & & & & . & . & . & . & . \end{array}\right] \end{array}.$$

An application.

The number of ways with 5 arcs going from A_7 to A_4 is equal to

$$\left[B^5(-1,1)\right]_{7,4} = C_5^1 = 5.$$

The number of ways with 5 arcs going from A_3 to A_2 is equal to

$$\left[B^5\left(-1,1\right)\right]_{3,2} = C_5^3 - 1 = 9.$$

The number of ways with 20 arcs going from A_{11} to A_9 is given by

$$\left[B^{20}\left(-1,1\right)\right]_{11,9} = C_{20}^{11} - 1.$$

Remark 3. We can extend the definition of $\varphi\left(z^k\right)$ to the case when $k \in \mathbb{Z}$ and define $\overline{\varphi}\left(z^k\right)$ as the matrix whose nonzero entries are equal to 1 and are on the diagonal $m - n = k$. We then have

$$\left(z^i + z^j\right)^N = C_N^0 z^{iN} + C_N^1 z^{iN+d} + ... + C_N^k z^{iN+kd} + ... + C_N^N z^{jN}.$$

Putting $\chi = \max\left(|i|, |j|\right)$ we can do the following conjecture, for each n, m satisfying $m - n = iN + kd$ $k = 0, 1, ..., N$ and $n + m > \chi N$ we have

$$\left[\overline{\varphi}\left(\left(z^i + z^j\right)^N\right)\right]_{nm} = \left[B^N\left(i,j\right)\right]_{nm} = C_N^k.$$

For instance we obtain

$$\left[B^5\left(-1,1\right)\right]_{7,6} = \left[B^5\left(-1,1\right)\right]_{7,8} = C_5^3 = 10.$$

Indeed we have $m-n = 1 = -5+3.2$ and $n+m = 13 > 5$. In the same way we easily see that $\left[B^{100}\left(-1,1\right)\right]_{300,260} = C_{100}^{30}$, since $m - n = -40 = -100 + 2k$ with $k = 30$.

4.1.4. *Case of the tridiagonal boolean matrix* $B\left(0,1,2\right)$.. We can explicitly calculate the number of ways with N arcs from A_n to A_m associated with the matrix $B\left(0,1,2\right)$ defined by

$$B\left(0,1,2\right) = \begin{array}{c c} & \begin{array}{ccccc} A_1 & A_2 & . & A_m & . \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ . \\ A_n \\ . \\ . \end{array} & \begin{bmatrix} 1 & 1 & 1 & & & \mathbf{0} \\ & 1 & 1 & 1 & & \\ & & . & . & & \\ \mathbf{0} & & & 1 & 1 & 1 \\ & & & & . & . \\ & & & & & . \end{bmatrix} \end{array}.$$

State the next result.

Proposition 4. (i) The number of ways with N arcs from A_n to A_m associated with the matrix $B\left(0,1,2\right)$ is given by the next formula where $z_0 = \left(-1 - \mathbf{i}\sqrt{3}\right)/2$ and $\mathbf{i} = \sqrt{-1}$,

$$\left[B^N\left(0,1,2\right)\right]_{nm} = \begin{cases} \left(-1\right)^k \sum_{\substack{i+j=k, \\ 0 \le i,j \le N}} C_N^i C_N^j z_0^{j-i} & \text{for } m = n + k,\ k = 0, 1, ..., 2N, \\ 0 & \text{otherwise.} \end{cases}$$

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(ii) We have

$$\left\|B^N(0,1,2)\right\|_{S_r} = \left(1+r+r^2\right)^N$$

and

$$\left[B^N(0,1,2)\right]_{nm} \leq \inf_{r>0}\left\{r^{n-m}\left(1+r+r^2\right)^N\right\} \text{ for } n+2N \geq m \geq n.$$

Proof. (i) We have $B(0,1,2) = \varphi\left(1+z+z^2\right)$, and since $1+z+z^2 = (z-z_0)(z-\overline{z_0})$ we deduce that

$$\begin{aligned}\left(1+z+z^2\right)^N &= \sum_{j=0}^{N}\sum_{i=0}^{N} C_N^i C_N^j z_0^{-i}\overline{z_0^{-j}}(-z)^{i+j} \\ &= \sum_{k=0}^{2N}\left((-1)^k \sum_{\substack{i+j=k,\\ 0\leq i,j\leq N}} C_N^i C_N^j z_0^{j-i}\right) z^k.\end{aligned}$$

This concludes the proof of (i).

(ii) is a direct consequence of Lemma 2. □

4.2. **Case of the matrices B_∞^+ and $\left(B_\infty^+\right)^T$.**. In this part we consider infinite matrices which have infinitely many diagonals with nonzero entries. So we consider the matrix B_∞^+ which is denoted by Σ^T in the literature and we explicitly calculate B_∞^{+N}. Then we deal with its transpose.

4.2.1. *The matrix B_∞^+.*. Define the infinite matrix B_∞^+ by

$$B_\infty^+ = \begin{bmatrix} 1 & 1 & 1 & 1 & . & . \\ & 1 & 1 & 1 & . & . \\ & & 1 & 1 & . & . \\ & & & 1 & . & . \\ & & & & . & . \\ \mathbf{0} & & & & & . \end{bmatrix}.$$

We have

$$B_\infty^+ = \varphi\left(\sum_{k=0}^{\infty} z^k\right) = \varphi\left(\frac{1}{1-z}\right) \text{ for } |z|<1. \tag{4.11}$$

Put

$$\widehat{C_1^+} = \left\{\alpha \in U^+ \bigcap cs \ : \quad r_n(\alpha) = O(1) \ \ (n\to\infty)\right\},$$

where $r_n(\alpha) = \left(\sum\limits_{m=n}^{\infty}\alpha_m\right)/\alpha_n$, (cf. [9]). We can state the following result.

Proposition 5. Let $N \geq 1$ be an integer.

(i) a) $(B_\infty^+)^N \in S_\alpha$ for $\alpha \in \widehat{C_1^+}$,
and

$$\left\| (B_\infty^+)^N \right\|_{S_\alpha} \leq \left(\sup_n r_n(\alpha) \right)^N . \tag{4.12}$$

b) $(B_\infty^+)^N \in S_r$ for $r < 1$
and

$$\left\| (B_\infty^+)^N \right\|_{S_r} = \| B_\infty^+ \|_{S_r}^N = \frac{1}{(1-r)^N}. \tag{4.13}$$

.

(ii) The number $\left[(B_\infty^+)^N \right]_{nm}$ of ways with N arcs going from A_n to A_m is given by

$$\left[(B_\infty^+)^N \right]_{nm} = \begin{cases} C_{N+m-n-1}^{m-n} & \text{for } m \geq n, \\ 0 & \text{for } m < n. \end{cases} \tag{4.14}$$

Proof. (i) a) It is immediate that $B_\infty^+ \in S_\alpha$ means that $\alpha \in \widehat{C_1^+}$. Since S_α is a Banach algebra we conclude that $(B_\infty^+)^N \in S_\alpha$. Inequality (4.12) is a direct consequence of the identity $\| B_\infty^+ \|_{S_\alpha} = \sup_n r_n(\alpha)$. We can show (i) b) and (ii) together. By (4.11) and Lemma 4.1 (ii) we successively have $(B_\infty^+)^N \in S_r$ for $r < 1$ and

$$(B_\infty^+)^N = \varphi \left[\left(\frac{1}{1-z} \right)^N \right] = \varphi \left(\sum_{m=0}^{\infty} C_{N+m-1}^m z^m \right), \tag{4.15}$$

which gives (4.14). Then by Lemma 3.2 (ii) c) we conclude that

$$\begin{aligned} \left\| (B_\infty^+)^N \right\|_{S_r} &= \sum_{m=0}^{\infty} C_{N+m-1}^m r^m \\ &= \frac{1}{(1-r)^N} = \| B_\infty^+ \|_{S_r}^N \end{aligned}$$

which shows equalities given in (4.13). (ii) comes from identity (4.15). □

Remark 4. Note that we have

$$\left[(B_\infty^+)^N \right]_{nm} = C_{N+m-n-1}^{m-n} \leq \inf_{r<1} \left\{ r^{n-m} \frac{1}{(1-r)^N} \right\} \text{ for } m \geq n.$$

For the next result define the set

$$\widehat{C_1} = \left\{ \alpha \in U^+ : \quad s_n(\alpha) = O(1) \quad (n \to \infty) \right\},$$

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where $s_n(\alpha) = \left(\sum_{m=1}^{n} \alpha_m\right) / \alpha_n$, (cf. [6, 9]). We deduce from the preceding the following corollary where we put $B_\infty = (B_\infty^+)^T$.

Corollary 2. *Let $N \geq 1$ be an integer.*
(i) a) $B_\infty^N \in S_\alpha$ for $\alpha \in \widehat{C_1}$,
b) $B_\infty^N \in S_r$ for $r > 1$
and

$$\|B_\infty^N\|_{S_r} = \left(\frac{r}{r-1}\right)^N. \tag{4.16}$$

.

(ii) The number $\left[B_\infty^N\right]_{nm}$ of ways with N arcs going from A_n to A_m is given by

$$\left[B_\infty^N\right]_{nm} = \begin{cases} C_{N+n-m-1}^{n-m} & \text{for } m \leq n, \\ 0 & \text{for } m > n. \end{cases} \tag{4.17}$$

Proof. (i) a) The condition $B_\infty \in S_\alpha$ means that $\sup_n s_n(\alpha) < \infty$, that is $\alpha \in \widehat{C_1}$. Since S_α is a Banach algebra we deduce that $B_\infty^N \in S_\alpha$.

b) By Lemma 3.2 and Proposition 5 for each r with $0 < 1/r < 1$ we have

$$\|B_\infty\|_{S_r} = \left\|\left(B_\infty^+\right)^T\right\|_{S_{1/r}} = \frac{1}{1-\frac{1}{r}} = \frac{r}{r-1}.$$

From (4.13) we easily deduce that

$$\|B_\infty^N\|_{S_r} = \left\|\left(\left(B_\infty^+\right)^T\right)^N\right\|_{S_{1/r}} = \left(\frac{r}{r-1}\right)^N.$$

(ii) is a direct consequence of Proposition 5 (ii). □

In the next section we will use the matrix $B^N(i,0)$ to study another problem on matrix transformations.

5. Matrix transformations between $c\left(B^N(i,0)\right)$ and c where $N \geq 1$ is an integer

In this part we focus on matrix transformations between $c\left(B^N(i,0)\right)$ and c. This means that we give necessary and sufficient conditions for an infinite matrix to satisfy the property

$$B^N(i,0)\,x_n = \sum_{k=0}^{N} C_N^k x_{n+ik} \to l \text{ implies } \mathcal{M}_n(x) \to l' \ \ (n \to \infty)$$

for some scalars l, l' and for all sequences x. We need to know the inverse of $B^N(i,0)$. So from (4.8) we have $B^N(i,0) = \left[B^N(0,-i)\right]^T = \left[\varphi\left(1+z^{-i}\right)^N\right]^T$. From Lemma 3.2 (iii) we have

$$\left[B^N(i,0)\right]^{-1} = \left[\varphi\left(\frac{1}{1+z^{-i}}\right)^N\right]^T.$$

5.1. **Characterization of** $\left(c\left(B^N(i,0)\right),c\right)$**.** First recall the Silverman -Toeplitz condition for the class (c,c), [18, Th. 1.3.6, p. 6].

Lemma 5.1. *$\mathcal{M} = (a_{nm})_{n,m\geq 1} \in (c,c)$ if and only if*
i) $\sup_{n\geq 1}\sum_{m=1}^{\infty}|a_{nm}| < \infty$,
ii) $\lim_{n\to\infty}\sum_{m=1}^{\infty} a_{nm} = l$ *for some* $l \in \mathbb{C}$
iii) $\lim_{n\to\infty} a_{nm} = l_m$ *for some* $l_m \in \mathbb{C}$ *and for all* $m \geq 1$.

As a direct consequence of a Theorem due to Malkowsky and Rakočević [15] it can easily be deduced the following lemma where $D_{(a_{sn})_n}$ for given s is the diagonal matrix with $\left[D_{(a_{sn})_n}\right]_{nn} = a_{sn}$ for all n and

$$B_\infty D_{(a_{sn})_n} = \begin{pmatrix} a_{s1} & & & & \\ a_{s1} & a_{s2} & & \mathbf{0} & \\ . & . & . & & \\ a_{s1} & a_{s2} & . & a_{sn} & \\ . & . & . & . & . \end{pmatrix}.$$

We have

Lemma 5.2. *Let T be a triangle. We have*

$$\mathcal{M} \in (c(T),c)$$

if and only if the series intervening in the product $\mathcal{M}T^{-1}$ are convergent and

$$\mathcal{M}T^{-1} \in (c,c)$$

and

$$B_\infty D_{(a_{sn})_n} T^{-1} \in (c,c) \text{ for all } s = 1,2,....$$

In the following we will use the notation $[N,k]$ for each $k \geq 0$

$$\begin{aligned} [N,k] &= C_{N+k-1}^{k} \\ &= \frac{(N+k-1)(N+k-2)\ldots(N+k-1-k+1)}{k!} \\ &= \frac{N(N+1)\ldots(N+k-1)}{k!}, \end{aligned}$$

and consider the conditions

$$\sup_n \left(\sum_{m=1}^{\infty} \left| \sum_{k=0}^{\infty} (-1)^k a_{n,m-ik} [N,k] \right| \right) < \infty, \tag{5.1}$$

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} \left(\sum_{k=0}^{\infty} (-1)^k a_{n,m-ik} [N,k] \right) = l \text{ for some scalar } l, \tag{5.2}$$

$$\lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k a_{n,m-ik} [N,k] = l_m \text{ for } m = 1, 2, ... \tag{5.3}$$

$$\sup_n \left(\sum_{m=1}^{n} \left| \sum_{k=0}^{E\left(-\frac{n-m}{i}\right)} (-1)^k a_{s,m-ik} [N,k] \right| \right) < \infty \text{ for all } s, \tag{5.4}$$

$$\lim_{n\to\infty} \sum_{m=1}^{n} \left(\sum_{k=0}^{E\left(-\frac{n-m}{i}\right)} (-1)^k a_{s,m-ik} [N,k] \right) = l \tag{5.5}$$

for some scalar l and for all s,

$$\lim_{n\to\infty} \sum_{k=0}^{E\left(-\frac{n-m}{i}\right)} (-1)^k a_{s,m-ik} [N,k] = l_m \tag{5.6}$$

for some scalar l_m, $m = 1, 2, ...$ and for all s.

We have

Theorem 5.3. *$\mathcal{M} \in \left(c\left(B^N(i,0)\right), c\right)$ if and only if (5.1), (5.2), (5.3), (5.4), (5.5) and (5.6) hold.*

Proof. The matrix $\left(B^N(i,0)\right)^{-1} = B^{-N}(i,0)$ can be explicitly calculated since

$$\begin{aligned} & \left(1 + z^{-i}\right)^{-N} \\ = \;& 1 - N z^{-i} + \frac{N(N+1)}{2!} z^{-2i} - ... + (-1)^k \frac{N(N+1)...(N+k-1)}{k!} z^{-ki} + ... \\ = \;& 1 + \sum_{k=1}^{\infty} (-1)^k [N,k] z^{-ik} \text{ for } |z| < 1. \end{aligned}$$

We immediately get

$$\left[\left(B^{-N}(i,0)\right)^T\right]_{nm} = \varphi\left(\frac{1}{(1+z^{-i})^N}\right)^T$$

$$= \begin{cases} (-1)^k [N,k] & \text{for } n-m=-ik,\ k=0,1,... \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathcal{M}B^{-N}(i,0) = \left(\sum_{k=0}^{\infty} a_{n,m-ik}(-1)^k [N,k]\right)_{n,m\geq 1},$$

and $\mathcal{M}B^{-N}(i,0) \in (c,c)$ is equivalent to (5.1), (5.2) and (5.3). Then we easily obtain for each s

$$B_\infty D_{(a_{sn})_n} B^{-N}(i,0) = \left(\sum_{k=0}^{E\left(-\frac{n-m}{i}\right)} (-1)^{k-m} a_{s,m-ik}[N,k]\right)_{n,m\geq 1}.$$

So $B_\infty D_{(a_{sn})_n} B^{-N}(i,0) \in (c,c)$ if and only if (5.4), (5.5) and (5.6) hold. □

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