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# On Families of Continuous Vector Fields Over Spheres

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# ON FAMILIES OF CONTINUOUS VECTOR FIELDS OVER SPHERES

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It is well known that[1], for $n \equiv 3 \pmod 4$, the sphere $S^n$ admits a set of three continuous vector fields independent at each point, i.e., a 3-field. G. W. Whitehead [3] has proved that, for $n \equiv 3 \pmod 8$, $S^n$ does not admit a 4-field, but his proof assumes the assertion of Pontrjagin [6] that $\pi_5(S^3) = 0$. But it was proved independently by Pontrjagin [7] and G. W. Whitehead [4] that $\pi_5(S^3)$ is cyclic of order two.

In this note we will prove that the result of G. W. Whitehead is true.

1. Let $R_{n+1}$ be the group of rotations of $(n+1)$-dimensional Euclidean space $E^{n+1}$ and $S^n$ the unit sphere of $E^{n+1}$. Then $R_{n+1}$ is the bundle space over $S^n$ with group and fibre $R_n$. Let $T_{n+1}: S^{n-1} \to R_n$ be the characteristic map[2] of its normal form[2]. The next lemma is known.

**Lemma 1**[3]. *The following two properties of $S^n$ are equivalent*: (i) $T_{n+1}$ *is homotopic in* $R_n$ *to a map of* $S^{n-1}$ *into* $R_k$, *and* (ii) $S^n$ *admits a continuous* $(n-k)$*-field.*

Let $Sp_{m+1}$ be the symplectic group operating on the space of $m+1$ quaternion variables $(q_0, \cdots\cdots, q_m)$. Then $Sp_{m+1}$ is the bundle space over $S^{4m+3}$ with group $Sp_m$, and let $T''_{m+1}: S^{4m+2} \to Sp_m$ be the characteristic map of its normal form. It is known that[4] $T''_{m+1}$ is represented by the equation

$$(1) \qquad T''_{m+1}(x) = \| \delta^i_j - 2q_i(1+q_m)^{-2}\bar{q}_j \|, \qquad i, j = 0, 1, \cdots\cdots, m-1;$$

where $x = (q_0, \cdots\cdots, q_m)$, $\sum_{i=0}^{m} |q_i|^2 = 1$ and the real part of $q_m$ is 0 and $\delta^i_j$ is the Kronecker $\delta$. It holds the following lemma concerning $T$ and $T''$.

1) Cf. Reference [1], p. 142, 27.10. Theorem.
2) Cf. [1], pp. 96 - 97, 18.1.
3) Cf. [1], p. 141, 27.6. Theorem.
4) Cf. [1], p. 130, 24.11.

**Lemma 2**[1]. *$T_{4m+4}: S^{4m+2} \to R_{4m+3}$ is homotopic in $R_{4m+3}$ to $T''_{m+1}: S^{4m+2} \to Sp_m$.*

Let $x_{4m-1} = (q_0, \cdots\cdots, q_m) \in S^{4m+2}$ be the point such that $q_i = 0$ for $i < m-1$, $q_{m-1} = 1$ and $q_m = 0$, and $p'': Sp_m \to S^{4m-1}$ the bundle projection defined by $p''(s) = s(x_{4m-1})$, $s \in Sp_m$. Then we have the theorem:

**Theorem 1.** *If $m$ is even, $p'' T''_{m+1}: S^{4m+2} \to S^{4m-1}$ is essential and homotopic to $(4m-5)$-hold suspension*[2] *of a map of $S^7$ on $S^4$ with odd Hopf invariant*[3].

*Proof.* Let $(r_0, \cdots\cdots, r_{m-1}) \in S^{4m-1}$ be the coordinates of $p'' T''_{m+1}(x)$, then, from (1), $p'' T''_{m+1}$ is given by the equations

$$(2) \qquad \begin{aligned} r_i &= -2q_i(1+q_m)^{-2}\bar{q}_{m-1}, \qquad i = 0, 1, \cdots\cdots, m-2, \\ r_{m-1} &= 1 - 2q_{m-1}(1+q_m)^{-2}\bar{q}_{m-1}; \end{aligned}$$

where $\sum_{i=1}^{m} q_i \bar{q}_i = 1$ and the real part of $q_m$ is 0. G. W. Whitehead[4] proved that the map given by (2) has the property that it is homotopic to the $(4m-5)$-hold suspension of a map of $S^7$ on $S^4$ with odd Hopf invariant. Hence the theorem holds.

2. We obtain the theorem concerning to the property of the boundary operation $\partial$ of homotopy sequence.

**Theorem 2.** *If $n \equiv 0 \pmod 8$ and $\neq 0$, and consider the composition of homomorphism and isomorphism*

$$\pi_{n+3}(R_{n+3}, R_n) \xrightarrow{\partial} \pi_{n+2}(R_n, R_{n-1}) \xrightarrow{p_*} \pi_{n+2}(S^{n-1}),$$

*where $p_*$ is the induced isomorphism of the projection $p: (R_n, R_{n-1}) \to S^{n-1}$. Then the element of $p_* \partial(\pi_{n+3}(R_{n+3}, R_n)) \subset \pi_{n+2}(S^{n-1})$ is represented by a map of $S^{n+2}$ into $S^{n-1}$ which is the $(n-5)$-hold suspension of a map of $S^7$ into $S^4$ with even Hopf invariant.*

To prove this theorem, we use three lemmas. Let $\xi_0$ be the identity of $R_n/R_p$ and $\pi^n_{p,q}$ be the natural projection of $R_n/R_{n-q}$ into $R_n/R_{n-p}$. In addition, let $I^n$ be the $n$-cube and $\dot{I}^n$ its boundary, then $\dot{I}^n$ is homeomorphic with $S^{n-1}$.

**Lemma 3**[5]. *If $n \equiv 0 \pmod 8$ and $\neq 0$, there is a map $\phi_0^*: (I^{n+3},$*

1) Cf. [1], p. 128, 24.5. Corollary.
2) Cf. [1], pp. 111 - 112, 21.3.
3) Cf [1], p. 123, 21.6.
4) See the proof of Lemma 1 of [3].
5) Lemmas 3 and 5 are proved by G. W. Whitehead, see [3], Lemmas 2 and 3.

$\dot{I}^{n+3}) \to (R_{n+3}/R_{n-1}, \xi_0)$ *such that* $\pi^{n+3}_{4,1}\phi^*_0 : (I^{n+3}, \dot{I}^{n+3}) \to (R_{n+3}/R_{n+2}, \xi_0) \cong (S^{n+2}, x_0)$ *is essential.*

**Lemma 4.** *Under the same assumption for* $n$, *there is a map* $\phi^*_1$: $(I^{n+3}, \dot{I}^{n+3}) \to (R_{n+2}/R_{n-1}, \xi_0)$ *such that* $\pi^{n+2}_{3,1}\phi^*_1\ (I^{n+3}, \dot{I}^{n+3}) \to (R_{n+2}/R_{n+1}, \xi_0) \cong (S^{n+1}, x_0)$ *is essential.*

**Lemma 5**[1]. *Under the same assumption for* $n$, *if* $\phi_2$ *maps* $(I^{n+3}, \dot{I}^{n+3})$ *into* $(R_{n+1}/R_{n-1}, R_n/R_{n-1})$ *and we consider the map* $\phi_2 \mid \dot{I}^{n+3}$ *is defined on* $S^{n+2}$, *then the last map is homotopic to the* $(n-5)$*-hold suspension of a map of* $S^7$ *on* $S^4$ *with even Hopf invariant.*

*Proof of Lemma* 4. It is easy to see that $R_n/R_{n-p}$ can be considered as the set of all $n \times p$ matrices $A$ such that $A'A = I$, the $p \times p$ identity matrix. Let $S^{n+3}$ be represented by coordinates $(x_0, x_1, \cdots\cdots, x_m)$, where $m = n/8$, $x_0$ is a quaternion and $x_1, \cdots\cdots, x_m$ are Cayley numbers such that $\sum_{i=0}^m |x_i|^2 = 1$. The matrices of the linear transformations $y \to xy$ and $y \to yx$ are denoted by $L(x)$ and $R(x)$ respectively for a quaternion, and $L_1(x)$ and $R_1(x)$ for a Cayley number. If $x_0 \neq 0$, let $f_0(x)$ be the $3 \times 3$ matrix obtained from $L(x_0)R(\bar{x}_0)\,|x_0|^{-1}$ by deleting the first row and the first column; while $x_0 = 0$, $f_0(x)$ be the $3 \times 3$ matrix of zeros. For $i = 1, \cdots\cdots, m-1$, let $f_i(x)$ be the $8 \times 3$ matrix formed from $L_1(x_i)$ by deleting the last five columns. If $x_m = 0$, let $f_m(x)$ be the $7 \times 3$ matrix of zeros; while if $x_m \neq 0$, $f_m(x)$ be the $7 \times 3$ matrix obtained from $L_1(x_m)R_1(\bar{x}_m)\,|x_m|^{-1}$ by deleting the first row and the first four and the last columns. Let $f$ be the map defined by $f'(x) = (f_0'(x), f_1'(x), \cdots\cdots, f_m'(x))$, then $f(x)$ is a $(n+2) \times 3$ matrix and it is easy to see that $f'(x)f(x) = I$, the $3 \times 3$ identity matrix. Hence $f$ maps $S^{n+3}$ into $R_{n+2}/R_{n-1}$ and $f'(1, 0, \cdots\cdots, 0) = (I, 0, \cdots\cdots, 0)$. Let $g$ maps $I^{n+3}$ on $S^{n+3}$ with degree 1 so that $g(\dot{I}^{n+3}) = (1, 0, \cdots\cdots, 0)$, and let $\phi^*_1 = fg$. Clearly $\phi^*_1(\dot{I}^{n+3}) = \xi_0$.

To prove the last assertion, we shall show that $h = \pi^{n+2}_{3,1}f : S^{n+3} \to S^{n+1}$ is essential. The map $h$ is given in real coordinates by

$$\left.\begin{aligned} h_1(x) &= (y_1^2 + y_2^2 - y_3^2 - y_4^2)\,/\,|y|, \\ h_2(x) &= 2(y_2y_3 + y_1y_4)\,/\,|y|, \\ h_3(x) &= 2(y_2y_4 - y_1y_3)\,/\,|y|, \end{aligned}\right\} \quad \text{where } |y| \neq 0,$$

$$h_i(x) = 0, \qquad i = 1, 2, 3, \qquad \text{where } |y| = 0.$$

1) See the footnote 5) of p. 50.

$$(3)\qquad \begin{aligned} h_i(x) &= y_{i+1}, & i &= 4, 5, \cdots\cdots, 8m-5,\\ h_{8m-5+j}(x) &= 2(z_{j+1}z_5 - z_1 z_{j+5}) / |z|, \quad j = 1, 2, 3, & &\\ h_{8m-1}(x) &= [z_1^2 + z_5^2 - \sum_{j=2}^{4}(z_j^2 + z_{j+4}^2)] / |z|, & &\text{where } |z| \neq 0,\\ h_{8m-1+j}(x) &= 2(z_{j+5}z_5 + z_1 z_{j+1}) / |z|, \quad j = 1, 2, 3, & &\\ h_i(x) &= 0, \quad i = 8m-4, \cdots\cdots, 8m+2, & &\text{where } |z| = 0; \end{aligned}$$

where $x_0 = y_1 + y_2 i + y_3 j + y_4 k$, $|y| = (\sum_{i=1}^4 y_i^2)^{\frac{1}{2}}$, $x_i = \sum_{\alpha=1}^8 y_{8i-1+\alpha}e_\alpha$ $(i = 2, \cdots\cdots, m-1)$, $x_m = \sum_{\alpha=1}^8 z_\alpha e_\alpha$ and $|z| = (\sum_{i=1}^8 z_i^2)^{\frac{1}{2}}$. This map is a composition of two maps $h^{(1)}: S^{8m+3} \to S^{8m+2}$ defined by

$$(4)\qquad \begin{aligned} h_1^{(1)}(y) &= (y_1^2 + y_2^2 - y_3^2 - y_4^2) / |y|, & &\\ h_2^{(1)}(y) &= 2(y_2 y_3 + y_1 y_4) / |y|, & &\text{where } |y| \neq 0,\\ h_3^{(1)}(y) &= 2(y_2 y_4 - y_1 y_3) / |y|, & &\\ h_i^{(1)}(y) &= 0, \quad i = 1, 2, 3, & &\text{where } |y| = 0,\\ h_i^{(1)}(y) &= y_{i+1}, & i &= 4, 5, \cdots\cdots, 8m+3; \end{aligned}$$

and $h^{(2)}: S^{8m+2} \to S^{8m+1}$ defined by

$$(5)\qquad \begin{aligned} h_i^{(2)}(y) &= y_i, & i &= 1, 2, \cdots\cdots, 8m-5,\\ h_{8m-5+j}^{(2)}(y) &= 2(z_{j+1}z_5 - z_1 z_{j+5}) / |z|, \quad j = 1, 2, 3, & &\\ h_{8m-1}^{(2)}(y) &= [z_1^2 + z_5^2 - \sum_{j=2}^{4}(z_j^2 + z_{j+4}^2)] / |z|, & &\text{where } |z| \neq 0,\\ h_{8m-1+j}^{(2)}(y) &= 2(z_{j+5}z_5 + z_1 z_{j+1}) / |z|, \quad j = 1, 2, 3, & &\\ h_i^{(2)}(y) &= 0, \quad i = 8m-4, \cdots\cdots, 8m+2, & &\text{where } |z| = 0. \end{aligned}$$

The map $h^{(1)}$ is the $8m$-hold suspension of the map of $S^3$ on $S^2$ obtained by setting $y_i = 0$ for $i = 4, 5, \cdots\cdots, 8m+3$ in (4), and the last map is the Hopf map[1] $H: S^3 \to S^2$. Hence $h^{(1)} \sim E^{8m}H$. The map $h^{(2)}$ is the $(8m-5)$-hold suspension of the map of $S^7$ on $S^6$ obtained by setting $y_i = 0$ for $i = 1, 2, \cdots\cdots, 8m-5$ in (5), and this map is essential[2], and so the 4-hold suspension of the Hopf map $H$. Hence $h^{(2)} \sim E^{8m-1}H$. Thus $h \sim (E^{8m-1}H)\cdot(E^{8m}H) \sim E^{8m-1}(H\cdot EH)$. As $H\cdot EH$ represents a non zero element of $\pi_4(S^2)$[3] and $E: \pi_{k+2}(S^k) \to \pi_{k+3}(S^{k+1})$ is the isomorphism onto[4], $h$ is essential. Thus the proof of Lemma 4 is complete.

1) Cf. [1], p. 126, 24.3, equation (9).
2) See [2], p. 140, equation (8).
3) See [1], p. 113, 21.7.
4) Cf. [7] or [4] for $k = 2$, [5] for $k = 3$ and [8] for $k > 3$.

*Proof of Theorem* 2. Since $^*\pi^n_{p,q}: \pi_i(R_n/R_{n-q}, R_{n-p}/R_{n-q}) \to \pi_i(R_n/R_{n-p})$ is an isomorphism onto, we shall show that *an element of the image of the map* $\partial: \pi_{n+3}(R_{n+3}/R_{n+1}, R_n/R_{n-1}) \to \pi_{n+2}(R_n/R_{n-1}) \cong \pi_{n+2}(S^{n-1})$ *is represented by a map of* $S^{n+2}$ *into* $S^{n-1}$ *being the* $(n-5)$*-hold suspension of a map of* $S^7$ *into* $S^4$ *with even Hopf invariant.*

Let $\phi_0$ be a map of $(I^{n+3}, \dot{I}^{n+3})$ into $(R_{n+3}/R_{n-1}, R_n/R_{n-1})$. We assert that *there exists a map* $\phi_1: (I^{n+3}, \dot{I}^{n+3}) \to (R_{n+2}/R_{n-1}, R_n/R_{n-1})$ *such that* $\phi_0 \mid \dot{I}^{n+3} = \phi_1 \mid \dot{I}^{n+3}$. If $\pi^{n+3}_{4,1}\phi_0$ is inessential, let $h: (I^{n+3} \times I, \dot{I}^{n+3} \times I) \to (S^{n+2}, x_0)$ be a homotopy of $\pi^{n+3}_{4,1}\phi_0$ into a point. Since $\pi^{n+3}_{4,1}: R_{n+3}/R_{n-1} \to R_{n+3}/R_{n+2}$ is a bundle projection, there exists a covering homotopy[1] $h^*: (I^{n+3} \times I, \dot{I}^{n+3} \times I) \to (R_{n+3}/R_{n-1}, R_n/R_{n-1})$ of $\phi_0$ such that $\pi^{n+3}_{4,1}h^* = h$ and $h^*(y, t) = \phi_0(y)$ for $(y, t) \in \dot{I}^{n+3} \times I$. Let $\phi_1 = h^* \mid (I^{n+3} \times 1, \dot{I}^{n+3} \times 1)$, then $\phi_1$ maps $(I^{n+3}, \dot{I}^{n+3})$ into $(R_{n+2}/R_{n-1}, R_n/R_{n-1})$ and $\phi_1 \mid \dot{I}^{n+3} = \phi_0 \mid \dot{I}^{n+3}$. If $\pi^{n+3}_{4,1}\phi_0$ is essential, let $\phi_0^*$ be the map of Lemma 3. Since both $\pi^{n+3}_{4,1}\phi_0$ and $\pi^{n+3}_{4,1}\phi_0^*$ are essential, they represent the same element of $\pi_{n+3}(S^{n+2})$. As $^*\pi^{n+3}_{4,1}: \pi_{n+3}(R_{n+3}/R_{n-1}, R_{n+2}/R_{n-1}) \to \pi_{n+3}(S^{n+2})$ is an isomorphism, $\phi_0$ and $\phi_0^*$ represent the same element of $\pi_{n+3}(R_{n+3}/R_{n-1}, R_{n+2}/R_{n-1})$, hence $\phi_0 \mid \dot{I}^{n+3}$ is homotopic in $R_{n+2}/R_{n-1}$ to $\phi_0^* \mid \dot{I}^{n+3} = \xi_0$. By this homotopy, we obtain the map $\phi_1: (I^{n+3}, \dot{I}^{n+3}) \to (R_{n+2}/R_{n-1}, R_n/R_{n-1})$ such that $\phi_0 \mid \dot{I}^{n+3} = \phi_1 \mid \dot{I}^{n+3}$.

For this $\phi_1$, *there is a map* $\phi_2: (I^{n+3}, I^{n+3}) \to (R_{n+1}/R_{n-1}, R_n/R_{n-1})$ *such that* $\phi_1 \mid \dot{I}^{n-3} = \phi_2 \mid \dot{I}^{n+3}$. If $\pi^{n+2}_{3,1}\phi_1$ is inessential, we can construct $\phi_2$ by the analoguous process of the first case of the above. If $\pi^{n+2}_{3,1}\phi_1$ is essential, we can also construct $\phi_2$ by Lemma 4 and the analoguous process of the second case of the above.

The last map $\phi_2$ maps $(I^{n+3}, \dot{I}^{n+3})$ into $(R_{n+1}/R_{n-1}, R_n/R_{n-1})$, and so, by Lemma 5, $\phi_2 \mid \dot{I}^{n+3}$, considered as defined on $S^{n+2}$, is homotopic to the $(n-5)$-hold suspension of a map of $S^7$ on $S^4$ with even Hopf invariant. By the construction, $\phi_2 \mid \dot{I}^{n+3} = \phi_0 \mid \dot{I}^{n+3}$, and hence $\phi_0 \mid \dot{I}^{n+3}$ has the same property. This completes the proof of Theorem 2.

3. Here we have the principal result.

**Theorem 3.** *If* $n \equiv 3 \pmod 8$, $S^n$ *does not admit a continuous 4-field.*

By Lemma 1, to prove this theorem, it is sufficient to prove

**Theorem 4.** *If* $n \equiv 0 \pmod 8$ *and* $\neq 0$, *then the characteristic map* $T_{n+4}: S^{n+2} \to R_{n+3}$ *is not homotopic to a map of* $S^{n+2}$ *into* $R_{n-1}$.

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1) Cf. [1], p. 50, 11.3. Theorem.

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*Proof.* Consider the diagram

$$\begin{array}{ccccc} & & \pi_{n+2}(R_n) & \xrightarrow{i_*} & \pi_{n+2}(R_{n+3}) \\ & & \downarrow{j_*} & & \downarrow{k_*} \\ \pi_{n+3}(R_{n+3},\ R_n) & \xrightarrow{\partial} & \pi_{n+2}(R_n,\ R_{n-1}) & \xrightarrow{m_*} & \pi_{n+2}(R_{n+3},\ R_{n-1}), \end{array}$$

where $i_*$, $j_*$, $k_*$ and $m_*$ are the induced homomorphisms of the inclusion maps $i, j, k$ and $m$ respectively, and the lower line is from the homotopy sequence of the triple $(R_{n+3},\ R_n,\ R_{n-1})$.

Let $q = n/4$ so that $q$ is even. By Lemma 2, $T_{n+4}$ is homotopic to $T''_{q+1}: S^{n+2} \to Sp_q$. Since $Sp_q \subset R_n$, $T''_{q+1}$ represents an element $\alpha$ of $\pi_{n+2}(R_n)$ such that $i_*\alpha$ is represented by $T_{n+4}$. As the composition.

$$S^{n+2} \xrightarrow{T''_{q+1}} Sp_q \xrightarrow{l} R_n \xrightarrow{j} (R_n,\ R_{n-1}) \xrightarrow{p} S^{n-1},$$

where $l$ is the inclusion map and $p$ is the projection, is just the map $p''T''_{q+1}$, $p''T''_{q+1}$ represents the element $p_*j_*\alpha$. As $q$ is even, Theorem 1 implies that $p''T''_{q+1}$ is the $(n-5)$-hold suspension of the map of $S^7$ on $S^4$ with odd Hopf invariant. Hence, by Theorem 2, $p_*j_*\alpha$ is not contained in $p_*\partial(\pi_{n+3}(R_{n+3},\ R_n))$, and so $j_*\alpha$ is not contained in $\partial(\pi_{n+3}(R_{n+3},\ R_n))$. Exactness of homotopy sequence implies that the kernel of $m_*$ does not contain $j_*\alpha$, and so $m_*j_*\alpha \neq 0$. From $mj = ki$, it follows that $k_*i_*\alpha \neq 0$. Therefore $kT_{n+4}$ represents a non-zero element of $\pi_{n+2}(R_{n+3},\ R_{n-1})$. This is equivalent to the desired conclusion. The proof of Theorem 4 is completed.

4. For a field of tangent hyperplanes of $S^n$, it is known that[1], for $2k \leqslant n$, $S^n$ admits a continuous field of tangent $k$-planes if and only if it admits a continuous $k$-field. Theorem 3 implies immediately

**Theorem 5.** *If* $n \equiv 3 \pmod 8$, *the n-sphere does not admit a continuous field of k-planes for* $4 \leqslant k \leqslant n-4$.

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