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I0-Rings and I0-Modules

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I_0-RINGS AND I_0-MODULES

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The main purpose of this paper is to study I_0 - rings (introduced by Nicholson [7]) and I_0-modules. In section 1, we investigate the polynomial ring over an I_0-ring (Theorem 1.2) and we give a new characterization of I_0-ring by means of the notion of cyclic flat modules (Theorem 1.4). In section 2, we give the conditions for the endomorphism ring of a module to be an I_0- ring (Theorem 2.2). In particular, we show that the endomophism ring of a regular module is an I_0-ring (Theorem 2.5). In section 3, first we give a characterization of a finitely generated semiperfect module (Theorem 3.1). Next we show that every projective module over an I_0-ring is an I_0-module (Theorem 3.2). Finally we show that if R, S (Morita equivalent) and R is an I_0-ring, then S is an I_0-ring (Theorem 3.9).

Throughout this paper R means an associative ring with identity and modules mean unitary R-modules. Also we denote the Jacobson radical of a module M by $J(M)$.

1. I_0-RINGS.

Lemma 1.1. [7, Lemma 1.1] *For a ring R, the following conditions are equivalent:*

(1) *Every left ideal $L \not\subset J(R)$ contains a nonzero idempotent.*
(2) *Every right ideal $L \not\subset J(R)$ contains a nonzero idempotent.*
(3) *If $a \notin J(R)$, then $xax = x$ for some $(0 \neq)x \in R$.*

Following [7], *we call a ring R I_0-ring if it satisfies equivalent conditions of Lemma 1.1. All local rings and all (von Neumann) regular rings are typical examples of I_0-rings.*

An element $a \in R$ is said to be regular if there exists an element $b \in R$ such that $a = aba$.

Lemma 1.2. *Let $\Gamma = R[x]$ be the polynomial ring over a ring R in the commuting indeterminant x, and $a \in R$. Then the following statements are equivalent:*

(1) *The element a is regular.*
(2) $\Gamma a + \Gamma x$ *is a projective right ideal of* Γ.
(3) *R/Ra is a flat left R-module.*
(3+i) *the left-right symmetry of* $(1+i)$, $i = 2, 3$.

The equivalence of (1) and (2) was proved in the proof of [8], *and the equivalence of (1) and (3) was essentially proved in* [4, 11.24,p.434].

The following theorem is an immediate consequence of the lemma above.

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Theorem 1.3. *Let R be a ring and $\Gamma = R[x]$. Then the following statements are equivalent:*

(1) *R is an I_0-ring.*
(2) *For each $a \in R - J(R)$, there exists a nonzero $a^* \in aR$ such that $a^*\Gamma + x\Gamma$ is a projective right ideal of Γ.*
(3) *For each $a \in R - J(R)$, there exists a nonzero $a^* \in aR$ such that R/Ra^* is a flat as a left R-module.*
(3+i) *The left-right symmetry of $(1+i)$, $i = 2, 3$.*

Proof. (1)⇒(2). Suppose that R is an I_0-ring. Then if $a \in R$ and $a \notin J(R)$,$\acute{a}a\acute{a} = \acute{a}$ for some $\acute{a} \in R$ and $\acute{a} \neq 0$. Since $\acute{a} \notin J(R)$, $a''\acute{a}a'' = a''$ for some $a'' \in R$ and $a'' \neq 0$. We put $a^* = a\acute{a}a''$ and $e = a^*\acute{a}$. Then $a^* = a^*\acute{a}a^*$ and e is an idempotent element of R. Thus we have $a^* = (e + (1-e)x)a^*$, $e + (1-e)x = a^*\acute{a} + x(1-e)$ and $x = (e + (1-e)x)(1 - e + ex)$ and so $a^*\Gamma + x\Gamma = (e + (1-e)x)\Gamma$. We put $g = e + (1-e)x$. As is easily seen, g is a non zero divisor of Γ. Hence $\Gamma \cong g\Gamma$,that is, $a^*\Gamma + x\Gamma$ is a projective right ideal of Γ.
(2)⇒(1). Let $a \in R$ and $a \notin J(R)$. Then there exist a nonzero element a^* of R such that $a^*\Gamma + x\Gamma$ is a projective right ideal of Γ. We put $K = a^*\Gamma + x\Gamma$. By Dual Basis Lemma, there exist Γ-homomorphisms α and β from K into Γ such that $y = a^*\alpha(y) + x\beta(y)$ for each $y \in K$. In particular, $a^* - a^*\alpha(a^*) = x\beta(a^*)$. Since x is in center of Γ and α is a Γ-homorophism, $x\alpha(a^*) = \alpha(a^*)x = \alpha(x)a^*$, and so $a^*x = a^*\alpha(x)a^* + x^2\beta(a^*) \cdots (\#)$. We put $\alpha(x) = b_0 + b_1x + b_2x^2 + \cdots + b_lx^l$, where $b_i \in R$ $(i = 1, 2, \dots, l)$. Then we have $a^* = a^*b_1a^*$, comparing with coeficients of x of 1 both sides in equality $(\#)$. Thus a^*b_1 is idempotent and is in a^*R. Hence R is an I_0-ring.
(1)⇒(3). Let $a \in R$ and $a \notin J(R)$. Then there exists a nonzero idempotent $e \in aR$ by assumption. Thus $R = Re \oplus R(1-e)$, that is, R/Re is flat as a left R-module.
(3)⇒(1). Let $a \in R$ and $a \notin J(R)$. Then there exists a^* in aR and $a^* \neq 0$ such that R/Ra^* is flat as a left R-module. Thus for an exact sequence $0 \to Ra^* \to R \to R/Ra^* \to 0$ of left R-modules $Ra^* \cap a^*R = a^*RRa^* = a^*Ra^*$ by [5, Theorem 10.5.1]. Since $a^* \in Ra^* \cap a^*R$, there exists $b \in R$ such that $a^* = a^*ba^*$, that is, a^*b is idempotent and $a^*b \in aR$. Hence, R is an I_0-ring. □

Let M be module and N a submodule of M. We call N is small in M if for submodule X of M such that $M = N + X$ implies that $X = M$. Also we call an exact sequence $0 \to \operatorname{Ker} f \to P \xrightarrow{f} M \to 0$ of modules a projective cover of M if P is projective and $\operatorname{Ker} f$ is small in P.
Following [2],we call a ring R semiperfect if every cyclic R-module has projective cover.

Proposition 1.4. *For a ring R, the following conditions are equivalent:*

(1) *R is a local ring.*
(2) *R is an I_0-ring and 1 is a primitive idempotent.*

(3) *R is semiperfect and* 1 *is a primitive idempotent.*

Proof. (1)⇒(2) is obvious. (2)⇒(1). Let r be in R. If r is in $J(R)$, then $1-r$ is a unit. If $r \notin J(R)$, then there exists a nonzero idempotent $e \in rR$. Since $1=(1-e)+e$ and 1 is a primitive idempotent, $e=1$, that is, r is a unit. (1)⇔(3) are obvious from [9]. □

2. I_0-ENDOMORPHISM RINGS.

R. Ware showed the following:

Lemma 2.1 ([9, Corollary 3.2]). *Let M be a right R-module ,$S=\mathrm{End}_R(M)$ and $f \in S$. Then f is regular if and only if for each $f \in S$,* $\mathrm{Im}\, f$ *and* $\mathrm{Ker}\, f$ *are direct summands of M.*

The following theorem easily follows from this lemma.

Theorem 2.2. *Let M be a right R-module and $S=\mathrm{End}_R(M)$. Then the following conditions are equivalent:*

(1) *S is an I_0-ring.*
(2) *For each $f \in S$ and $f \notin J(S)$, there exists $g \in S$ and $g \neq 0$ such that* $\mathrm{Ker}\, fg$ *and* $\mathrm{Im}\, fg$ *are direct summands of M.*
(3) *For $f \in S$ and $f \notin J(S)$, there exists $g \in S$ and $g \neq 0$ such that* $\mathrm{Ker}\, gf$ *and* $\mathrm{Im}\, gf$ *are direct summands of M.*

Corollary 2.3. *Let P_R be a projective module and $S=\mathrm{End}_R(P)$. Then the following conditions are equivalent:*

(1) *S is an I_0-ring.*
(2) *For any $f \in S$ and $f \notin J(S)$, there exists a non-zero $\psi \in S$ such that $f\psi(P)$ is a nonzero direct summand of P.*
(3) *For any $f \in S$ and $f \notin J(S)$, there exists a non-zero $\psi \in S$ such that $\psi f(P)$ is a nonzero direct summand of P.*

Proof. (1)⇒(2). It suffices to proof (2)⇒(1). If $\mathrm{Im}(f\psi)$ is a nonzero direct summand of P, then it is projective. Hence the exact sequence $0 \to \mathrm{Ker}(f\psi) \to P \to \mathrm{Im}(f\psi) \to 0$ splits, and so our claim follows from Theorem 2.2. □

3. I_0-MODULES.

Let P_R be a projective module. As is well-known, $J(P)=PJ(R)$ and $P \neq PJ(R)$.

A projective module P_R is called an I_0-module if every submodule which is not contained in $J(R)$ contains a direct summand of P.

A projective module P_R is called semiperfect if every factor module of P has a projective cover.

For a finitely generated projective module, we have the following result.

Theorem 3.1. *Let P_R be a finitely generated projective module. Then the following conditions are equivalent:*

(1) *P is a semiperfect module.*
(2) *P is an I_0-module and $P/J(P)$ is semisimple.*
(3) *P is an I_0-module with maximum condition for direct summands.*
(4) *If A is a submodule of P, then $A = P_0 + D$, where P_0 is a direct summand of P and D is a submodule of $J(P)$.*

Proof. (1)⇒(2). It is easy to see that P is an I_0-module. By [5, Theorem 11.3.1] $P/J(P)$ is semisimple.

(2)⇒(3) is trivial.

(3)⇒(4). Let A be a submodule of P. We may assume that $A \not\subset J(P)$. By hypothesis, there exists a direct summands P_0 of P which maximal with respect to the property that $P_0 \subseteq P$. If $P = P_0 \oplus L$, then and $A = P_0 \oplus (L \cap A)$. Since P is an I_0-module and since P_0 is maximal with respect to the above property, we conclude that $L \cap A \subseteq J(P)$.

(4)⇒(1) follows from [5, Theorem 11.3.1]. □

Theorem 3.2. *Let R be an I_0-ring and P_R projective module. Then P is an I_0-module.*

Proof. First we prove that any finitely generated free module is an I_0-module.

Let F_R be a free module with basis $\{x_1, \cdots, x_n\}$. Then $F = x_1R \oplus \cdots \oplus x_nR$.
Let A be a submodule of F such that $A \not\subset J(F)$. Then there exists $a \in A$ such that $a \notin J(A)$. As is easily seen, $J(F) = x_1J(R) \oplus \cdots \oplus x_nJ(R)$. We put $a = x_1r_1 + \cdots + x_nr_n$, $r_i \in R (i = 1, \cdots, n)$.Without loss of generality, we may assume that $r_1 \notin J(R)$.Then there exists non-zero idempotent $e \in r_1R$.We put $e = r_1s$ for some s in R. Then $ase = x_1e + x_2r_2se + \cdots + x_nr_nse$.We can easily see that $F = aseR \oplus (x_1(1-r)R \oplus x_2R \oplus \cdots \oplus x_nR)$. Hence F is an I_0 - module. Second we prove that every free module is an I_0-module. Let G_R be a free module with basis $\{x_\lambda\}_{\lambda \in \Lambda}$ and A a submodule of G such that $A \not\subset J(F)$. Then there exists $a \in A$ and $a \notin J(F) = FJ(R)$. We put $a = x_{i1}x_{i1} + \cdots + s_{in}r_{in}$, $r_{ij} \in R$ $(j = 1, \cdots, n)$ and $G_n = x_{i1}R \oplus \cdots \oplus x_{in}R$. Since $aR \subseteq G_n$ and $aR \not\subset J(G_n)$, there exists a submodule H of aR such that it is a direct summand of G_n by first case. Also since G_n is a direct summand of G, H is a direct summand of G. Hence G is an I_0-module. Final we shall complete the proof of this theorem. Let P be a projective module. Then P is a direct summand of a free module F. We put $F = P \oplus F'$, where F' is a submodule of F. Let C be a submodule of P such that $C \not\subset J(P)$. Then $J(F) = J(P) \oplus J(F')$, and so $P \cap J(F) = J(P)$. Since $C \not\subset J(F)$, there exists a direct summand Q of F such that $Q \subseteq C$. We put $F = Q \oplus Q'$. Then $P = Q \oplus (P \cap Q')$ by modular law, that is, Q is a direct summand of P. Hence P is an I_0-module. □

Now we investigate the endomorphism ring of an I_0-module.

Lemma 3.3. *Let P_R be a projective module and $S = \mathrm{End}_R(P)$. If S is an I_0-ring, then $J(P)$ is small in P.*

Proof. From [9, Proposition 1. 1], $J(S) \subseteq \mathrm{Hom}_R(P, J(P))$. Let $f \in \mathrm{Hom}_R(P, J(P))$. If $f \notin J(S)$, then there exists a non-zero idempotent e of S such that $e \in fS$. Thus $e(P)$ is a direct summand of P. We put $P = e(P) \oplus P'$. Then $J(e(P)) = e(P) \cap J(P)$. Also since $e \in fS$, there exists $\psi \in S$ such that $e = f\psi$. Then we have $e(P) = f\psi(P) \subseteq f(P) \subseteq J(P)$ and so $e(P) = J(e(P))$. Hence $e(P) = 0$, that is, $e = 0$. This is a contradiction. Thus $f \in J(S)$ and so $J(S) = \mathrm{Hom}_R(P, J(P))$. Hence $J(P)$ is small in P by [1, Proposition 2.4]. □

Proposition 3.4. *Let P_R be a projective module and $S = \mathrm{End}_R(P)$. Then the following conditions equivalent:*

(1) *P is an I_0-module.*

(2) *If $f \in S$ such that $f \notin \mathrm{Hom}_R(P, J(P))$, then $\mathrm{Im} f$ contains a non- zero direct summand of P.*

Proof. (1)⇒(2) follows from the definition of I_0-modules. (2)⇒(1). Let A be a submodule of P such that $A \not\subseteq J(P)$. Then there exists a maximal submodule D of P which is not contained in D and so $P = A + D$. By [1, Lemma 2.2], $S = \hat{A} + \hat{D}$, where $\hat{A} = \mathrm{Hom}_R(P, A)$ and $\hat{D} = \mathrm{Hom}_R(P, D)$. Thus there exist $\psi \in \hat{A}$ and $\varphi \in \hat{D}$ such that $1 = \psi + \varphi$ and $\psi \notin \mathrm{Hom}_R(P, J(P))$. In fact, if $\psi \in \mathrm{Hom}_R(P, J(P))$, $P = \psi(P) + \varphi(P) = J(P) + D = P$. This is a contradiction. Hence $\psi \notin \mathrm{Hom}_R(P, J(P))$. By assumption, $\psi(P)$ contains a non-zero direct summand of P. Thus P is an I_0-module. □

Theorem 3.5. *Let P_R be a projective module and $S = \mathrm{End}_R(P)$. Then the following conditions are equivalent:*

(1) *P is an I_0-module, and $J(P)$ is small in P.*

(2) *If $f \in S$ such that $f \notin \mathrm{Hom}_R(P, J(P))$, then $\mathrm{Im} f$ contains a non-zero direct summand of P and $J(P)$ is small in P.*

(3) *S is an I_0-ring.*

Proof. (1)⇔(2) follows from Proposition 3.4. (2)⇒(3). Since $J(P)$ is small in P, $J(S) = \mathrm{Hom}_R(P, J(P))$ by [1, Proposition 2.4]. Let $f \in S$ and $f \notin J(S)$. Since $f(P) \not\subseteq J(P)$, $f(P)$ contains a non-zero direct summand N of P. Let e be the projection from P to N. Then $e = e^2 \in S$ and $e(P) \subseteq f(P)$. Thus $eS \subseteq fS$ by [1, Lemma 2.1], that is, S is an I_0-ring. (3)⇒(1).By Lemma 3.3, $J(P)$ is small in P and P is an I_0-module from Proposition 3.4. □

Following [10], a module M_R is called regular if for each m, there exists $f \in \mathrm{Hom}_R(M, R)$ such that $mf(m) = m$.

R.Ware gave an example of a regular module which does not have a regular endomophism ring [9, Example 3.4.].

It is well-know that the Jacobcon radical of a regular module is zero. Hence by Theorem 3.5, we have

Corollary 3.6. *Let M_R be a regular module and $S = \mathrm{End}_R(M)$. Then S is an I_0-ring and $J(S) = 0$.*

As is well-known, if P is a finitely generated module, then $J(P)$ is small in P. By Theorem 3.5, we have

Corollary 3.7. *Let P_R be a finitely generated projective module and $S = \mathrm{End}_R(P)$. Then the following conditions are equivalent:*

(1) *P is an I_0-module.*
(2) *S is an I_0-ring.*

Following [2], we call an ideal A of R left T-nilpotent if given any sequence $\{a_i\}$ of elements in A, there exists an n such that $a_1 \cdots a_n = 0$.

Theorem 3.8. *The following conditions are equivalent:*

(1) *R is an I_0-ring and $J(R)$ is left T-nilpotent.*
(2) *$\mathrm{End}_R(P)$ is an I_0-ring for each projective module P_R.*
(3) *$\mathrm{End}_R(F)$ is an I_0-ring for each free module F_R.*

Proof. (1)⇒(2). Let P_R be a projective module. Since $J(R)$ is left T-nilpotent, $J(P)$ is small in P by [5, Corollary 11.5.6]. From Theorem 3.5, $\mathrm{End}_R(P)$ is an I_0-ring. (2)⇒(1) is clear from Theorem 3.5. and [5, Theorem 11.5.5]. The proof (1)⇒(3) is analogous. □

Theorem 3.9. *Let R_1 and R_2 are rings with identities and R_1 an I_0-ring. If R_2 is Morita equivalent to R_1, then R_2 is an I_0-ring.*

Proof. Since R_2 is Morita equivalent to R_1, there exists a finitely generated projective module P as a right R_1-module such that $R_2 \cong \mathrm{End}_{R_1}(P)$. Also since R_1 is an I_0-ring, P_{R_1} is an I_0-module. Thus $\mathrm{End}_{R_1}(P)$ is an I_0-ring that is, R_2 is an I_0-ring. □

Proposition 3.10. *Let P_R be a projective module, $S = \mathrm{End}_R(P)$ and $S^* = \mathrm{End}_R(P/J(P))S$. If R is an I_0-ring, then S^* is an I_0-ring and $J(S^*) = 0$.*

Proof. By [9, Proposition 1.1], there exists a ring epimorphism $\varphi : S \to S^*$ with $\mathrm{Ker}\,\varphi = \mathrm{Hom}_R(P, J(P))$. Let $f^* \in S^*$ and $f^* \neq 0$. Then there exists $f \in S$ and $f \neq 0$ such that $\varphi(f) = f^*$. Since $f \notin \mathrm{Ker}\,\varphi, f(P) \not\subset J(P)$. Thus $f(P)$ contains a non-zero direct summand N of P. Let e be the projection from P to N. Since $e = e^2 \in S$ and $e(P) \subseteq f(P), eS \subseteq fS$ by [1, Lemma 2.1]. We put $e^* = \varphi(e)$. As is easily seen, e^* is an idempotent of S^* and $e^*S^* \subseteq f^*S^*$. Hence S^* is an I_0-ring. If $J(S^*) \neq 0$, then there exists $\varphi^* \in S^*$ and $\varphi^* \neq 0^*$. Thus there exists a non-zero idempotent in $\varphi^*S \subseteq J(S^*)$. This is a contradiction. Hence $J(S^*) = 0$. □

References

[1] G. Azumaya: F-semiperfect modules, J. Algebra, 136 (1991), 73-85.
[2] H. Bass : Finitistic dimension and a homological generalization of semi-primary rings, Trans. Amer. Math. Soc. 95(1960), 466 - 488.
[3] H. Cartan and S. Eilenberg : Homological Algebra, Princeton Univ. Press, 19(1956).
[4] C.Faith: Algebra, Rings, Modules and Categories,T.2, Springer-Verlag 190(1981).
[5] F. Kasch : Modules and Rings, London Math. Soc. Mono. 17 (1982).
[6] E. A. Mares : Semi-perfect modules, Math. Zeit. 82(1963), 347-360.

[7] W.K. Nicholson : I-rings, Trans Amer. Math. Soc. 207 (1975), 361-373.
[8] P.Pillay: On semihereditary noncommutative palynomial rings, Proc.Amer. Math.Soc. 78 (1980),473-474.
[9] R. Ware : Endomorphism ring of projective module, Trans. Amer. Math. Soc. 155 (1971), 233-256.
[10] J.Zelmanowitz : Regular modules, Trans. Amer. Math. Soc. 163 (1972), 1-18.

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