Mathematical Journal of Okayama University

Volume 22, *Issue* 2 1980 *Article* 5

JUNE 1981

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Math. J. Okayama Univ. 22 (1980), 141—143

ON THE NILPOTENCY INDEX OF THE RADICAL OF A GROUP ALGEBRA. II

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Throughout this paper, let G be a p-solvable group with a p-Sylow subgroup P of order p^a, K an algebraically closed field of characteristic p, and $t(G)$ the nilpotency index of the radical of the group algebra KG of G over K.

The next inequality has been shown by D. A. R. Wallace [10].

$$t(G) \geq a(p-1)+1.$$

Along with this inequality, the following question arises: *If* $t(G)=a(p-1)+1$, *then is* P *elementary*? This question has an affirmative answer for any group of p-length 1 (S. A. Jennings [4] and K. Morita [5]) and for any group with a regular p-Sylow subgroup (Y. Tsushima [9]). However, this question has not always an affirmative answer; $t(S_4)=4$ for $p=2$ (see [6]). In the present paper, we shall give an affirmative answer to this question in case $p=2$, P is meta-cyclic and $G/O(G) \neq S_4$ (Proposition 1). Moreover, we shall construct an example which gives a negative answer to Tsushima's problems [8, Problems 3, 4, 5].

Proposition 1. *Assume that* $p=2$, P *is meta-cyclic, and* $G/O(G) \neq S_4$. *Then* $t(G)=a+1$ *if and only if* P *is elementary abelian.*

Proof. If P is elementary, then $t(G)=a+1$ (see [6]). Conversely, assume that $t(G)=a+1$. Since the center of P is contained in $O_{2',2}(G)$ ([1, Theorem 6. 3. 3]), G is of p-length 1 if $|P| \leq 4$. Hence we may assume $|P| \geq 8$. We shall proceed by induction on $|G|$. If $O(G) \neq 1$, then $a+1=t(G) \geq t(G/O(G)) \geq a+1$, and so P is elementary. If $O(G)=1$, then $N=O_2(G) \neq 1$ and the inequality $t(G) \geq t(N)+t(G/N)-1$ ([10, Theorem 2. 4]) forces N elementary. Since N is meta-cyclic and elementary, we must have $|N| \leq 4$ and $C_G(N)=N$ (see [1, Theorem 6. 3. 3]). In view of $C_G(N)=N$, G/N is isomorphic to a subgroup of the automorphism group of N, and so $|G/N| \leq 6$. Since $|P| \geq 8$ and $|G| \leq 24$, the order of G is either 8 or 24. $|G|=8$ is contrary to the assumption that P is meta-cyclic. Thus $|G|=24$, $N=C_G(N)$ is elementary abelian of type (2, 2), and G/N acts faithfully on N by conjugation. Let $S=\langle a \rangle$ be a 3-Sylow subgroup of G, and $N=\{1, x, y, xy\}$. Then by $C_G(N)=N$, we may assume that $x^a=y$ and $y^a=$

xy. Consequently $N_G(S) \cap N = 1$ and $|N_G(S)| = 6$ by Sylow's theorem. We conclude therefore that G is a semi-direct product of N and $N_G(S)$, and so $N_G(S)$ is the automorphism group of N. Thus, G is the holomorph of N and hence $G = S_4$, a contradiction.

Remark. If p is odd and P is meta-cyclic, then P is regular by [2, Satz 10. 2]. Thus, by the result of Tsushima [9], $t(G) = a(p-1)+1$ forces P elementary.

In the remaining part of this paper, we consider a group G such that $G/O(G) = S_4$. Let $\{e_1, e_2, \cdots, e_t\}$ be the set of all centrally primitive idempotents of $KO(G)$, $I(e_s) = \{x \in G \mid xe_sx^{-1} = e_s\}$, and $H_s = I(e_s)/O(G)$.

Proposition 2. *Assume that* $p = 2$ *and* $G/O(G) = S_4$. *Then*

$$t(G) = \begin{cases} 5 & \text{if } |H_s| = 8 \text{ for some } s. \\ 4 & \text{otherwise.} \end{cases}$$

Proof. By Morita's theorem [5, Theorem 2], $t(G) = \mathrm{Max}_s \{t(T_s)\}$, where T_s is a twisted group algebra of H_s over K with a factor set and $t(T_s)$ denotes the nilpotency index of the radical of T_s. If $|H_s|$ is neither 8 nor 24, then $t(T_s) \leq 4$ by [7, Theorem 1. 6]. If $|H_s| = 8$, then $t(T_s) = t(P) = 5$. If $|H_s| = 24$, then $T_s = KS_4$ by [3, Corollary], and so $t(T_s) = 4$ by [6]. This completes the proof.

Example 1. Assume that $p = 2$. Let $\langle d \rangle$ be a cyclic group of order 3. Let G be the semi-direct product of $\langle d \rangle$ by S_4 such that $d^{(12)} = d^{-1}$ and A_4 acts trivially on $\langle d \rangle$. Then $t(G) = 4$ by Proposition 2.

Example 2. Assume that $p = 2$. Let $\langle c \rangle$ and $\langle d \rangle$ be cyclic groups of order 3. Let U be a homomorphism of S_3 into $\mathrm{GL}(2, 3)$ defined by

$$a = (1\ 2\ 3) \longrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } b = (1\ 2) \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then U can be regarded as a homomorphism of S_4 into $\mathrm{GL}(2, 3)$. Let G be the semi-direct product of $M = \langle c \rangle \times \langle d \rangle$ by S_4 with respect to U. Then the following relations hold:

$$c^a = c, \quad d^a = cd, \quad c^b = c \text{ and } d^b = d^{-1}.$$

On the other hand, $e = (1 + \omega c + \omega^2 c^2)(1 + d + d^2)$ is a primitive idempotent of KM, where ω is a primitive 3rd root of 1 in K. We obtain that

$e^a \neq e$ and $e^b = e$. Thus $|I(e)/M| = 8$, and consequently $t(G) = 5$ by Proposition 2. In Example 1, we have already showed that $t(G/\langle c\rangle) = 4$. Since $\langle c\rangle$ is contained in the center of G, this gives a negative answer to Tsushima's Problems [8, Problems 3, 4, 5].

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(*Received December 1, 1979*)