

Mathematical Journal of Okayama University

Volume 42, Issue 1

2000

Article 7

JANUARY 2000

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Abstract

Let R be a semiprime Noetherian PI-ring and $Q(R)$ the semisimple Artinian ring of fractions of R . We shall prove the following conditions are equivalent: (1) the Krull dimension of R is at most one, (2) Any ring between R and $Q(R)$ is again right Noetherian, (3) Let a, b be central regular elements of $Q(R)$. Then the subring $R + aR[b]$ of $Q(R)$ is right Noetherian.

Math. J. Okayama Univ. **42** (2000), 55-59

ON SEMIPRIME NOETHERIAN PI-RINGS

Dedicated to Professor Yukio Tsushima on his 60th birthday

KATSUO CHIBA

ABSTRACT. Let R be a semiprime Noetherian PI-ring and $\mathcal{Q}(R)$ the semisimple Artinian ring of fractions of R . We shall prove the following conditions are equivalent: (1) the Krull dimension of R is at most one, (2) Any ring between R and $\mathcal{Q}(R)$ is again right Noetherian, (3) Let a, b be central regular elements of $\mathcal{Q}(R)$. Then the subring $R + aR[b]$ of $\mathcal{Q}(R)$ is right Noetherian.

Throughout this note all rings will have a unit element. Let R be a ring. We denote by $\dim(R)$ the Krull dimension of R , i.e., the supremum of the lengths of chains of distinct prime ideals in R , and by $Z(R)$ the center of R . Cauchon showed that a semiprime PI-ring with the ascending chain condition on two-sided ideals is left and right Noetherian (See [5, II, p.174]). Therefore we simply say semiprime Noetherian PI-rings for semiprime left and right Noetherian PI-rings. Let R be a semiprime Noetherian PI-ring. As is well known, the ring of fractions of R with respect to the set of central regular elements of R is a semisimple Artinian ring (See [5, II, p.174]). The main result of this note is the following:

Theorem. *Let R be a semiprime Noetherian PI-ring and $\mathcal{Q} = \mathcal{Q}(R)$ the semisimple Artinian ring of fractions of R . Then the following conditions are equivalent:*

- (1) $\dim(R) \leq 1$.
- (2) *Any ring between R and $\mathcal{Q}(R)$ is again right Noetherian.*
- (3) *Let a, b be central regular elements of $\mathcal{Q}(R)$. Then the subring $R + aR[b]$ of $\mathcal{Q}(R)$ is right Noetherian.*

Remark. Let R and $\mathcal{Q}(R)$ be as in the above Theorem, and let T be a ring between R and $\mathcal{Q}(R)$. If T satisfies the conditions of the above theorem, then T is semiprime, therefore T is left and right Noetherian.

Let S be a ring and R a subring of S . We say that S is an *extension* of R if $S = RS^R$, where $S^R = \{s \in S ; sr = rs \text{ for all } r \in R\}$ and that S is *integral* over R if each element s of S satisfies an equation of the form $s^n + r_1s^{n-1} + r_2s^{n-2} + \cdots + r_n = 0$, where $r_i \in R$ for all $i = 1, 2, \dots, n$.

1991 *Mathematics Subject Classification.* 16R20, 16P40.

Let R be a prime PI-ring and $\mathcal{Q}(R)$ the central simple ring of fractions of R . The ring obtained by adjoining to R all elements of $Z(\mathcal{Q}(R))$ which are integral over R is called centrally integral closure of R . The proof of Theorem is similar to that in the commutative case as given by Kaplansky [2, Theorem 93 and Exercise 20, p.64]. For the proof of Theorem we need several lemmas.

Lemma 1 ([6, Theorem 1]). *Let S be a PI-ring and R a subring of S . If S is an extension of R , i.e. $S = RS^R$ and integral over R , then the following hold:*

- (1) *For any prime ideal P in R there exists a prime ideal \mathcal{Q} in S with $\mathcal{Q} \cap R = P$ (lying over).*
- (2) *For any pair of prime ideals $P \subset P_1$ in R and a prime ideal \mathcal{Q} in S with $\mathcal{Q} \cap R = P$, then there exists a prime ideal \mathcal{Q}_1 in S satisfying $\mathcal{Q} \subset \mathcal{Q}_1$ and $\mathcal{Q}_1 \cap R = P_1$ (going up).*
- (3) *Two different primes in S with the same contraction in R cannot be comparable (incomparability).*

Lemma 2 ([7, Theorem 3]). *If R is a prime PI-ring and integral over an integrally closed central subring A of R , then $A \subset R$ has going down, i.e. , given prime ideals $P_0 \subset P$ in A and a prime ideal \mathcal{Q} in R with $\mathcal{Q} \cap A = P$ then there exists a prime ideal \mathcal{Q}_0 in R satisfying $\mathcal{Q}_0 \subset \mathcal{Q}$ and $\mathcal{Q}_0 \cap A = P_0$.*

Lemma 3 ([6, Theorem 2]). *If R is a prime PI-ring with ascending chain condition on centrally generated ideals, then the coefficients of the reduced characteristic polynomial of any element of R are integral over R .*

As a corollary of Lemma 3 we shall prove the following lemma.

Lemma 4. *Let R be a Noetherian prime PI-ring with the central simple ring of fractions $\mathcal{Q}(R)$ and let R^* be its centrally integral closure, the ring obtained by adjoining to R all elements of $Z(\mathcal{Q}(R))$ which are integral over R , then:*

- (1) *R^* is integral over R .*
- (2) *R^* is integral over $Z(R^*)$.*
- (3) *$Z(R^*)$ is integrally closed in its field of fractions.*

Proof. (1) Let $x \in R^*$ then there are finite elements $t_1, \dots, t_k \in Z(\mathcal{Q}(R))$ which are integral over R and $x \in R[t_1, \dots, t_k]$. Clearly, $R[t_1, \dots, t_k]$ is a finitely generated R -module. Since R is Noetherian, x is integral over R . (2) If $x \in R^*$, then there are finite elements $t_1, \dots, t_k \in Z(\mathcal{Q}(R))$ such that $x \in R[t_1, \dots, t_k]$ as in the proof of (1). Let θ be a coefficient of the reduced characteristic polynomial of x . It is enough to show that θ is an element of $Z(R^*)$. Since $R[t_1, \dots, t_k]$ is a Noetherian prime PI-ring, by

Lemma 3, θ is integral over $R[t_1, \dots, t_k]$, hence $R[t_1, \dots, t_k, \theta]$ is a finite R -module, therefore θ is integral over R . This shows that θ is an element of $Z(R^*)$. (3) Let t be an element of $Z(\mathcal{Q}(R))$ and integral over $Z(R^*)$. Then, as in (1), t is integral over $R[t_1, \dots, t_k]$, where $t_1, \dots, t_k \in Z(\mathcal{Q}(R))$ are integral over R , hence t is integral over R , therefore, by the definition of R^* , $t \in Z(R^*)$. \square

Lemma 5. *Let R be a PI-ring. Then R is a right Noetherian ring with $\dim(R) = 0$ if and only if R is a right Artinian ring.*

Proof. Suppose R is a right Noetherian ring with $\dim(R) = 0$. Then all its prime ideals are both minimal and maximal prime ideals and there are only finitely many such prime ideals, say M_1, M_2, \dots, M_n . We have $J(R) = M_1 \cap M_2 \cap \dots \cap M_n$, where $J(R)$ is the Jacobson radical of R . Since $J(R)$ is nil, by Levitzki's theorem [1, Theorem 1.4.5], $J(R)$ is nilpotent and hence $(M_1 \cdots M_n)^k = 0$ for some k . Since R/M_i ($i = 1, 2, \dots, n$) are simple Artinian rings, we can refine the series of R -modules $R \supset M_1 \supset M_1 M_2 \supset \dots \supset M_1 \cdots M_n \supset \dots \supset 0$ and then we have a composition series of the right R -module R . Thus R is a right Artinian ring. The converse is well known. \square

Lemma 6. *Let R be a semiprime Noetherian PI-ring with finite Krull dimension $\dim(R)$. Let a be a non-unit central regular element of R . Then $\dim(R/aR) < \dim(R)$.*

Proof. Since R is a semiprime Noetherian PI-ring, R has a finite set of minimal prime ideals, say P_1, P_2, \dots, P_n . We show that the canonical image of a in the factor ring R/P_i is regular for each i . Suppose \bar{a} is not regular in R/P_i for some i , where \bar{a} denote the canonical image of a in R/P_i , then there is an element b in R such that $\bar{a}\bar{b} = 0$ and $\bar{b} \neq 0$ in R/P_i . Since $\bigcap_{j \neq i} P_j \not\subseteq P_i$, there is a non-zero central element \bar{c} in R/P_i such that $\bar{c} \in \bigcap_{j \neq i} P_j - P_i$ by [4, Theorem 2]. Then $abc \in \bigcap P_i = 0$ so $bc = 0$, implying $\bar{b} = 0$, a contradiction. Now, let $\mathcal{Q}_0 \subset \mathcal{Q}_1 \subset \dots \subset \mathcal{Q}_n$ be a chain of prime ideals of R such that $a \in \mathcal{Q}_0$, then \mathcal{Q}_0 is not a minimal prime ideal of R . This shows that $\dim(R/aR) < \dim(R)$. \square

Now we shall prove Theorem:

Proof of Theorem. (1) implies (2). Let T be a ring between R and \mathcal{Q} and let X be a right ideal of T . Then $X\mathcal{Q} \oplus Y = \mathcal{Q}$ for some right ideal Y of \mathcal{Q} . Since \mathcal{Q} is the central localization of R , we have $xa^{-1} + ya^{-1} = 1$, for suitable elements $a \in Z(R)$, $x \in X \cap R$ and $y \in Y \cap R$. Hence the right ideal $X \oplus (Y \cap T)$ of T contains a regular central element a of R . It suffices to show that T/aT is a finite generated right R -module. Since the Krull

dimension of R is at most 1, by Lemma 6 it follows that the Krull dimension of R/aR is 0. By Lemma 5, R/aR is right Artinian. Thus the descending chain of ideals $\{a^m T \cap R + aR \mid m = 1, 2, \dots\}$ in R becomes stable, say at $a^n T \cap R + aR$. For this n we assert that $T/aT \subseteq (a^{-n}R + aT)/aT$. Let $t = zc^{-1}$ be an element of T , where $z \in R$ and c is a regular central element of R . Since R/cR is right Artinian, then $a^k R \subseteq a^{k+1}R + cR$, for some k , so $a^k t = a^{k+1}rt + cr_1t$ ($r, r_1 \in R$), whence $t \in a^{-h}R + aT$ for some h . Let us suppose that the equation $t \in a^{-h}R + aT$ has been arranged with the smallest possible value of h . We shall prove that $h \leq n$. Suppose $h > n$. We write $t = a^{-h}u + at_1$, $u \in R$ and $t_1 \in T$. Then $u = a^h(t - at_1) \in a^h T \cap R \subseteq a^{h+1}T \cap R + aR$. So we can write $u = a^{h+1}t_2 + au_1$, where $t_2 \in T$ and $u_1 \in R$. Thus we have $t = a^{-(h-1)}u_1 + a(t_1 + t_2)$. This contradicts the minimal choice of h .

(2) implies (3). Trivial.

(3) implies (1). Assume first that R is prime. Let R^* be the centrally integral closure of R , then $\dim(R^*) = \dim(R)$ by Lemma 4 (1) and [6, Corollary 1, p.247]. It suffices to show that $\dim(R^*) = 1$. By Lemma 4 (3), R^* is integral over $Z(R^*)$ and $Z(R^*)$ is integrally closed. Therefore, we may assume that R is integral over $Z(R)$ and $Z(R)$ is integrally closed. If $\dim(R) > 1$ then there exist prime ideals $0 \neq Q \subset P$ in R . By Lemma 1 (3), $Z(R) \cap Q \subset Z(R) \cap P$ are distinct primes in $Z(R)$. Take $x \in Z(R) \cap Q$, $x \neq 0$. Since R is Noetherian, there are only finitely many prime ideals minimal over xR , say P_1, \dots, P_n by [3, Corollary 2.4, p.108]. By Lemma 2, $P_k \cap Z(R)$ is a minimal prime ideal over x in $Z(R)$ and thus $P \cap Z(R) \not\subseteq P_k \cap Z(R)$ for any k . Hence we have $P \cap Z(R) \not\subseteq (P_1 \cap Z(R)) \cup \dots \cup (P_n \cap Z(R))$ by [2, Theorem 81]. Take $y \in P \cap Z(R)$ with $y \notin (P_1 \cap Z(R)) \cup \dots \cup (P_n \cap Z(R))$. Let $T = R + xR[y^{-1}]$ and $I = xR[y^{-1}]$. We assert that I is not a finitely generated ideal in T . If I is a finitely generated ideal then I is generated by xy^{-i} for some i . Then we have $xR[y^{-1}] = xy^{-i}T$, and so $R[y^{-1}] = T$. Let $y^{-1} = a + xby^{-j}$, $a, b \in R$, $j \geq 1$. Then $y^{j-1}(1 - ay) = xb \in xR \subset P_k$ for all k . Therefore $1 - ay \in P_k$ for all k , and then, we have $(1 - ay)^m \in xR$ for some m . Expanding $(1 - ay)^m = xc$, $c \in R$, we have the relation $1 = yd + xc$ where $d \in R$, which leads a contradiction, $1 \in P$.

Suppose R is semiprime. Since R is Noetherian, there are finitely many minimal prime ideals, say Q_1, \dots, Q_r . Put $R_i = R/Q_i$. We show that the Krull dimension of R_i is at most 1 for any i . Let $\mathcal{Q}(R_i)$ ($i = 1, 2, \dots, r$) be the ring of fractions of R_i . Suppose $\dim(R/\mathcal{Q}_k) > 1$ for some k . By the above argument, there are regular central elements \bar{x}, \bar{y} of R_k such that the subring $R_k + \bar{x}R_k[\bar{y}^{-1}]$ in $\mathcal{Q}(R_k)$ is non-Noetherian, where \bar{x}, \bar{y} are the canonical image of $x, y (\in R)$ in R_k . Using the injections

$R \rightarrow \prod_{i=1}^r R_i \subseteq \prod_{i=1}^r \mathcal{Q}(R_i)$, the ring $\prod_{i=1}^r \mathcal{Q}(R_i)$ is considered as the ring of fractions of R (See [5, I, Theorem 3.2.27 and II, p.174]). Let $x_k = (1 + \mathcal{Q}_1, \dots, x + \mathcal{Q}_k, \dots, 1 + \mathcal{Q}_r)$, $y_k = (1 + \mathcal{Q}_1, \dots, y + \mathcal{Q}_k, \dots, 1 + \mathcal{Q}_r)$ be the elements of $\prod_{i=1}^r \mathcal{Q}(R_i)$. By the hypotheses, the subring $R + x_k R[y_k^{-1}]$ of $\prod_{i=1}^r \mathcal{Q}(R_i)$ is Noetherian and any homomorphic image of $R + x_k R[y_k^{-1}]$ is Noetherian, hence the subring $R_k + \bar{x}R[\bar{y}^{-1}]$ of $\mathcal{Q}(R_k)$ is Noetherian, which is a contradiction. This completes the proof. \square

ACKNOWLEDGEMENTS

The author thanks Professor Y. Hirano of Okayama University for his helpful suggestions.

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(Received November 15, 2000)