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On artinian rings of right local type

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ON ARTINIAN RINGS OF RIGHT LOCAL TYPE

Dedicated to Professor Hisao Tominaga on his 60th birthday

TAKESHI SUMIOKA

Let A be an artinian ring with identity. Then A is called of right local type if any finitely generated indecomposable right A-module is local and a ring of left colocal type is defined as the dual version. Yoshii [9] and Tachikawa [8] studied finite dimensional algebras of right local type (or equivalently of left colocal type). In the following theorems, for a module L $|L|$ denotes the composition length of L, $\overline{L}$ the top of L and $p(A)$ the set of primitive idempotents of A.

Theorem A (Tachikawa [8], Yoshii [9]). *For a finite dimensional algebra A with Jacobson radical J, the following conditions* (a)~(c) *are equivalent.*

(a) *A is of right local type.*

(b) *A is of left colocal type.*

(c) (1) *A is left serial.*

(2) *eJ is a serial module (i.e. a direct sum of uniserial modules) with $|\overline{eJ}| \leqq 2$ for each $e \in p(A)$.*

But in [8] and [9] they essentially used the property "A has the self duality" (and in [9] there are some mistakes; see the footnote of [8, p. 227]). On the other hand in [7], Tachikawa studied artinian rings of left colocal type (see [6]). Hence in this paper we study artinian rings of right local type as the dual notion of rings of left colocal type and in particular we prove the following (see Section 4):

Theorem B. *Let A be an artinian ring with Jacobson radical J. Consider the following conditions*:

(a) *A is of right local type.*

(b) *A is of left colocal type.*

(c) (1) *A is left serial.*

(2) *eJ is serial with $|\overline{eJ}| \leqq 2$ for any $e \in p(A)$.*

(d) (1) *A is left serial.*

(2) *J is serial as a right A-module.*

(3) *If $\overline{eJ}$ is not homogeneous, then $|\overline{eJ}| \leqq 2$, where $e \in p(A)$.*

(4) *If U is a uniserial left A-module with $|U| \geqq 2$, then $[D_1(U) : D_2(U)]_l \leqq 2$.*

Then it holds the following (i) *and* (ii) (*see Section* 4 *for the definition* $[D_1(U) : D_2(U)]_l$ *and the condition* (D)).

(i) (a) *implies* (d) *and under the condition* (D) (d) *implies* (a).

(ii) (a) *and* (b) *are satisfied if and only if so are* (c) *and* (D).

In Section 3 we give necessary and sufficient conditions for a left serial ring A to have the right serial Jacobson radical. Moreover, in Section 5 we give a simple proof of [6, Example 3] using the dual assertion to one obtained by Asashiba.

Throughout this paper A is a left and right artinian ring with identity, J is the Jacobson radical of A and all modules are finitely generated unitary right A-modules (except for Section 5) unless otherwise stated. We call a module M *serial* if M is a direct sum of uniserial modules, and call an element a of A *right local* (resp. *local*) if $a = af$ (resp. $a = eaf$) for some $e, f \in p(A)$. For a module M and a subset I of A we use the following notations (refer [6] for the other definitions and notations):

$\overline{M}$: the top of M (i.e. $\overline{M} = M/MJ$).

$Soc(M)$: the socle of M.

$|M|$: the composition length of M.

$p(A)$: the set of primitive idempotents of A.

$t(eJ^rf) = eJ^rf \setminus eJ^{r+1}f$, where r is an integer $\geqq 0$ and e and f are idempotents of A.

$loc(I)$ (resp. $loc'(I)$): the set of local (resp. right local) elements in I.

$[D : D']_l$ (resp. $[D : D']_r$): the dimension of the left (resp. right) vector space D over D', where D is a division ring and D' is a division subring of D.

The author has useful communication (in particular for Proposition 1.3 and Lemma 1.4) with Doctor H. Asashiba. In [2] Baba and Harada have independently obtained the same results as some ones in Section 2.

1. Preliminaries. Let M_i be a module and T_i its submodule; $i = 1, 2$. If a homomorphism $\theta : T_1 \to T_2$ has some extension map $\phi : M_1 \to M_2$ (i.e. $\phi(a) = \theta(a)$ for any $a \in T_1$), then θ is called (M_1, M_2)-*extendible*. On the other hand θ is called (M_1, M_2)-*maximal* if θ is not (M_1', M_2)-extendible for any submodule M_1' with $T_1 \subsetneqq M_1' \subset M_1$.

Let $(\alpha_1, \alpha_2)^T : T \to M_1 \oplus M_2$ be a monomorphism with a monomorphism α_1 (where $(\alpha_1, \alpha_2)^T$ denotes the transposed matrix of (α_1, α_2)) and put $T_i = Im\ \alpha_i$.

Clearly we have the map $\theta: T_1 \to T_2$ with $\theta\alpha_1 = \alpha_2$. Then we say that $[\alpha_1, \alpha_2]$ is *extendible* (resp. $[\alpha_1, \alpha_2]$ is *maximal*) provided θ is (M_1, M_2)-extendible (resp. (M_1, M_2)-maximal). The following lemma is immediate from [6, Lemma 1.2 (1)] (see [6, Remark 1]).

Lemma 1.1. *Let $0 \to T \xrightarrow{\alpha} M_1 \oplus M_2 \to M \to 0$ be an exact sequence of modules with $\alpha = (\alpha_1, \alpha_2)^T$ and α_1 a monomorphism. If $[\alpha_1, \alpha_2]$ is extendible, then $M \cong (M_1/Im\ \alpha_1) \oplus M_2$. In case α_2 is also a monomorphism, the converse holds.*

Let M be a module and S a non-empty subset of M. If $S \neq 0$ (resp. $S = 0$), we put $d_M(S) = \max\{n \mid S \subset MJ^n\}$ (resp. $d_M(S) = \infty$) and we call $d_M(S)$ the *depth of S in M*. In particular for $M = A$, we simply denote $d_M(S)$ by $d(S)$ and call $d(S)$ the *depth of S* (cf. [3, p. 76] for the definition of "depth").

We define a linear order " $<$ " for the set $\mathbf{Z} \times \mathbf{Z}$ (*where* $\mathbf{Z}$ is the set of integers) by the following:

$(r, s) < (r', s')$ if (1) $r < r'$ or (2) $r = r'$ and $s \geqq s'$.

For a local module L with a simple submodule S, we call (L, S) a *local-simple set*. We denote the isomorphism class of S by $C(S)$. Then for a local-simple set (L, S) we define its *degree* $deg(L, S)$ by setting $deg(L, S) = (C(S), d_L(S), |L|)$ and give a partial order " $\leqq$ " to the set $\{deg(L, S) \mid (L, S)$ is a local-simple set$\}$ as follows:

$deg(L, S) \leqq deg(L', S')$ if $C(S) = C(S')$ and $(d_L(S), |L|) < (d_{L'}(S'), |L'|)$.

For convenience we define $deg(L, S) < deg(L', 0)$ for any local module L'.

Lemma 1.2. *Let (L_i, S_i) be a local-simple set; $i = 1, 2$, and $\theta: S_1 \to S_2$ an isomorphism. If θ is extended to a map $\phi: L_1 \to L_2$, then the following hold.*

(1) $deg(L_1, S_1) \leqq deg(L_2, S_2)$.

(2) *In the case $d_{L_1}(S_1) = d_{L_2}(S_2)$, ϕ is an epimorphism.*

(3) *In the case $deg(L_1, S_1) = deg(L_2, S_2)$, ϕ is an isomorphism.*

Proof. (1) and (2). Put $n = d_{L_1}(S_1)$. Then $S_1 \subset L_1J^n$, so $S_2 = \phi(S_1) \subset \phi(L_1)J^n \subset L_2J^n$. Thus we have $d_{L_1}(S_1) \leqq d_{L_2}(S_2)$. If ϕ is not an epimorphism, then we have $\phi(L_1) \subset L_2J$, for L_2 is local. Hence $S_2 \subset \phi(L_1)J^n \subset L_2J^{n+1}$, which implies $d_{L_2}(S_2) \geqq n+1$. It follows (2) and (1).

(3) ϕ is an epimorphism by (2) and we have $|L_1| = |L_2|$. Thus ϕ is an isomorphism.

Remark 1. Let (L_i, S_i); $i = 1, 2$, be local-simple sets with $deg(L_1, S_1) = deg(L_2, S_2)$. If an isomorphism $\theta: S_1 \to S_2$ is not (L_1, L_2)-extendible, then $\theta^{-1}: S_2 \to S_1$ is not (L_2, L_1)-extendible by the above lemma.

Proposition 1.3. *Let A be an artinian ring. Then the following conditions are equivalent.*

(1) *A is of right local type.*

(2) *For any local-simple sets (L_i, S_i); $i = 1, 2$, with $deg(L_1, S_1) \leqq deg(L_2, S_2)$, any isomorphism $\theta: S_1 \to S_2$ is (L_1, L_2)-extendible.*

Proof. (1) $\Rightarrow$ (2). It is immediate from [6, Lemma 2.1].

(2) $\Rightarrow$ (1). Let C be the class of modules which are direct sums of local modules. Every module is a homomorphic image of a module in C. Hence we show that C is closed under taking factor modules. In order to do it, as easily seen, it suffices to show the following: If $L = \bigoplus_{i=1}^{n} L_i$; each L_i is a local module, i.e. L is in C, and S is a simple submodule of L, then L/S is also in C. Let $L = \bigoplus_{i=1}^{n} L_i$ and S be as above and $\alpha = (\alpha_1, \ldots, \alpha_n)^T: S \to L_1 \oplus \cdots \oplus L_n$ the inclusion map and put $S_i = \alpha_i(S)$. If we pick out j such that $deg(L_j, S_j) \leqq deg(L_i, S_i)$ for any $i = 1, \ldots, n$, then $[\alpha_j, \alpha_i]$ is extendible; $i = 1, \ldots, n$. Therefore $[\alpha_j, \alpha_j']$ is extendible where $\alpha_j': S \to \bigoplus_{i \neq j} L_i$ is the map induced from $\alpha: S \to \bigoplus_{i=1}^{n} L_i$, and by Lemma 1.1, L/S is in C.

Remark 2. For any local-simple sets (L_i, S_i); $i = 1, 2$, with $S_1 \cong S_2$, we have $deg(L_1, S_1) \leqq deg(L_2, S_2)$ or $deg(L_1, S_1) \geqq deg(L_2, S_2)$. Hence Proposition 1.3 implies the following: An artinian ring is of right local type if and only if for any monomorphism $(\alpha_1, \alpha_2)^T: S \to L_1 \oplus L_2$, $[\alpha_1, \alpha_2]$ or $[\alpha_2, \alpha_1]$ is extendible. This proposition was independently proved by H. Asashiba.

Let L_i be a local module satisfying the following condition (i) for each $i = 1, 2, 3$:

(1) $|\overline{L_1 J}| = 1$, $L_1 J^2 = U_1 \oplus U_2$, where U_i $(i = 1, 2)$ are simple modules.

(2) $L_2 J = V_1 \oplus V_2 \oplus V_3$, where V_i $(i = 1, 2, 3)$ are simple modules

with $V_2 \not\cong V_3$.

(3) $L_3J = W_1 \oplus W_2 \oplus W_3$, where W_i $(i = 1, 2)$ are simple modules and W_3 is a uniserial module with $|W_3| = 2$.

Then the local module L_i induces local-simple sets (L_i', S_i') and (L_i'', S_i'') with a canonical isomorphism $\theta_i : S_i' \to S_i''$ for each $i = 1, 2, 3$ as follows (see the diagram below):

(1') $L_1' = L_1J/U_2$, $S_1' = (U_1 \oplus U_2)/U_2 (\cong U_1)$; $L_1'' = L_1$, $S_1'' = U_1$,

(2') $L_2' = L_2/V_3$, $S_2' = (V_1 \oplus V_3)/V_3 (\cong V_1)$; $L_2'' = L_2/V_2$, $S_2'' = (V_1 \oplus V_2)/V_2 (\cong V_1)$,

(3') $L_3' = L_3/W_3J$, $S_3' = (W_1 \oplus W_3J)/W_3J (\cong W_1)$; $L_3'' = L_3/W_2$, $S_3'' = (W_1 \oplus W_2)/W_2 (\cong W_1)$,

and $\theta_i : S_i' \to S_i''$ $(i = 1, 2, 3)$ are canonical isomorphisms.

As easily seen it holds

(1'') $d_{L_1'}(S_1') < d_{L_1''}(S_1'')$ and the uniserial module L_1' can not be embedded in L_1''.

(2'', 3'') $deg(L_i', S_i') = deg(L_i'', S_i'')$ and $L_i' \not\cong L_i''$ for each $i = 2, 3$.

Hence θ_i is not (L_i', L_i'')-extendible for each $i = 1, 2, 3$ (see Lemma 1.2).

For each $i = 1, 2, 3$, we say a module L is of type-i if L is a local module which has the same form as the above L_i. Now summarizing the above

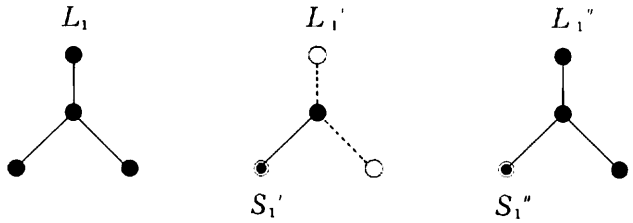
Lemma 1.4. *If there exists a module L_i of type-i, then L_i induces local-simple sets (L_i', S_i'), (L_i'', S_i'') and an isomorphism $\theta_i : S_i' \to S_i''$ such that θ is not (L_i', L_i'')-extendible and*

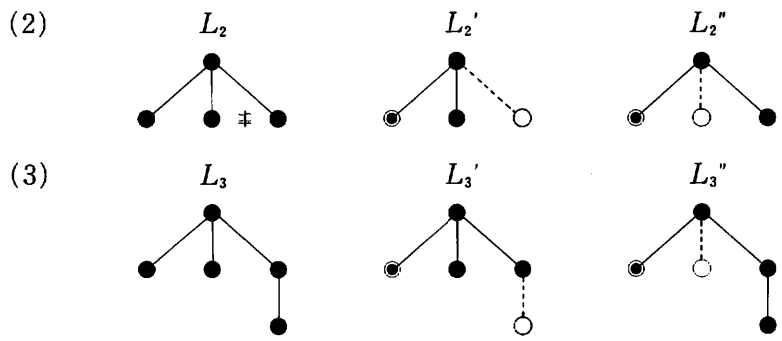
(1) $d_{L_1'}(S_1') < d_{L_1''}(S_1'')$ *(in the case $i = 1$).*

(2) $deg(L_i', S_i') = deg(L_i'', S_i'')$ *(in the case $i = 2, 3$).*

Remark 3. The assertion for $i = 3$ in the above lemma was communicated by H. Asashiba. He has independently given results [1, Lemmas 1.4 and 1.7] which include Lemma 1.4. We represent the above modules L_i, L_i', S_i' etc. and the isomorphisms $\theta_i : S_i' \to S_i''$ as the following diagrams:

(1)





Lemma 1.5. *If A is of right local type, then A is left serial.*

Proof. Let $u_i \in t(e_iJ^{r-1}f)$ $(e_i, f \in p(A))$: $i = 1,2$ for $r \geqq 2$, and α_i : $\overline{fA} \to e_iA/e_iJ^r$ the monomorphism such that $\alpha_i(x+fJ) = u_ix+e_iJ^r$ for each $x+fJ \in \overline{fA}$. By Remark 2 and [6, Lemma 1.4] $\overline{u}_2 \in A\overline{u}_1$ or $\overline{u}_1 \in A\overline{u}_2$ where $\overline{u}_i = u_i+J^rf \in J^{r-1}f/J^rf$. This shows the left A-module Af is uniserial.

2. Left serial rings. In this section we prove some results for left serial rings which are used later.

If u is a right local element with $u = uf$ $(f \in p(A))$, then uA is a local right A-module with $\overline{uA} \cong \overline{fA}$, and conversely if L is a local module with $\overline{L} \cong \overline{fA}$, then we can take such a right local element u as a generator of L. Hence for every local module L, we always take such a generator, and moreover for a several local modules L_i $(i = 1,\ldots,n)$ with $\overline{L_i} \cong \overline{fA}$ we take generators $u_i = u_if$ on a common primitive idempotent f. Let $a = ae$ and $b = bf$ be elements in A ; $e,f \in p(A)$. If we say that c is a local element with $a = bc$, then we mean $c = fce$, and in a similar case we take such elements. The following lemma is obvious.

Lemma 2.1. *Let u and v be elements of A.*

(1) *If $Au \supset Av$ (resp. $Au = Av$), then there exists an epimorphism (resp. isomorphism) $\hat{a} : uA \to vA$ which is a left multiplication map by some element a of A.*

(2) *If Au and Av are submodules of a uniserial left ideal of A with $d(u) \leqq d(v)$ (resp. $d(u) = d(v)$), then it holds $Au \supset Av$ (resp. $Au = Av$).*

Remark 4. Let A be left serial. If L_1 and L_2 are local right ideals with $\overline{L_1} \cong \overline{L_2}$ and $d(L_1) = d(L_2)$, then $L_1 \cong L_2$ by Lemma 2.1. As a

generalization of this assertion we have (2) in Lemma 2.2.

Lemma 2.2. *Let A be a left serial ring.*

(1) *If (L_i, S_i) are local-simple sets $(i = 1, 2)$ with $S_1 \cong S_2$ and $d_{L_1}(S_1) = d_{L_2}(S_2) = r$, then $\overline{L_1} \cong \overline{L_2}$.*

(2) *Let $u_i \in loc(A)$ with $d(u_1) = d(u_2)$. If $\overline{u_iJ^r}$ $(i = 1, 2)$ have simple submodules S_i such that $S_1 \cong S_2$ for some $r \geqq 0$, then $u_1A \cong u_2A$.*

Proof. (1) Let $\overline{L_i} \cong \overline{e_iA}$ $(i = 1, 2)$ and $S_1 \cong S_2 \cong \overline{fA}$ where e_i, $f \in p(A)$. Then we have an epimorphism $\phi_i : e_iA \to L_i$ for each i. Since $S_i \subset L_iJ^r = \phi(e_iJ^r)$ and $S_i \not\subset L_iJ^{r+1} = \phi(e_iJ^{r+1})$, there exists a local element $x_i = e_ix_if$ with $d(x_i) = r$. Hence $Ae_1x_1f = Ae_2x_2f$, which implies $Ae_1 \cong Ae_2$, i.e. $e_1A \cong e_2A$.

(2) is immediate from (1) and Remark 4.

The following lemma is a slight modification of [6, Lemma 4.1].

Lemma 2.3. *Let A be a left serial ring, M a direct summand of the right ideal J^r for an integer $r \geqq 0$. Let $u_1, \ldots, u_n$ be right local elements such that $u_i \in M$ and $d(u_i) = r$ for each $i = 1, \ldots, n$, and denote by $\bar{a}$ the residue class of a in M/MJ for any $a \in M$. Then the following hold.*

(1) *If $\overline{M} = \bar{u}_1A \oplus \cdots \oplus \bar{u}_nA$, then $M = u_1A \oplus \cdots \oplus u_nA$.*

(2) *If $\bar{u}_1A, \ldots, \bar{u}_nA$ are independent, then $\overline{M} = \bar{u}_1A \oplus \cdots \oplus \bar{u}_nA \oplus \bar{v}_1A \oplus \cdots \oplus \bar{v}_mA$ with some right local elements v_i in M; $i = 1, \ldots, m$.*

Proof. (1) The canonical epimorphism $\sigma : J^r \to J^r/J^{r+1}$ induces the isomorphism $\overline{M} \to \sigma(M)$, and so $\overline{M} = \bar{u}_1A \oplus \cdots \oplus \bar{u}_nA$ if and only if $\sigma(M) = \sigma(u_1)A \oplus \cdots \oplus \sigma(u_n)A$. Hence if $\overline{M} = \bar{u}_iA \oplus \cdots \oplus \bar{u}_nA$, then $u_1A, \ldots, u_nA$ are independent by [6, Lemma 4.1] and $M = u_1A \oplus \cdots \oplus u_nA$, for MJ is small in M. (2) It is clear from semi-simplicity of $\overline{M}$.

Let M be a direct summand of the right module J^r and L a submodule of M. Then L is a submodule of MJ^s which is a direct summand of J^{r+s}, and $d(L) = d(MJ^s) = r+s$, where $s = d_M(L)$. Hence it suffices to assume $d(L) = d(M)$ when we consider such a submodule L of M. As an immediate consequence of the above lemma we have

Corollary 2.4. *Let A, M and r be as in Lemma 2.3 and L_i $(i = 1, 2)$ local submodules of M with $d(L_1) = d(L_2) = r$.*

(1) *L_i is a direct summand of M.*

(2) *If* $L_1 \cap L_2 \neq 0$ *and* $M = L_1 \oplus N$ *for a module* N, *then* $M = L_2 \oplus N$, *in particular* $L_1 \cong L_2$.

Lemma 2.5. *Let* A *be a left serial ring and* M *a direct summand of the right ideal* J^r *for an integer* $r \geq 0$. *If* w *is a right local element in* M *with* $d_M(w) = s$, *then there exists a right local element* u *in* M *such that* $d(u) = r$ *and* $w \in uJ^s$.

Proof. Let $w = wf$; $f \in p(A)$. Since $J^s f$ is a uniserial left A-module, we have $J^s f = Aa$ for some $a \in t(eJ^s f)$; $e \in p(A)$. Hence $w \in MJ^s f = Ma$, and so $w = ua \in uJ^s$ for some $u = ue \in M$. Then $d(u) = r$ is clear, for $w \notin J^{r+s+1}$.

Lemma 2.6. *Let* u *and* v *be elements of* A. *Then the following conditions hold.*

(1) $d(u+v) \geq min\{d(u), d(v)\}$.

(2) *If* $d(u) < d(v)$, *then* $d(u+v) = d(u)$.

(3) *If* $J^r = M_1 \oplus M_2$, $u \in M_1$ *and* $v \in M_2$ *for some integer* $r \geq 0$ *and some modules* M_1 *and* M_2, *then* $d(u+v) = min\{d(u), d(v)\}$.

Proof. (1) and (2) are clear.

(3) If $u+v \in J^{r+s}$, then $u+v \in M_1J^s \oplus M_2J^s$. Hence $u \in M_1J^s \subset J^{r+s}$ and similarly $v \in J^{r+s}$.

Lemma 2.7. *Let* A *be a left serial ring,* $u \in t(J^r e)$ *and* $v \in t(eJ^s f)$, *where* e, $f \in p(A)$. *Then it holds* $Auv = J^{r+s} f$. *In particular*

(1) *If* $uv \neq 0$, *then* $uv \in t(J^{r+s} f)$ (*i.e.* $d(uv) = d(u) + d(v)$).

(2) *If* $uv = 0$, *then* $u'v' = 0$ *for any* $u' \in J^r e$ *and* $v' \in eJ^s f$.

Proof. Since A is left serial, $w \in t(J^n g)$ if and only if $Aw = J^n g \neq 0$ for any $g \in p(A)$. Thus it follows from $Au = J^r e$ and $Av = J^s f$ that $Auv = J^r v = J^{r+s} f$. The other assertions are immediate from this.

Corollary 2.8. *Let* A *be a left serial ring and assume* $eJ^r = M_1 \oplus \cdots \oplus M_n$ *for an integer* $r \geq 1$, *where each* M_i *is a right ideal and* $e \in p(A)$. *Then for any element* $a \in A$, *it holds* $aeJ^r = aM_1 \oplus \cdots \oplus aM_n$.

Proof. We may clearly assume that $a = ae$ and $n \geq 2$. If $ax_1 + \cdots + ax_n = 0$ and $ax_j \neq 0$ for some j $(1 \leq j \leq n)$, where $x_i = ex_i \in M_i$ $(i = 1, \ldots, n)$, then we have $a(x_1 f + \cdots + x_n f) = ax_1 f + \cdots + ax_n f = 0$ and $ax_j f \neq$

0 for some $f \in p(A)$. But by (3) in Lemma 2.6, $d(x_1f+\cdots+x_nf) \leqq d(x_jf)$. This is a contradiction by (2) in Lemma 2.7.

3. Left serial rings with right serial Jacobson radicals. The following lemma is immediate from Corollary 2.4 and Lemma 2.2.

Lemma 3.1. *Let A be a left serial ring, and M a serial direct summand of the right module J.*

(1) *Any local submodule of M is uniserial.*

(2) *If u and v are right local elements of M such that $d(u) = d(v)$ and $\overline{uJ^r} \cong \overline{vJ^r} \neq 0$ for some integer $r \geqq 0$, then $uA \cong vA$.*

Lemma 3.2. *Let A be a left serial ring such that the right ideal J is serial, and let $eJ = u_1A \oplus \cdots \oplus u_nA$ $(n \geqq 2)$ with local elements u_i ; $e \in p(A)$.*

(1) *If $a \in J$ and $au_1 \neq 0$, then $au_i = 0$ for any $i = 2, \ldots, n$.*

(2) *If $u_1A \cong u_2A$, then $Ju_1 = 0$. Therefore it holds $JeJ = 0$ in the case $u_1A \cong u_iA$ for each $i = 2, \ldots, n$.*

(3) *Assume $u_1A \cong u_2A$. If $\overline{u_1J^{r-1}} \cong \overline{fA}$; $f \in p(A)$, then it holds $J^{r+1}f = 0$ (i.e. $r = |Jf|$). In particular $\overline{u_1J^s} \cong \overline{u_1J^{s'}}$ $(\cong \overline{u_2J^{s'}}) \neq 0$ implies $s = s'$.*

Proof. (1) We may assume $a = ae$. By Corollary 2.8 $aeJ = au_1A \oplus \cdots \oplus au_nA$. But by Lemma 3.1 aeA is uniserial and so is aeJ. It follows (1).

(2) If $\overline{u_1A} \cong \overline{u_2A} \cong \overline{fA}$, we may assume $u_i = u_if$ for some $f \in p(A)$; $i = 1,2$. Then $Au_1 = Au_2 = Jf$. Suppose $Ju_1 \neq 0$. Then $au_1 \neq 0$ for some local element a of A, so $au_2 \neq 0$ by (2) of Lemma 2.7. This is a contradiction to (1).

(3) We have $J^{r+1}f = Ju_1J^{r-1}f = 0$ by (2) since $J^rf = Au_1J^{r-1}f$.

Lemma 3.3. *Let A be a left serial ring, M a direct summand of J_A, and I a submodule of M. If M is serial, then there exist right local elements $u_1, \ldots, u_n$ such that $M = u_1A \oplus \cdots \oplus u_nA$ and $I = u_1J^{r_1} \oplus \cdots \oplus u_nJ^{r_n}$ for some integers $r_1, \ldots, r_n \geqq 0$.*

Proof. Assume $I \neq 0$. Then we have $I \subset MJ^r$ and $I \not\subset MJ^{r+1}$ for some $r \geqq 0$, and we can choose a right local element w in I such that $w \notin MJ^{r+1}$.

It is immediate from Lemma 2.5 and Corollary 2.4 that $M = uA \oplus L$ and $wA = uJ^r$ for some $u \in loc'(A)$ and some right module L. It follows $I = uJ^r \oplus (L \cap I)$ from $I \subset MJ^r = uJ^r \oplus LJ^r \subset uJ^r \oplus L$. Since L is a direct summand of M, we can apply the same argument to L and $(L \cap I)$. Hence by iterating this, our assertion is verified.

Lemma 3.4. *Let A be a left serial ring, M a direct summand of J_A, and I and K submodules of M with a simple module K/I. If M is serial, then there exist right local elements $u_1, \ldots, u_n$ such that $M = u_1A \oplus \cdots \oplus u_nA$, $K = u_1J^{r_1-1} \oplus u_2J^{r_2} \oplus \cdots \oplus u_nJ^{r_n}$ and $I = u_1J^{r_1} \oplus u_2J^{r_2} \oplus \cdots \oplus u_nJ^{r_n}$ for some integers $r_1, \ldots, r_n$.*

Proof. By Lemma 3.3 there exist $u_i \in loc'(A)$ $(i = 1, \ldots, n)$ such that $M = u_1A \oplus \cdots \oplus u_nA$, $I = u_1J^{r_1} \oplus \cdots \oplus u_nJ^{r_n}$. We may assume $r_i \geqq 1$ and $u_iJ^{r_i-1} \neq 0$. Since K/I is simple and in particular $K/I \subset Soc(M/I)$, we have $K \subset u_1J^{r_1-1} \oplus \cdots \oplus u_nJ^{r_n-1}$ and $K = vA + I$ for some right local element $v = u_1x_1 + \cdots + u_nx_n$ where $x_i \in loc(J^{r_i-1})$. By exchanging the indices i, we may assume that $u_ix_i \notin I$ (i.e. $d(x_i) = r_i - 1$) if and only if $i \in \{1, \ldots, m\}$. Put $v' = u_1x_1 + \cdots + u_mx_m$, $M' = u_1A \oplus \cdots \oplus u_mA$, $I' = u_1J^{r_1} \oplus \cdots \oplus u_mJ^{r_m}$ and $K' = v'A + I'$. Considering M', I' and K' instead of M, I and K respectively, we may assume $u_ix_i \notin I$ for any i; $1 \leqq i \leqq n$, since $K = K' \oplus (\bigoplus_{i=m+1}^{n} u_iJ^{r_i})$. If $d(x_j) \leqq d(x_i)$ for any i, say $j = 1$, then x_i is in the uniserial module Ax_1, so $x_i = a_ix_1$ for $a_i \in loc(A)$ where $i = 2, \ldots, n$. Putting $u = u_1 + u_2a_2 + \cdots + u_na_n$, we have $d(u) = 1$ and $v = u_1x_1 + \cdots + u_nx_n = (u + u_2a_2 + \cdots + u_na_n)x_1 = ux_1$. If $uy_1 + u_2y_2 + \cdots + u_ny_n = 0$ for some $y_i \in A$, then $u_1y_1 + \sum_{i=2}^{n} u_i(a_iy_1 + y_i) = 0$, so $u_1y_1 = 0$ and by Lemma 2.7 $uy_1 = 0$. This shows $M = uA \oplus u_2A \oplus \cdots \oplus u_nA$. Since $v = u_1x_1 + \cdots + u_nx_n = ux_1$ and $d(x_i) = r_i - 1$ for any i $(1 \leqq i \leqq n)$, we have $I = u_1J^{r_1} \oplus \cdots \oplus u_nJ^{r_n} = u_1x_1J \oplus \cdots \oplus u_nx_nJ = ux_1J \oplus u_2x_2J \oplus \cdots \oplus u_nx_nJ = uJ^{r_1} \oplus u_2J^{r_2} \oplus \cdots \oplus u_nJ^{r_n}$ and $K = vA + I = ux_1A + I = uJ^{r_1-1} \oplus u_2J^{r_2} \oplus \cdots \oplus u_nJ^{r_n}$.

Let (L_i, S_i) $(i = 1, 2)$ be local-simple sets with $L_i \neq S_i$ and assume $\theta : S_1 \to S_2$ is an isomorphism. Then by Lemma 3.4, we may assume the following situation for $i = 1, 2$:

(St1) $L_i = e_iA/I_i$ $(e_i \in p(A))$, $S_i = K_i/I_i$, $e_iJ = u_iA \oplus N_i$, $K_i = x_iA \oplus M_i$ and $I_i = x_iJ \oplus M_i$, where $u_i \in t(e_iJf_i)$ $(f_i \in p(A))$, $x_i = e_ix_ig \in u_iA$ $(g \in p(A))$, $M_i \subset N_i$.

Under the situation, we have $\theta(x_1+I_1) = x_2c+I_2$ for some unit element c in gAg, since $\theta(S_1) = S_2$ is simple. Then $x_2cA = x_2A$, so we may assume the following situation by exchanging x_2c with x_2 :

(St2) $\theta(x_1+I_1) = x_2+I_2$ for $x_i+I_i \in K_i/I_i$ $(i = 1, 2)$.

Since e_iA is the projective cover of L_i for $i = 1, 2$, θ is (L_1, L_2)-extendible if and only if there exists an element $a \in e_2Ae_1$ such that $ax_1-x_2 \in I_2$ and $aI_1 \subset I_2$. But if $ax_1-x_2 \in I_2$, then $ax_1J \subset I_2$. Hence θ is (L_1, L_2)-extendible if and only if the following condition holds :

(E) There exists an element $a \in e_2Ae_1$ such that $ax_1-x_2 \in I_2$ and $aM_1 \subset I_2$.

Therefore we have the following lemma.

Lemma 3.5. *Let A be a left serial ring, (L_i, S_i) $(i = 1,2)$ local-simple sets with $L_i \ncong S_i$ and $\theta : S_1 \to S_2$ be an isomorphism, and assume the situations* (St1) *and* (St2). *Then θ is (L_1, L_2)-extendible if and only if it holds the condition* (E).

Theorem 3.6. *For a left serial ring A, the following conditions are equivalent.*

(1) *J is serial as a right module.*

(2) *For any local module L, LJ is serial.*

(3) *There exist no modules of type-*1.

(4) *If (L_i, S_i) $(i = 1, 2)$ are local-simple sets with $d_{L_1}(S_1) < d_{L_2}(S_2)$ and $S_1 \cong S_2$, then any isomorphism $\theta : S_1 \to S_2$ is (L_1, L_2)-extendible.*

(5) *For any uniserial left A-modules U_i $(i = 1, 2)$ with socles S_i and $|U_1| \leqq |U_2|$, any isomorphism $\theta : S_1 \to S_2$ is (U_1, U_2)-maximal or (U_1, U_2)-extendible.*

Proof. The equivalence of (1) and (5) follows from [6, Lemma 4.3]. The implication from (4) to (3) is immediate from Lemma 1.4. (1) is a special case of (2) and the converse is obtained by Lemma 3.3.

(3) $\Rightarrow$ (1). Assume eJ is not serial for some $e \in p(A)$. Since eJ is a direct sum of local modules, there exist a local module K and a module M such that $eJ = K \oplus M$, K/KJ^r is uniserial and $\overline{KJ}^r$ is not simple for some $r \geqq 1$. Thus we have a module L of type-1 as a factor module of KJ^{r-2}/KJ^{r+1} (resp. $eA/(KJ^2 \oplus M)$) in the case $r > 1$ (resp. in the case $r = 1$).

(1) $\Rightarrow$ (4). Let (L_i, S_i) $(i = 1,2)$ be as in (4). Then as easily seen we may assume that (L_i, S_i) $(i = 1, 2)$ are fulfilled the situations $L_i \ncong S_i$,

(St1) and (St2) in Lemma 3.5 since $d_{L_1}(S_1) < d_{L_2}(S_2)$. Then Ax_1 and Ax_2 are submodules of the uniserial module Ag with $d(x_1) = d_{L_1}(S_1) < d_{L_2}(S_2) = d(x_2)$, which implies $Jx_1 \supset Ax_2$. Hence there exists an element $a \in e_2Je_1$ with $ax_1 = x_2$. Then $au_1A \supset ax_1A = x_2A \neq 0$, and by Lemma 3.2 aM_1 $(\subset aN_1) = 0$. Thus by Lemma 3.5, θ is (L_1, L_2)-extendible.

4. Artinian rings of right local type. For any ring B with Jacobson radical $J(B)$, we denote $B/J(B)$ and the residue class $b+J(B) \in B/J(B)$ by $\overline{B}$ and $\overline{b}$, respectively.

Let $v \in t(eJf)$; $e,f \in p(A)$. Put $D_1(v) = \overline{eAe}$, $D_2(v) = \overline{fAf}$ and $D_1'(v)$ $= \{\overline{a} \in \overline{eAe} \mid av \in vA\}$, $D_2'(v) = \{\overline{b} \in \overline{fAf} \mid vb \in Av\}$. Then we have a ring isomorphism ϕ: $D_1'(v) \to D_2'(v)$; $\phi(\overline{a}) \to (\overline{b})$ such that $av = vb$; $a \in eAe$ and $b \in fAf$ (see [8, Section 3] and [4, Section 1] for the definition of division rings $D_i(v)$ and $D_i'(v)$; $i = 1, 2$). We identify $D_1'(v)$ with $D_2'(v)$ through the isomorphism ϕ: $D_1'(v) \to D_2'(v)$.

Assume A is left serial. Then it is clear $D_2(v) = D_2'(v)$ and in particular we have $D_2(v)$ $(= D_2'(v) = D_1'(v)) \subset D_1(v)$. Moreover, putting $U = Af/Jv$, we can identify $D_1(v)$ and $D_2(v)$ with division rings $D_1(U)$ and $D_2(U)$ defined in [6], respectively (see [6], Section 3). We quote the following lemma from [6, Lemma 3.2].

Lemma 4.1. *Let A be a left serial ring and $v \in t(eJf)$; $e, f \in p(A)$. Then $[D_1(v) : D_2(v)]_r \geqq n$ if and only if $\overline{eJ}$ contains $\overline{fA}^n$ (i.e., the direct sum of n-copies of $\overline{fA}$).*

Remark 5. Let A be a left serial ring and e a primitive idempotent of A. Consider the following conditions :

(1) $|\overline{eJ}| \leqq 2$.

(2) $|\overline{eJ}| \leqq 2$ provided $\overline{eJ}$ is not homogeneous.

(3) $[D_1(U) : D_2(U)]_r \leqq 2$ for any uniserial left A-module U with $|U| \geqq 2$ and $Soc(U) \cong \overline{Ae}$.

Then (1) is satisfied if and only if so are (2) and (3) by Lemma 4.1 (see [6, Lemma 3.2]).

Remark 6. Let $v \in t(eJf)$; $e,f \in p(A)$. If $\overline{a} \in D_2(v)$ for an element $a \in eAe$, then $av \in vA+J^2$ by the definition of $D_2(v)$. In case A is left serial, conversely we assume $av \in vA+J^2$, i.e., $av = vb+c$ for some $b = bf \in A$ and $c \in J^2$. Since $d(vb) \geqq d(v)$ and Avb and Av are submodules of a

uniserial module Af, we have $a'v = vb$ for some $a' \in eAe$. Thus $(a-a')v = c \in J^2$, and so $(a-a') \in J$. Hence $\overline{a} = \overline{a}' \in D_2(v)$. This shows that we may assume $J^2 = 0$ when we consider $[D_1(v) : D_2(v)]_l$ (or $[D_1(v) : D_2(v)]_r$) for any left serial ring A.

The following lemma is essentially shown in the proof of [6, Example 4] (cf. Examples in Section 5).

Lemma 4.2. *Let A be a left serial ring with $J^2 = 0$, and $v \in t(eJf)$; $e, f \in p(A)$. Then the following are equivalent.*

(1) $[D_1(v) : D_2(v)]_l \leqq n$.

(2) *If $L_i = eA/vA$ $(i = 1, \ldots, n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)^T : vA \to \bigoplus_{i=1}^{n} L_i$ is a monomorphism, then $[\alpha_j', \alpha_j]$ is extendible for some j, where $\alpha_j' : vA \to \bigoplus_{i \neq j} L_i$ is the map induced from α.*

Proof. Since A is left serial, the map $\alpha_i : vA \to L_i$ is a left multiplication map by some element $b_i \in eAe$: $\alpha_i(v) = b_i v + vA \in L_i$. Therefore it suffices to show the following equivalence : $\overline{b}_n \in D_2(v) + D_2(v)\overline{b}_1 + \cdots + D_2(v)\overline{b}_{n-1}$, where $\overline{b}_i = b_i + eJe \in D_2(v)$ (i.e. $\overline{b}_n - (\overline{a}_1\overline{b}_1 + \cdots + \overline{a}_{n-1}\overline{b}_{n-1}) \in D_2(v)$ for some $\overline{a}_i \in D_2(v)$; $i = 1, \ldots, n-1$) if and only if $[\alpha_n', \alpha_n]$ is extendible. But this is easily seen, since we can identify $D_2(v)$ with $\mathrm{End}(eA/vA)$.

Proposition 4.3. *Let A be an artinian ring of right local type. Then the following hold:*

(1) *A is a left serial ring.*

(2) *J is serial as a right A-module.*

(3) *If $\overline{eJ}$ is not homogeneous, then $|\overline{eJ}| \leqq 2$, where $e \in p(A)$.*

(4) *If U is a uniserial left A-module with $|U| \geqq 2$, then $[D_1(U) : D_2(U)]_l \leqq 2$.*

Proof. (1) follows from Lemma 1.5. For any local-simple sets (L_i, S_i) $(i = 1, 2)$ with $deg(L_1, S_1) \leqq deg(L_2, S_2)$, any isomorphism $\theta : S_1 \to S_2$ is (L_1, L_2)-extendible by Proposition 1.3. Hence (2) follows from Theorem 3.6. Moreover (3) is immediate from Lemma 1.4. Next A/J^2 is also of right local type, and so (4) follows from Lemma 4.2.

Proposition 4.4. *Let A be an artinian ring with the Jacobson radical J. If A satisfies the following conditions (1)~(4), then A is of right local type.*

(1) *A is a left serial ring.*

(2) *J is serial as a right A-module.*

(3) $|\overline{eJ}| \leqq 2$ *for any* $e \in p(A)$.

(4) *If U is a uniserial left A-module with* $|U| \geqq 2$, *then* $[D_1(U) : D_2(U)]_1 \leqq 2$.

Proof. Let (L_i, S_i) $(i = 1, 2)$ be local-simple sets with $deg(L_1, S_1) \leqq deg(L_2, S_2)$ and $\theta : S_1 \to S_2$ an isomorphism. Then it suffices to show that θ is (L_1, L_2)-extendible by Proposition 1.3. We assume $d_{L_1}(S_1) = d_{L_2}(S_2)$ since in the other case the assertion is verified by Theorem 3.6. Moreover we may assume $L_i \neq S_i$ $(i = 1, 2)$, and the situations (St1) and (St2) in Lemma 3.5. Then we have $N_i = v_iA$ and $M_i = y_iA$ for some $v_i, y_i \in loc(A)$ by (3). Now Ax_1 and Ax_2 are submodules of the uniserial left module Ag with $d(x_1) = d_{L_1}(S_1) = d_{L_2}(S_2) = d(x_2)$. Hence there exists an element a in e_2Ae_1 such that $ax_1 = x_2$ and $d(a) = 0$ by (1) in Lemma 2.7. Moreover we have $e_1A \cong e_2A$ and $u_1A \cong u_2A$ by Lemma 2.2 and we may assume $e_1 = e_2$. Put $e = e_1 = e_2$. It suffices to show the condition (E) by Lemma 3.5. Since $d(x_1) = d(x_2)$ i.e., $|u_1A/x_1A| = |u_2A/x_2A|$ by (1) in Lemma 3.1 and $|L_1| \geqq |L_2|$, we have $|v_1A/y_1A| \geqq |v_2A/y_2A|$ i.e., $d(y_1) \geqq d(y_2)$. Put $r = d(x_1)$ and $s = d(y_1)$. Then $u_2J^r = x_2J$, $v_1J^{s-1} = y_1A$ and $v_2J^{s-1} \subset y_2A$. If $r < s$, then $ax_1 = x_2$ and $ay_1 \in eJ^s \subset x_2J \oplus y_2A = I_2$, and (E) is satisfied. Thus we assume $r \geqq s$.

Now suppose the condition

(E′) $cx_1 - x_2 \in x_2J \oplus v_2A$ and $cv_1 \in x_2J \oplus v_2A$ for some element $c \in eAe$. (i.e. (E) in the case $I_i = x_iJ \oplus v_iA$; $i = 1, 2$).

Then $cx_1 - x_2 \in (x_2J \oplus v_2A) \cap eJ^r = (x_2J \oplus v_2A) \cap (u_2J^{r-1} \oplus v_2J^{r-1}) = x_2J \oplus v_2J^{r-1} \subset x_2J \oplus y_2A = I_2$ and $cy_1 \in cv_1J^{s-1} \subset x_2J^s \oplus v_2J^{s-1} \subset x_2J \oplus y_2A = I_2$, so (E) is satisfied. Thus it suffices only to show (E′), and so we may assume that $I_i = x_iJ \oplus v_iA$ $(i = 1, 2)$, i.e., L_i are uniserial.

(i) In the case $u_1A \not\cong v_1A$ (i.e. $u_2A \not\cong v_1A$) : Write $av_1 = u_2p + v_2q$ for some $p \in f_2Ah_1$, $q \in h_2Ah_1$; where $h_1, h_2 \in p(A)$. If $p \notin J$, then $\overline{av_1A} \cong \overline{u_2pA}$ and $d(av_1) = d(u_2p) = 1$. Hence $u_2A = u_2pA \cong av_1A \cong v_1A$ by Remark 4, which is a contradiction. Thus we have $p \in J$. If $u_2p = 0$, (E′) is clearly satisfied, for $ax_1 = x_2$ and $av_1 = v_2q \in I_2$. Hence we assume $u_2p \neq 0$. We have $cv_1 = u_2p \neq 0$ for some $c \in eJe$, since $Au_2p \subset Jv_1$ $(\subset Ah_1)$. Then by (1) in Lemma 3.2 $cu_1 = 0$ and consequently $cx_1 = 0$, for $x_1A \subset u_1A$. Thus we have $(a-c)x_1 = ax_1 = x_2$ and $(a-c)v_1 = av_1 - u_2p = v_2q \in I_2$.

(ii) In the case $u_1A \cong v_1A$ (i.e. $u_1A \cong u_2A \cong v_1A \cong v_2A$) : For some unit b of eAe, we have $bv_1 = v_2$ and $bI_1 = beJ^{r+1} + bv_1A = eJ^{r+1} + v_2A = I_2$, so $L_1 \cong L_2$. Thus it suffices to prove our assertion under the condition $L_1 = L_2$, $u_1A \cong v_1A$, and $J^{r+1} = 0$, since L is an A/J^{r+1}-module. In this case our assertion is verified from the following lemma.

Lemma 4.5. *Let A be as in the above proposition and $eJ = uA \oplus vA$ and $uA \cong vA$, where $e \in p(A)$ and $u, v \in loc(A)$. Put $L = eA/vA$ and $S = Soc(L)$. Then any isomorphism $\theta : S \to S$ is (L, L)-extendible.*

Proof. As in the proof of the above lemma, we may assume $S = (x_iA + vA)/vA$ $(i = 1, 2)$, $\theta(x_1 + vA) = x_2 + vA$ and $ax_1 = x_2$, where x_i, $a \in loc(A)$ with $x_i \in uA$, $d(x_i) = r$, $d(a) = 0$. Since $Soc(auA) = ax_1A = x_2A = Soc(uA)$ and $d(au) = d(u)$, we have $eJ = auA \oplus vA$ by Corollary 2.4 (more explicitly we can show $auA = uA$ by (3) in Lemma 3.2 and Lemma 2.5). Put $u_1 = u$ and $u_2 = au$ $(= au_1)$. Since $x_1 \in u_1A$, for some $z \in loc(J^{r-1})$ we have $x_1 = u_1z$, so $x_2 = ax_1 = au_1z = u_2z$. Put $L' = L/LJ^2$ and $S' = LJ/LJ^2$ and consider the isomorphism $\theta' : S' \to S'$ with $\theta'(\overline{u}_1) = \overline{au}_1$ $(= \overline{u}_2)$, where $\overline{u}_i = u_i + LJ^2 \in S'$. Since L' is a module over the ring A/J^2, θ' is (L', L')-extendible by Lemma 4.2. Hence there exists a unit c in eAe such that $cu_1 - u_2 \in eJ^2 + vA$ and $cv \in eJ^2 + vA$. On the other hand $d(z) = r-1$ and $eJ^{r+1} = 0$. Therefore $cx_1 - x_2 = (cu_1 - u_2)z \in eJ^{r+1} + vJ^{r-1} \subset vA$. Moreover since $d(cv) = 1$ and $cv \in u_2J \oplus vA$, we have $cv \in vA$ by (3) in Lemma 3.2. Thus θ is (L, L)-extendible by Lemma 3.5.

We consider the following condition (D) introduced in [6, Section 3] :

(D) $[D_l(U) : D_2(U)]_l = [D_1(U) : D_2(U)]_r$ for any uniserial left A-module U with $|U| \geqq 2$.

If the conditions (b-1, 3, 4) in the following theorem and (D) are satisfied, then it holds $|\overline{eJ}| \leqq 2$ for any $e \in p(A)$ (see Remark 5). Thus by Propositions 4.3 and 4.4 we have

Theorem 4.6. *Let A be an artinian ring with the Jacobson radical J. Consider the following conditions :*

(a) *A is of right local type.*

(b) (1) *A is a left serial ring.*

(2) *J is serial as a right A-module.*

(3) *If $\overline{eJ}$ is not homogeneous, then $|\overline{eJ}| \leqq 2$, where $e \in p(A)$.*

(4) *If U is a uniserial left A-module with $|U| \geqq 2$, then $[D_1(U)$:*

$D_2(U)]_l \leqq 2$.

Then (a) *implies* (b), *and under the condition* (D) *the converse holds.*

Representing [6, Theorem 4.5] as the dual version of Theorem 4.6, we have (see Remark 5)

Theorem 4.7 ([7, Theorem 5.3] and [6, Lemma 4.3]). *Let A be an artinian ring with the Jacobson radical J. Consider the following conditions:*

(c) *A is of left colocal type.*

(d) (1) *A is a left serial ring.*

(2) *J is serial as a right A-module.*

(3) *If $\overline{eJ}$ is not homogeneous, then $|\overline{eJ}| \leqq 2$, where $e \in p(A)$.*

(4) *If U is a uniserial left A-module with $|U| \geqq 2$, then $[D_1(U) : D_2(U)]_r \leqq 2$.*

Then (c) *implies* (d), *and under the condition* (D) *the converse holds.*

We note that in (b) and (d) of the above theorems there is only one difference between indeces l and r in $[D_1(U) : D_2(U)]_l$ and $[D_1(U) : D_2(U)]_r$. Moreover when (b-4) and (d-4) are satisfied, $D_1(U) \neq D_2(U)$ implies $[D_1(U) : D_2(U)]_l = [D_1(U) : D_2(U)]_r (= 2)$. Thus by the theorems above we have

Theorem 4.8. *Let A be an artinian ring with the Jacobson radical J. Consider the following conditions:*

(a) *A is of right local type.*

(b) *A is of left colocal type.*

(c) (1) *A is left serial.*

(2) *eJ is a right serial module with $|\overline{eJ}| \leqq 2$ for any primitive idempotent e of A.*

Then the following statements hold.

(i) *Under the condition* (D), *the conditions* (a), (b) *and* (c) *are equivalent.*

(ii) (a) *and* (b) *are satisfied if and only if so are* (c) *and* (D).

5. Some properties on local modules. In this section, all modules are (finitely generated unitary) left A-modules unless otherwise stated. Any homomorphism between left A-modules operates on the right.

Let A be of right local type. H. Asashiba showed that there exist no modules of type-3 (see Remark 3). As easily seen this implies

Proposition 5.1 (Asashiba). *If A is of right local type, then $|\overline{eJ}| \leqq 2$ for any $e \in p(A)$ with $eJ^2 \neq 0$.*

Let $e \in p(A)$. Then $eJ^2 \neq 0$ if and only if there exists a uniserial module U such that $|U| = 3$ and $Soc(U)$ $(= J^2U) \cong \overline{Ae}$ (i.e. $eJ^2g \neq 0 \Longleftrightarrow J^2(Ag/J^3g) \cong \overline{Ae}$ for some $g \in p(A)$). Hence the following conditions (1) and (2) are equivalent provided the conditions (b-1, 3) in Theorem 4.6 are fulfilled (see Remark 5).

(1) $|\overline{eJ}| \leqq 2$ provided $eJ^2 \neq 0$.

(2) $[D_1(U) : D_2(U)]_r \leqq 2$ for any uniserial module U with $|U| \geqq 3$ and $Soc(U) \cong \overline{Ae}$.

Thus as an immediate consequence of Proposition 5.1, we have

Proposition 5.2. *If A is of right local type, then $[D_1(U) : D_2(U)]_r \leqq 2$ for any uniserial module U with $|U| \geqq 3$.*

Now we show the dual version to Proposition 5.2. In order to do it, we consider the dual notions to some ones in Section 1. As the dual notions to "extension", "extendible" and "maximal" we define "coextension", "coextendible" (or "liftable") and "comaximal", respectively. That is, for epimorphisms $\alpha_i : L_i \to K\,(i = 1, 2)$, $[\alpha_1, \alpha_2]$ has a *coextension* $\phi : L_1 \to L_2$ (i.e. $[\alpha_1, \alpha_2]$ is *coextendible*) if and only if $\alpha_1 = \phi\alpha_2$, and $[\alpha_1, \alpha_2]$ is *comaximal* if and only if $[\alpha_1, \overline{\alpha_2}]$ has no coextension for any submodule N with $N \subsetneqq Ker\,\alpha_2$ and the map $\overline{\alpha_2} : L_2/N \to K$ which is induced from α_2.

Let $\alpha_i : L_i \to K$ be epimorphisms and $M = Ker\,\alpha$, where α is an epimorphism $\alpha = (\alpha_1, \alpha_2)^T : L_1 \oplus L_2 \to K$. If $\beta = (\beta_1, \beta_2) : M \to L_1 \oplus L_2$ is the inclusion map, then we have the exact sequence

(E) $0 \to M \xrightarrow{\beta} L_1 \oplus L_2 \xrightarrow{\alpha} K \to 0$.

Then we should note that β_1 and β_2 are also epimorphisms and the sequence (E) induces the following exact sequences : $0 \to M \cap L_i \to L_i \xrightarrow{\alpha_i} K \to 0$ and $0 \to M \cap L_j \to M \xrightarrow{\beta_i} L_i \to 0$, where $\{i, j\} = \{1, 2\}$. The following lemma is the dual one to [6, Lemma 1.2] but we give a proof for the sake of readers ((1) in Lemma 5.3 is the dual one to Lemma 1.1 and we only use it in this paper).

Lemma 5.3. *Let $0 \to M \xrightarrow{\beta} L_1 \oplus L_2 \xrightarrow{\alpha} K \to 0$ be an exact sequence with epimorphisms α_i and an inclusion map β, where $\alpha = (\alpha_1, \alpha_2)^T$ and $\beta =$*

(β_1, β_2). *Then the following hold.*

(1) $[\alpha_1, \alpha_2]$ *is coextendible if and only if* $M = L' \oplus (L_2 \cap M)$ *for some module* L'. *In this case* $L' \cong L_1$.

(2) $[\alpha_1, \alpha_2]$ *is comaximal if and only if* $\overline{M} \cong \overline{L}_1$.

Proof. (1). Assume $[\alpha_1, \alpha_2]$ has a coextension ϕ. Then putting $L' = \{(x, -x\phi) \mid x \in L_1\}$, it holds $M \subset L' \oplus L_2$ and $L' \subset M$, so $M = L' \oplus (L_2 \cap M)$. Conversely, if $M = L' \oplus (L_2 \cap M)$ for some module L', then the epimorphism β_1 and the projection $\psi: M = L' \oplus (M \cap L_2) \to L'$ induce the isomorphisms $\overline{\beta_1} : M/(M \cap L_2) \to L_1$ and $\overline{\psi}: M/(M \cap L_2) \to L'$. As easily seen the composition map $\phi = \overline{\beta_1}^{-1}\overline{\psi}\gamma\beta_2$ is an coextension of $[\alpha_1, \alpha_2]$, where $\gamma: L' \to M$ is the map with $x\gamma = -x : x \in L'$.

(2) For any submodule N of $Ker\ \alpha_2 = M \cap L_2$, the above exact sequence induces the exact sequence $0 \to M/N \to L_1 \oplus (L_2/N) \xrightarrow{(\alpha_1, \overline{\alpha}_2)^T} K \to 0$ by identifying $(L_1 \oplus N)/N$ with L_1. Then we should note that $[\alpha_1, \overline{\alpha}_2]$ is coextendible if and only if $M/N = M'/N \oplus (M \cap L_2)/N$, and $M'/N \cong L_1$ for some submodule M' of M by (1).

"If" part : Assume $M/N = M'/N \oplus (M \cap L_2)/N$ for some modules N and M' as above. Since $\overline{M/N}$ is a homomorphic image of $\overline{M}$, $\overline{M} \cong \overline{L}_1$ (i.e. $\overline{M} \cong \overline{M'/N}$) implies $(M \cap L_2)/N = 0$, so $N = Ker\ \alpha_2$. This shows $[\alpha_1, \alpha_2]$ is comaximal.

"Only if" part : The exact sequence $0 \to M \cap L_2 \to M \to L_1 \to 0$ induces the exact sequence $0 \to \sigma(M \cap L_2) \to \overline{M} \to \overline{L}_1 \to 0$, where $\sigma : M \to \overline{M}$ is a canonical epimorphism. Since $\overline{M}$ is semi-simple, there exists a submodule M' of M such that $M = M' + (M \cap L_2)$ and $M' \cap (M \cap L_2) \subset MJ$. Putting $N = M' \cap (M \cap L_2)$, $M/N = M'/N \oplus (M \cap L_2)/N$ and $N \subset MJ$. Hence the comaximality of $[\alpha_1, \alpha_2]$ implies $M \cap L_2 = N\ (\subset MJ)$ and consequently $\sigma(M \cap L_2) = 0$. Thus we have $\overline{M} \cong \overline{L}_1$.

For a colocal module C and a simple module S which is a component of $\overline{C}$, we call (C, S) a *colocal-simple set.*

Let M be a module with socle $S_1 \oplus \cdots \oplus S_n$. Then M is embedded in $E(S_1) \oplus \cdots \oplus E(S_n)$, where $E(S_i)$ is the injective hull of S_i. Since M is finitely generated, we have $M \subset C_1 \oplus \cdots \oplus C_n$ for some finitely generated submodules C_i of $E(S_i)$ (i.e. some finitely generated and colocal modules C_i) ; $i = 1, \ldots, n$. Hence by the dual method to the proof of Proposition 1.3 we have

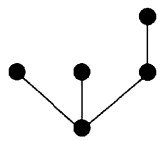
Proposition 5.4. *Let A be an artinian ring. Then the following conditions are equivalent.*

(1) *A is of left colocal type.*

(2) *For any colocal-simple sets (C_i, S_i) ; $i = 1, 2$, and any isomorphism $\theta: S_1 \to S_2$, θ is (C_1, C_2)-coextendible or θ^{-1} is (C_2, C_1)-coextendible.*

Proposition 5.5. *If A is of left colocal type, then it holds $[D_1(U) : D_2(U)]_l \leqq 2$ for any uniserial module U with $|U| \geqq 3$.*

Proof. Assume there exists a uniserial module U with $|U| \geqq 3$ and $[D_1(U) : D_2(U)]_l \geqq 3$. Then by [6, Lemma 5.3] we can construct a colocal module represented by the following diagram (which is a dual one to (3) in Remark 3).



By the dual method to the proof (in the case $i = 3$) of Lemma 1.4, there exists a colocal-simple sets (C_i, S_i) $(i = 1, 2)$ and an isomorphism $\theta: S_1 \to S_2$ which does not satisfy (2) in Proposition 5.4. Therefore A is not of left colocal type, which is a contradiction. Thus the proof is complete.

Examples . Let D and E (resp. F and G) be division rings such that E is a subring of D and $[D : E]_l = 2, \infty > [D : E]_r \geqq 3$ (resp. G is a subring of F and $\infty > [F : G]_l \geqq 3, [F : G]_r = 2$) (see [5] for the existence of these division rings). Let

$$A_n = \begin{pmatrix} E & & & \\ EE & & & \\ \cdots & & & \\ EEE\cdots E & & & \\ DDD\cdots DD \end{pmatrix}, \qquad B_n = \begin{pmatrix} G & & & \\ GG & & & \\ \cdots & & & \\ GGG\cdots G & & & \\ FFF\cdots FF \end{pmatrix}$$

be subrings of $M_n(D)$ and $M_n(F)$, where $M_n(D)$ and $M_n(F)$ denote the full-matrix rings over D and F with degree $n \geqq 2$, respectively. Generally if a ring A is of right local type (or left colocal type), then so is any factor ring of A. On the other hand, for integers $m \geqq n \geqq 2$, A_n (resp. B_n) is clearly a factor ring of A_m (resp. B_m). For any $n \geqq 2$, A_n (resp. B_n) satisfies (b) in Theorem 4.6 (resp. (d) in Theorem 4.7) (see [6, Example 3]). In [6,

Examples 3 and 4] we showed that B_4 and B_2 do not satisfy (c) and (a), respectively (i.e. B_4 is not of left colocal type and B_2 is not of right local type). But it is immediate from Propositions 5.2 (or 5.1), 5.5 and Theorems 4.6, 4.7 that A_3 and B_2 are not of right local type and A_2 and B_3 are not of left colocal type.

REFERENCES

[1] H. ASASHIBA : On algebras of second local type, III, Osaka J. Math. **24** (1987), 107−122.

[2] Y. BABA and M. HARADA : Generalizations of Nakayama ring IV, Osaka J. Math. **24** (1987), 139−145.

[3] P. GABRIEL : The universal cover of a representation-finite algebra, Springer Lecture Notes **903** (1981), 68−105.

[4] M. HARADA : Generalizations of Nakayama ring I, Osaka J. Math. **23** (1986), 181−200.

[5] A. H. SCHOFIELD : Representation of rings over skew fields, London Math. Soc. Lecture Notes **92** (1985).

[6] T. SUMIOKA : Tachikawa's theorem on algebras of left colocal type, Osaka J. Math. **21** (1984), 692−648.

[7] H. TACHIKAWA : On rings for which every indecomposable right module has a unique maximal submodule, Math. Z. **71** (1959), 200−222.

[8] H. TACHIKAWA : On algebras of which every indecomposable representation has a unique irreducible one as the top or the bottom Loewy constituent, Math. Z. **75** (1961), 215−227.

[9] T. YOSHII : On algebras of left cyclic representation type, Osaka Math. J. **10** (1958), 231−237.

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