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Belyi Function whose Grothendieck Dessin is a Flower Tree with Two Ramification Indices

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Abstract

In this paper we present an explicit construction of Belyi functions whose dessins are flower trees (i.e., graphs of diameter 4) with two ramification indices. We also give a method for obtaining Belyi functions defined over the moduli fields of the dessins.

KEYWORDS: Belyi function, Grothendieck dessin.

Math. J. Okayama Univ. **47** (2005), 119–131**BELYI FUNCTION WHOSE GROTHENDIECK DESSIN IS
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1. INTRODUCTION

For a compact connected Riemann surface R and a finite covering $\beta : R \rightarrow \mathbb{P}^1$ one calls β a Belyi function on R if β is unramified outside the three points $0, 1$ and $\infty \in \mathbb{P}^1$. Belyi [1] shows that for a complete nonsingular algebraic curve R defined over a field of characteristic zero, R can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a covering $R \rightarrow \mathbb{P}^1$ with three ramification points. There are various studies on properties of Belyi functions (cf. [2], [11], [13], [14], ...). The main result in this paper is a unified method for constructing Belyi functions of a certain family. In the following we assume $R = \mathbb{P}^1 = \mathbb{P}_{\mathbb{C}}^1$ and denote by \mathcal{B} the set of Belyi functions $\beta : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$. One usually identifies $\mathbb{P}_{\mathbb{C}}^1$ with $\mathbb{C} \cup \{\infty\}$. Let $[0, 1] \subset \mathbb{P}_{\mathbb{C}}^1$ be the segment on the real line with end points 0 and 1 not through ∞ , that is, $[0, 1] = \{z \in \mathbb{R} \mid 0 \leq z \leq 1\}$. We denote by D_{β} the inverse image $\beta^{-1}([0, 1])$ of $[0, 1]$ for $\beta \in \mathcal{B}$, and call D_{β} a dessin due to Grothendieck. Here β is a polynomial in $\mathbb{C}[X]$ if and only if D_{β} is a graph of tree type, i.e., a graph with no cycles. It is easily seen that D_{β} is a connected graph. Let A_0 and A_1 be the sets of points whose images by β are 0 and 1 , respectively. Then $A_0 \amalg A_1$ coincides with the set V of vertices of the graph D_{β} . On the graph D_{β} one draws \bullet and \times at points of A_0 and A_1 , respectively. Then D_{β} is a bipartite connected graph with two partitions A_0 and A_1 of V . Let \mathcal{G} be the set of bipartite connected graphs on $\mathbb{P}_{\mathbb{C}}^1$ with finite edges. We define an equivalence relation in \mathcal{G} such that $g_1 \sim g_2$ if g_1 is equivalent to g_2 as bipartite graphs on $\mathbb{P}_{\mathbb{C}}^1$. On the other hand, we denote $\beta_1 \sim \beta_2$ for $\beta_1, \beta_2 \in \mathcal{B}$ if there exists $\rho \in \mathrm{PSL}_2(\mathbb{C})$ such that $\beta_2 = \beta_1 \circ \rho$. It is an equivalence relation in \mathcal{B} . The following is known as Grothendieck's correspondence (cf. [12]). In fact, it follows from Riemann existence theorem and Weil descent theorem.

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Proposition 1.1. *There exists a one-to-one correspondence between two quotient sets \mathcal{B}/\sim and \mathcal{G}/\sim such that $\beta \mapsto D_\beta$. Moreover, we have $\mathcal{B}/\sim = \mathcal{B}_{\overline{\mathbb{Q}}}/\sim$ where $\mathcal{B}_{\overline{\mathbb{Q}}} \subset \mathcal{B}$ is the set of Belyi functions defined over $\overline{\mathbb{Q}}$. In particular, every graph $g \in \mathcal{G}$ can be realized as the dessin D_β of a Belyi function β defined over $\overline{\mathbb{Q}}$.*

In this paper we study an explicit construction of Belyi functions whose dessins are graphs in a family of plane trees. For every case we construct a Belyi function over a number field whose degree is as small as possible, so called the moduli field.

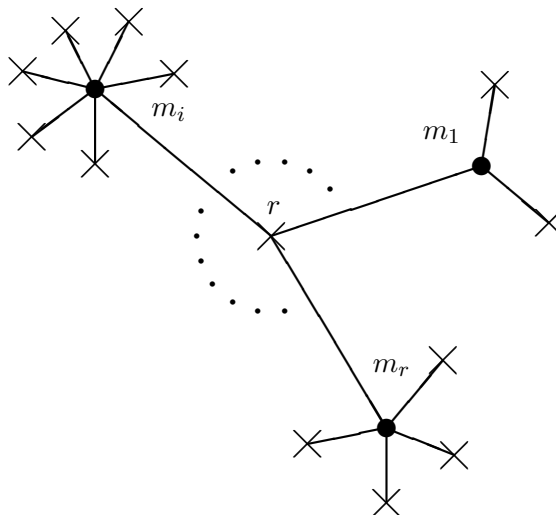


Figure 1.2 (flower tree with ramification $\langle m_1, m_2, \dots, m_r \rangle$)

Let us call a tree of diameter 4 a flower tree. The flower tree in the Figure 1.2 above is called a flower tree with ramification $\langle m_1, m_2, \dots, m_r \rangle$; or a flower tree of type (m_1, m_2, \dots, m_r) (cf. [20]). Here $\langle m_1, m_2, \dots, m_r \rangle$ is considered to be a multi-set, that is, a set of numbers up to ordering. We denote $\langle m_1, m_2, \dots, m_r \rangle$ by $\langle m_1, \dots, m_i \rangle + \langle m_{i+1}, \dots, m_r \rangle$. For example, we have $2\langle 4 \rangle + 3\langle 5 \rangle = \langle 4, 4, 5, 5, 5 \rangle$. In the Figure 1.2 each point \bullet has m_i edges, respectively. Here the edge connecting to the center point \times is also counted for m_i . The center point \times has r edges. Schneps [12], Shabat-Zvonkin [17] and Zapponi [20] study many properties of flower trees. The Belyi functions for flower trees with ramification $\langle 2, 3, 4, 5, 6 \rangle$ are computed in [12]. Shabat-Zvonkin [17] calculate the Belyi functions for flower trees with the following ramifications:

- (f.1) $i_1\langle m \rangle + i_2\langle n \rangle$,
- (f.2) $j_1\langle m \rangle + j_2\langle n \rangle + j_3\langle p \rangle$,

for $(i_1, i_2) = (2, 3), (2, 5)$ and $(j_1, j_2, j_3) = (1, 1, 1), (2, 1, 1), (3, 1, 1)$ where m, n and p are distinct positive integers. The Belyi function for a flower tree over a finite field and over a complete field are also studied (cf. [18],[19]). The main result in this paper is to present a complete solution for the case (f.1).

Let k, l, m and $n \in \mathbb{Z}$ be positive integers with $m \neq n$. Let S be the set $\{s_i | i = 1, 2, \dots, k\}$ of k variables s_i . We may assume $s_0 = 1$ for convenience' sake. Let K be the field $\mathbb{Q}(S)$ adjoining to \mathbb{Q} all of the elements in S , and \mathcal{O} the ring of polynomials in K with \mathbb{Q} coefficients, that is, $\mathcal{O} = \mathbb{Q}[S]$. For an $\mathfrak{s} = (s_1, s_2, \dots, s_k) \in \mathcal{O}^k$ let $f(\mathfrak{s})(X)$ be a polynomial in $\mathcal{O}[X]$ such that

$$f(\mathfrak{s})(X) = \sum_{i=0}^k s_i X^i$$

where $s_0 = 1$. Then for each rational number $q \in \mathbb{Q}$ there exists a unique power series $g(q, \mathfrak{s})(X) \in K[[X]]$ such that

$$g(q, \mathfrak{s})(X) = f(\mathfrak{s})(X)^q$$

with the branch condition $g(q, \mathfrak{s}) \equiv 1 \pmod{XK[[X]]}$. For every non-negative integer $j \in \mathbb{Z}, j \geq 0$ let $c_j(q, \mathfrak{s}) \in K$ denote the coefficient of X^j in $g(q, \mathfrak{s})$, i.e.

$$g(q, \mathfrak{s})(X) = \sum_{j=0}^{\infty} c_j(q, \mathfrak{s}) X^j.$$

Here $c_j(q, \mathfrak{s}) \in \mathcal{O}$ holds for every $j \geq 0$ (cf. Lemma 2.1). We define a polynomial $\beta_{k,l}(m, n; \mathfrak{s})(X) \in \mathcal{O}[X]$ by

$$\beta_{k,l}(m, n; \mathfrak{s})(X) = \left(\sum_{i=0}^k s_i X^i \right)^m \left(\sum_{j=0}^l c_j(-m/n, \mathfrak{s}) X^j \right)^n.$$

Let us denote $\mathbb{C}^{k*} = \mathbb{C}^k - \{(0, 0, \dots, 0)\}$ and

$$\begin{aligned} \mathcal{T} &= \mathcal{T}(k, l, m, n) \\ &= \{t \in \mathbb{C}^{k*} | c_j(-m/n, t) = 0 \text{ for every } j \in \mathbb{Z} \text{ with } l < j < k + l\}. \end{aligned}$$

Theorem 1.3. *For each $t \in \mathcal{T}$, $\beta_{k,l}(m, n; t)(X) \in \mathbb{C}[X]$ is a Belyi function whose dessin is a flower tree with ramification $k\langle m \rangle + l\langle n \rangle$.*

For $t \in \mathcal{T}$ let D_t denote the dessin which is obtained from $\beta_{k,l}(m, n; t)$. Let $\mathcal{F} = \mathcal{F}(k, l, m, n)$ be the set of flower trees with ramification $k\langle m \rangle + l\langle n \rangle$ up to the graph equivalence \sim . Proposition 1.1 implies that for a graph $D \in \mathcal{F}$ there exists a Belyi function β over $\overline{\mathbb{Q}}$ corresponding to D . The action on the graph D of an element σ in the absolute Galois group $\Gamma_{\mathbb{Q}}$ of \mathbb{Q} is defined via that on the coefficients of β , that is, $D^\sigma = D_{\beta^\sigma}$. Let Γ_D be

the subgroup of $\Gamma_{\mathbb{Q}}$ such that $\Gamma_D = \{\sigma \in \Gamma_{\mathbb{Q}} \mid D^\sigma \sim D\}$. We denote the fixed field $\overline{\mathbb{Q}}^{\Gamma_D}$ by $\mathcal{M}(D)$ and call it the moduli field of D .

Theorem 1.4. *There exists a finite subest \mathcal{T}_1 of \mathcal{T} satisfying the following two properties (i) and (ii) :*

- (i) *the map $\mathcal{T}_1 \rightarrow \mathcal{F}$, $\mathfrak{t} \mapsto D_{\mathfrak{t}}$ gives a bijection,*
- (ii) *for each $\mathfrak{t} \in \mathcal{T}_1$, the Belyi function $\beta_{k,l}(m, n; \mathfrak{t})(X)$ is defined over $\mathcal{M}(D_{\mathfrak{t}})$.*

Remark. See §2 for the explicit definition of the \mathcal{T}_1 . The definition field of any Belyi function realizing a dessin D is an extension of the moduli field $\mathcal{M}(D)$.

Remark. Main construction in this paper generalizes our construction in [8] and contains Examples 5.2 and 5.3 in [17] as special cases.

Remark. In Theorems 1.3 and 1.4 we may take $l = 0$, which yields Belyi functions for the case with ramification $k\langle m \rangle$ (see Proposition 3.5).

2. CONSTRUCTION OF BELYI FUNCTIONS

Let k, l, m and $n \in \mathbb{Z}$ be positive integers with $m \neq n$. We first show that $c_j(q, \mathfrak{s}) \in \mathcal{O}$ holds. One can calculate $c_j(q, \mathfrak{s}) \in K$ explicitly as follows. The branch condition implies $c_0(q, \mathfrak{s}) = 1$. For a positive integer $j \in \mathbb{Z}$, $j \geq 1$ we define

$$\mathcal{R}_j = \{(r_1, r_2, \dots, r_k) \in \mathbb{Z}^k \mid r_i \geq 0 \text{ and } \sum_{i=1}^k r_i i = j\}.$$

For $\mathfrak{r} = (r_1, r_2, \dots, r_k) \in \mathcal{R}_j$ let $\mathfrak{s}^{\mathfrak{r}}$ denote $\prod_{i=1}^k s_i^{r_i}$, and $M(q, \mathfrak{r})$ the multinomial coefficient $P(q, \sum_{i=1}^k r_i) / \prod_{i=1}^k (r_i!)$ where $P(q, r) = q(q-1) \cdots (q-r+1)$.

Lemma 2.1. *For a rational number $q \in \mathbb{Q}$ we have*

$$c_j(q, \mathfrak{s}) = \sum_{\mathfrak{r} \in \mathcal{R}_j} M(q, \mathfrak{r}) \mathfrak{s}^{\mathfrak{r}}.$$

In particular, $c_j(q, \mathfrak{s}) \in \mathcal{O}$ holds for every $j \in \mathbb{Z}$ with $j \geq 0$.

Proof. One can check that two power series $\sum_{j=0}^{\infty} \sum_{\mathfrak{r} \in \mathcal{R}_j} M(q, \mathfrak{r}) \mathfrak{s}^{\mathfrak{r}} X^j$ and $g(q, \mathfrak{s})(X)$ satisfy a partial differential equation $f(\mathfrak{s})\partial Y/\partial X - qY\partial f(\mathfrak{s})/\partial X = 0$. It follows from $g(q, \mathfrak{s})(X) \equiv \sum_{\mathfrak{r} \in \mathcal{R}_0} M(q, \mathfrak{r}) \mathfrak{s}^{\mathfrak{r}} \equiv 1 \pmod{XK[[X]]}$ that $g(q, \mathfrak{s})(X) = \sum_{j=0}^{\infty} \sum_{\mathfrak{r} \in \mathcal{R}_j} M(q, \mathfrak{r}) \mathfrak{s}^{\mathfrak{r}} X^j$. □

Let us fix a $\mathfrak{t} \in \mathcal{T} = \mathcal{T}(k, l, m, n)$ and denote $\beta_{k,l}(m, n; \mathfrak{t})$ simply by β . Note that the map $\beta : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$ is non-constant for $\mathfrak{t} \in \mathbb{C}^{k*}$. Let e_x be the ramification index of β at $x \in \mathbb{P}_{\mathbb{C}}^1$. Let A_z be the set $\beta^{-1}(z) = \{x \in$

$\mathbb{P}_{\mathbb{C}}^1 \setminus \{\beta(x) = z\}$ for $z = 0, 1$ and ∞ , and put $A = A_0 \amalg A_1 \amalg A_{\infty}$. We will calculate the indices e_a for all $a \in A$.

Lemma 2.2. *We have $0 \in A_1$ and $e_0 \geq k + l$. In particular, $\sharp A_1 \leq \deg \beta - (k + l) + 1$.*

Proof. It follows from the definitions of $\beta(X)$ and $\mathfrak{t} \in \mathcal{T}$ that

$$\begin{aligned} \beta(X) &= \left(\sum_{i=0}^k c_i(\mathfrak{t}) X^i \right)^m \left(\sum_{j=0}^l c_j(-m/n, \mathfrak{t}) X^j \right)^n \\ &= \left(\sum_{i=0}^k c_i(\mathfrak{t}) X^i \right)^m \left(\sum_{j=0}^{k+l-1} c_j(-m/n, \mathfrak{t}) X^j \right)^n. \end{aligned}$$

This implies

$$\begin{aligned} \beta(X) &\equiv \left(\sum_{i=0}^k c_i(\mathfrak{t}) X^i \right)^m \left(\sum_{j=0}^{\infty} c_j(-m/n, \mathfrak{t}) X^j \right)^n \pmod{X^{k+l} \mathbb{C}[[X]]} \\ &= 1. \end{aligned}$$

Since $\beta(X) \in \mathbb{C}[X]$, we have $\beta(X) - 1 \in X^{k+l} \mathbb{C}[X]$ and $e_0 \geq k + l$. □

Proof of Theorem 1.3. Lemma 2.2 implies that

$$\begin{aligned} \sharp A_0 + \sharp A_1 + \sharp A_{\infty} &\leq k + l + \deg \beta - (k + l) + 1 + 1 \\ &= \deg \beta + 2. \end{aligned} \tag{1}$$

Let us consider the following conditions.

(c.0) $(\sum_{i=0}^k c_i(\mathfrak{t}) X^i)(\sum_{j=0}^l c_j(-m/n, \mathfrak{t}) X^j) = 0$ has $k + l$ distinct roots in \mathbb{C} ,

(c.1) $(\beta(X) - 1)/X^{k+l-1} = 0$ has $\deg \beta - (k + l) + 1$ distinct roots in \mathbb{C} .

Then both (c.0) and (c.1) are satisfied if and only if the equality in (1) holds.

It is clear that

$$\sum_{x \in \mathbb{P}_{\mathbb{C}}^1 - A} (e_x - 1) \geq 0. \tag{2}$$

By using (1) and (2) we have

$$\begin{aligned} \sum_{x \in \mathbb{P}_{\mathbb{C}}^1} (e_x - 1) &= \sum_{x \in A} (e_x - 1) + \sum_{x \in \mathbb{P}_{\mathbb{C}}^1 - A} (e_x - 1) \\ &= 3 \deg \beta - (\sharp A_0 + \sharp A_1 + \sharp A_{\infty}) + \sum_{x \in \mathbb{P}_{\mathbb{C}}^1 - A} (e_x - 1) \\ &\geq 3 \deg \beta - (\deg \beta + 2) \\ &= 2 \deg \beta - 2. \end{aligned}$$

On the other hand, Riemann-Hurwitz formula shows $\sum_{x \in \mathbb{P}_{\mathbb{C}}^1} (e_x - 1) = 2 \deg \beta - 2$ since β is a non-constant separable map from $\mathbb{P}_{\mathbb{C}}^1$ to $\mathbb{P}_{\mathbb{C}}^1$. This means that the inequalities in (1) and (2) are, in fact, equalities. The equality $\sum_{x \in \mathbb{P}_{\mathbb{C}}^1 - A} (e_x - 1) = 0$ verifies that $\beta(X)$ is a Belyi function.

By the above argument we see that both (c.0) and (c.1) hold. It follows from (c.0) that $c_k(\mathfrak{t})$ and $c_l(-m/n, \mathfrak{t})$ are non-zero. Thus we have $\deg \beta = km + ln$. Let $A_{0,1}$ and $A_{0,2}$ be subsets of A_0 such that $A_{0,1} =$

$\{x \in \mathbb{C} \mid \sum_{i=0}^k c_i(\mathbf{t})x^i = 0\}$ and $A_{0,2} = \{x \in \mathbb{C} \mid \sum_{j=0}^l c_j(-m/n, \mathbf{t})x^j = 0\}$, respectively. Then $A_0 = A_{0,1} \amalg A_{0,2}$. By (c.0) we have $e_a = v_i$ for every $a \in A_{0,i}$ where $v_1 = m$ and $v_2 = n$. The condition (c.1) means that $e_0 = k + l$ and $e_a = 1$ for each $a \in A_1 - \{0\}$. It is clear that $A_\infty = \{\infty\}$ and $e_\infty = \deg\beta = km + ln$. We now have a complete list of the ramification indices e_a for all $a \in A$. This data concludes that the dessin of $\beta(X)$ is a flower tree with ramification $k\langle m \rangle + l\langle n \rangle$. \square

In the above proof, we have shown the following lemma which will be used later.

Lemma 2.3. *For $\mathbf{t} \in \mathcal{T}$ neither $c_k(\mathbf{t})$ nor $c_l(-m/n, \mathbf{t})$ vanishes.*

We define the action of $\alpha \in \mathbb{C}^\times$ on $\mathbf{t} = (t_1, t_2, \dots, t_k) \in \mathcal{T}$ by

$$\alpha\mathbf{t} = (\alpha t_1, \dots, \alpha^i t_i, \dots, \alpha^k t_k).$$

In fact, one sees that $\alpha\mathbf{t} \in \mathcal{T}$ since $c_j(-m/n, \alpha\mathbf{t}) = \alpha^j c_j(-m/n, \mathbf{t})$ holds for every positive integer $j \in \mathbb{Z}$. Let $\overline{\mathcal{T}}$ denote the quotient $\mathbb{C}^\times \backslash \mathcal{T}$ of \mathcal{T} by the action of \mathbb{C}^\times . Now recall that $\mathcal{F} = \mathcal{F}(k, l, m, n)$ is the set of flower trees with ramification $k\langle m \rangle + l\langle n \rangle$ up to the graph equivalence.

Proposition 2.4. *There exists a one-to-one correspondence between $\overline{\mathcal{T}}$ and \mathcal{F} by $\mathbf{t} \mapsto D_{\mathbf{t}}$.*

Proof. For $\mathbf{t} \in \mathcal{T}$ and $\alpha \in \mathbb{C}^\times$ we have $\beta_{k,l}(m, n; \alpha\mathbf{t})(X) = \beta_{k,l}(m, n; \mathbf{t})(\alpha X)$. Thus $\beta_{k,l}(m, n; \alpha\mathbf{t}) \sim \beta_{k,l}(m, n; \mathbf{t})$. Proposition 1.1 shows that $D_{\alpha\mathbf{t}} = D_{\mathbf{t}}$ in \mathcal{F} . Thus the map $\varphi : \overline{\mathcal{T}} \rightarrow \mathcal{F}, \mathbf{t} \mapsto D_{\mathbf{t}}$ is well-defined. We first see that φ is injective. For $\mathbf{t}_1, \mathbf{t}_2 \in \mathcal{T}$ let us denote $\beta_{k,l}(m, n; \mathbf{t}_i)$ simply by β_i , respectively. Now assume $D_{\mathbf{t}_1} = D_{\mathbf{t}_2}$ in \mathcal{F} . Then Proposition 1.1 implies that there exists an automorphism $\rho \in \text{PSL}_2(\mathbb{C})$ of $\mathbb{P}_{\mathbb{C}}^1$ such that $\beta_2(X) = \beta_1(\rho(X))$. Here, $\beta_1^{-1}(\infty) = \beta_2^{-1}(\infty) = \{\infty\}$. This means that $\rho(X) = \alpha_1 X + \alpha_0$ with $\alpha_1 \in \mathbb{C}^\times$ and $\alpha_0 \in \mathbb{C}$. By the argument in the proof of Theorem 1.3, it satisfies that $\{x \in \mathbb{P}_{\mathbb{C}}^1 \mid \beta_i(x) = 1 \text{ and } e_x = k + l\} = \{0\}$ for each $i = 1$ and 2 . This implies $\alpha_0 = 0$. Thus we have $\beta_2(X) = \beta_1(\alpha_1 X)$ and $\beta_{k,l}(m, n; \mathbf{t}_2)(X) = \beta_{k,l}(m, n; \mathbf{t}_1)(\alpha_1 X) = \beta_{k,l}(m, n; \alpha_1 \mathbf{t}_1)(X)$. For $m \neq n$, one sees $\mathbf{t}_2 = \alpha_1 \mathbf{t}_1$. Hence $\mathbf{t}_1 = \mathbf{t}_2$ holds in $\overline{\mathcal{T}}$.

We next show that φ is surjective. Let D be a graph in $\mathcal{F} = \mathcal{F}(k, l, m, n)$. Then there exists a Belyi function $\beta \in \mathcal{B}$ whose dessin is equivalent to D . Let $y \in \mathbb{P}_{\mathbb{C}}^1$ be a unique point such that $\beta(y) = 1$ and $e_y = k + l$. We denote $\beta(X + y)$ by $\beta_y(X)$. Note that $D_\beta \sim D_{\beta_y}$. Let $A_{0,1}$ (resp. $A_{0,2}$) be the sets of points $x \in \mathbb{C}$ such that $e_x = m$ (resp. $e_x = n$) and $\beta_y(x) = 0$. Then

$$\beta_y(X) = \gamma_1 \left(\prod_{a \in A_{0,1}} (X - a) \right)^m \left(\prod_{a \in A_{0,2}} (X - a) \right)^n$$

where γ_1 is the coefficient of the highest degree in $\beta_y(X)$. Since $\beta_y(0) = 1$, we have $a \neq 0$ for every $a \in A_0 = A_{0,1} \amalg A_{0,2}$. Thus there exists a constant $\gamma_2 \in \mathbb{C}^\times$ satisfying

$$\beta_y(X) = \gamma_2 \left(\prod_{a \in A_{0,1}} (1 - a^{-1}X) \right)^m \left(\prod_{a \in A_{0,2}} (1 - a^{-1}X) \right)^n.$$

Indeed, $\gamma_2 = 1$ from $\beta_y(0) = 1$. Let t_i and u_j be complex numbers such that

$$\sum_{i=0}^k t_i X^i = \prod_{a \in A_{0,1}} (1 - a^{-1}X) \quad \text{and} \quad \sum_{j=0}^l u_j X^j = \prod_{a \in A_{0,2}} (1 - a^{-1}X).$$

Then we have

$$\beta_y(X) = \left(\sum_{i=0}^k t_i X^i \right)^m \left(\sum_{j=0}^l u_j X^j \right)^n.$$

Now put $\mathbf{t} = (t_1, t_2, \dots, t_k)$. One notes that $\mathbf{t} \in \mathbb{C}^{k*}$ since $a^{-1} \neq 0$ for $a \in A_{0,1}$. It is easily seen that $\beta_y(X) \equiv 1 \pmod{X^{k+l}\mathbb{C}[X]}$ implies

$$c_j(-m/n, \mathbf{t}) = \begin{cases} u_j & \text{if } 0 \leq j \leq l, \\ 0 & \text{if } l < j < k+l. \end{cases}$$

This shows that $\mathbf{t} \in \mathcal{T}$ and $\beta_y(X) = \beta_{k,l}(m, n; \mathbf{t})(X)$. Hence φ is surjective. □

We will find a suitable subset of \mathcal{T} which is a complete system of representatives for $\overline{\mathcal{T}}$. Let us define the period $\text{pd}(\mathbf{t})$ of $\mathbf{t} \in \mathcal{T}$ to be $\text{gcd}\{1 \leq i \leq k \mid c_i(\mathbf{t}) \neq 0\}$.

Lemma 2.5. *For every $\mathbf{t} \in \mathcal{T}$ the period $\text{pd}(\mathbf{t})$ is a common divisor of k and l .*

Proof. By Lemma 2.3 we have $c_k(\mathbf{t}) \neq 0$. Thus $\text{pd}(\mathbf{t})$ is a divisor of k . Let ζ be a primitive $\text{pd}(\mathbf{t})$ -th root of unity. Then we have $\zeta \mathbf{t} = \mathbf{t}$. This implies that $c_j(-m/n, \mathbf{t}) = c_j(-m/n, \zeta \mathbf{t}) = \zeta^j c_j(-m/n, \mathbf{t})$. Since $c_l(-m/n, \mathbf{t}) \neq 0$, $\text{pd}(\mathbf{t})$ is a divisor of l . Hence $\text{pd}(\mathbf{t}) \mid \text{gcd}(k, l)$ holds. □

For an element $\mathbf{t} \in \mathcal{T}$ of period p we define the non-vanishing index set $I(\mathbf{t})$ of \mathbf{t} by

$$I(\mathbf{t}) = \{i \in \mathbb{Z} \mid 1 \leq i \leq k, c_i(\mathbf{t}) \neq 0\} \\ = \{i_1 < i_2 < \dots < i_\kappa\}.$$

Then there exist non-negative integers $\lambda_j \in \mathbb{Z}$ such that $\lambda_1 i_1 - \sum_{j=2}^\kappa \lambda_j i_j = p$. The integers λ_j depending on $I(\mathbf{t})$ can be determined uniquely in the following way. For an integer $j_1 \in \mathbb{Z}$ with $1 \leq j_1 \leq \kappa$ let μ_{j_1} denote $\text{gcd}\{i_j \mid 1 \leq j \leq j_1 - 1\}$. For each integer j_1 decreasing from κ to 2, we define λ_{j_1} to be the smallest non-negative integer such that $p + \sum_{j=j_1}^\kappa \lambda_j i_j \equiv 0$

(mod μ_{j_1}) inductively. Then one puts $\lambda_1 = (p + \sum_{j=2}^{\kappa} \lambda_j i_j) / i_1$. We call such $(\lambda_1, \lambda_2, \dots, \lambda_{\kappa})$ the minimization operator of $I(\mathbf{t})$. Let us define the direction $\text{dir}(\mathbf{t}) \in \mathbb{C}^{\times}$ of $\mathbf{t} \in \mathcal{T}$ by

$$\text{dir}(\mathbf{t}) = c_{i_1}(\mathbf{t})^{\lambda_1} \prod_{j=2}^{\kappa} c_{i_j}(\mathbf{t})^{-\lambda_j},$$

where $(\lambda_1, \lambda_2, \dots, \lambda_{\kappa})$ is the minimization operator of $I(\mathbf{t})$. Let α be a p -th root of $\text{dir}(\mathbf{t})$, that is, $\alpha^p = \text{dir}(\mathbf{t})$. We denote $\alpha^{-1}\mathbf{t}$ by $\text{nom}(\mathbf{t})$, and call it the normalized element of \mathbf{t} . Here $\zeta\alpha$ is also a p -th root of $\text{dir}(\mathbf{t})$ for a p -th root ζ of unity. Then $(\zeta\alpha)^{-1}\mathbf{t} = \alpha^{-1}\zeta^{-1}\mathbf{t} = \alpha^{-1}\mathbf{t}$. Thus $\text{nom}(\mathbf{t}) \in \mathcal{T}$ is well-defined. Let us define $\mathcal{T}_1 = \{\mathbf{t} \in \mathcal{T} \mid \text{dir}(\mathbf{t}) = 1\}$. Then the following lemma is easily seen.

Lemma 2.6. *There exists a bijective map from \mathcal{T} to the direct product of two sets \mathbb{C}^{\times} and \mathcal{T}_1 such that*

$$\begin{aligned} \mathcal{T} &\xrightarrow{\sim} \mathbb{C}^{\times} \times \mathcal{T}_1 \\ \mathbf{t} &\mapsto (\text{dir}(\mathbf{t}), \text{nom}(\mathbf{t})). \end{aligned}$$

In particular, every normalized element has direction 1.

Let ψ be the composite map of the canonical inclusion map $\mathcal{T}_1 \rightarrow \mathcal{T}$ and the projection $\mathcal{T} \rightarrow \overline{\mathcal{T}}$.

Lemma 2.7. *The map $\psi : \mathcal{T}_1 \rightarrow \overline{\mathcal{T}}$ is bijective, that is, \mathcal{T}_1 is a complete system of representatives for $\overline{\mathcal{T}}$.*

Proof. For every $\mathbf{t} \in \mathcal{T}$ we have $\text{nom}(\mathbf{t}) \in \mathcal{T}_1$ and $\psi(\text{nom}(\mathbf{t})) = \mathbf{t}$ in $\overline{\mathcal{T}}$, which means that ψ is surjective. Now assume $\psi(\mathbf{t}_1) = \psi(\mathbf{t}_2)$ for $\mathbf{t}_1, \mathbf{t}_2 \in \mathcal{T}_1$. Then there exists an $\alpha \in \mathbb{C}^{\times}$ such that $\mathbf{t}_2 = \alpha\mathbf{t}_1$. Here the period p of \mathbf{t}_1 is equal to that of \mathbf{t}_2 . It follows from the definition that $\text{dir}(\mathbf{t}_2) = \text{dir}(\alpha\mathbf{t}_1) = \alpha^p \text{dir}(\mathbf{t}_1)$. The assumption $\mathbf{t}_1, \mathbf{t}_2 \in \mathcal{T}_1$ implies that $\alpha^p = 1$. Since $\text{pd}(\mathbf{t}_1) = p$, one sees $\alpha\mathbf{t}_1 = \mathbf{t}_1$. Hence we conclude $\mathbf{t}_1 = \mathbf{t}_2$, which shows the injectivity of ψ . \square

Proposition 2.4 and Lemma 2.7 imply the first assertion of Theorem 1.4.

Corollary 2.8. *There exists a one-to-one correspondence between \mathcal{T}_1 and \mathcal{F} by $\mathbf{t} \mapsto D_{\beta}$ where $\beta = \beta_{k,l}(m, n; \mathbf{t})$.*

Remark. For a $\mathbf{t} \in \mathcal{T}$ with $c_1(\mathbf{t}) \neq 0$, the condition $\mathbf{t} \in \mathcal{T}_1$ is equivalent to $c_1(\mathbf{t}) = 1$.

Let $\mathcal{T}_{\overline{\mathbb{Q}}}$ be the algebraic subset of \mathcal{T} , i.e.,

$$\mathcal{T}_{\overline{\mathbb{Q}}} = \{\mathbf{t} \in \mathcal{T} \mid c_i(\mathbf{t}) \in \overline{\mathbb{Q}} \text{ for every } 0 \leq i \leq k\}.$$

Lemma 2.9. *We have $\mathcal{T}_1 \subset \mathcal{T}_{\overline{\mathbb{Q}}}$.*

Proof. It follows from Corollary 2.8 that $\#\mathcal{T}_1 = \#\mathcal{F} < \infty$. Let us fix an element $\mathbf{t}_1 \in \mathcal{T}_1$ and put $\mathcal{T}_2 = \{\mathbf{t} \in \mathcal{T}_1 | I(\mathbf{t}) = I(\mathbf{t}_1)\}$. For an integer $i \in \mathbb{Z}$ with $c_i(\mathbf{t}) \neq 0$ we define a polynomial $f_i(\mathfrak{s}) \in \mathbb{C}[S]$ such that

$$f_i(\mathfrak{s}) = \prod_{\mathbf{t} \in \mathcal{T}_2} (c_i(\mathfrak{s}) - c_i(\mathbf{t})) \in \mathbb{C}[s_i].$$

Then $f_i(\mathbf{t}) = 0$ holds for all $\mathbf{t} \in \mathcal{T}_2$. Note that \mathcal{T}_2 is equal to the set of zeros of simultaneous equations

$$\begin{aligned} c_i(-m/n, \mathfrak{s}) &= 0 \text{ for } l < i < k + l, \\ c_i(\mathfrak{s}) &= 0 \text{ for } 1 \leq i \leq k \text{ and } i \notin I(\mathbf{t}_1), \\ c_{i_1}(\mathfrak{s})^{\lambda_1} - \prod_{j=2}^k c_{i_j}(\mathfrak{s})^{\lambda_j} &= 0, \end{aligned}$$

where $I(\mathbf{t}_1) = \{i_1 < i_2 < \dots < i_\kappa\}$ is the non-vanishing index set of \mathbf{t}_1 and $(\lambda_1, \lambda_2, \dots, \lambda_\kappa)$ is the minimization operator of $I(\mathbf{t}_1)$. Here the above simultaneous equations consist of polynomials in $\mathbb{Q}[S]$. Thus Hilbert zero point theorem implies that $f_i(\mathfrak{s})^r \in \mathbb{Q}[S]$ for some positive integer $r \in \mathbb{Z}$. Since $f_i(\mathfrak{s})^r \in \mathbb{Q}[S] \cap \mathbb{C}[s_i] = \mathbb{Q}[s_i]$, it holds that $c_i(\mathbf{t}) \in \overline{\mathbb{Q}}$ for every $\mathbf{t} \in \mathcal{T}_2$. Hence we have $\mathcal{T}_1 \subset \mathcal{T}_{\overline{\mathbb{Q}}}$. \square

Let $\Gamma_{\mathbb{Q}}$ be the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of \mathbb{Q} . The action of $\sigma \in \Gamma_{\mathbb{Q}}$ on $\mathbf{t} = (t_1, t_2, \dots, t_k) \in \mathcal{T}_{\overline{\mathbb{Q}}}$ is defined by $\sigma\mathbf{t} = (\sigma t_1, \sigma t_2, \dots, \sigma t_k)$. For a fixed $\mathbf{t} = (t_1, t_2, \dots, t_k) \in \mathcal{T}_{\overline{\mathbb{Q}}}$ let us denote by $\mathbb{Q}(\mathbf{t})$ the field $\mathbb{Q}(t_1, t_2, \dots, t_k)$, and by $\mathbb{Q}(\beta)$ the definition field of the polynomial $\beta(X) = \beta_{k,l}(m, n; \mathbf{t})(X)$. The moduli field $\mathcal{M}(D)$ of the dessin $D = D_\beta$ is the fixed field $\overline{\mathbb{Q}}^{\Gamma_D}$ where $\Gamma_D = \{\sigma \in \Gamma_{\mathbb{Q}} | D^\sigma \sim D\}$. Then we have $\mathcal{M}(D) \subseteq \mathbb{Q}(\beta) \subseteq \mathbb{Q}(\mathbf{t})$ in general.

Proposition 2.10. *If $\mathbf{t} \in \mathcal{T}_1$, then $\mathcal{M}(D) = \mathbb{Q}(\beta) = \mathbb{Q}(\mathbf{t})$.*

Proof. Let us note that $\beta_{k,l}(m, n; \mathfrak{s})(X) \in \mathbb{Q}[S][X]$. Thus $\sigma\beta_{k,l}(m, n; \mathbf{t})(X) = \beta_{k,l}(m, n; \sigma\mathbf{t})(X)$ for $\sigma \in \Gamma_{\mathbb{Q}}$ and $\mathbf{t} \in \mathcal{T}_1$. Let $\sigma \in \Gamma_{\mathbb{Q}}$ be an element in Γ_D , that is, $D_{\beta^\sigma} \sim D_\beta$. By the same argument as in the proof of Proposition 2.4 we have $\sigma\beta_{k,l}(m, n; \mathbf{t})(X) = \beta_{k,l}(m, n; \mathbf{t})(\alpha X)$ for an $\alpha \in \mathbb{C}^\times$. It is easy to see that $\sigma\mathbf{t} = \alpha\mathbf{t}$ since $m \neq n$. It follows from the definition that $\text{dir}(\sigma\mathbf{t}) = \sigma(\text{dir}(\mathbf{t})) = \sigma(1) = 1$ for $\mathbf{t} \in \mathcal{T}_1$. On the other hand, we have $\text{dir}(\alpha\mathbf{t}) = \alpha^p \text{dir}(\mathbf{t}) = \alpha^p$ where $p = \text{pd}(\mathbf{t})$. This means that $\alpha^p = 1$ and $\sigma\mathbf{t} = \alpha\mathbf{t} = \mathbf{t}$. Hence we have $\mathbb{Q}(\mathbf{t}) \subseteq \mathcal{M}(D)$, which concludes $\mathcal{M}(D) = \mathbb{Q}(\beta) = \mathbb{Q}(\mathbf{t})$. \square

Proposition 2.10 verifies the second assertion of Theorem 1.4.

Remark. The notion of the normalized element $\mathbf{t} \in \mathcal{T}_1$ is essentially similar to that of a normalized model in [20].

3. SOME NUMERICAL EXAMPLES

In this section we calculate some Belyi functions by using Theorem 1.4. Let us consider the case of the flower tree with ramification $\langle m \rangle + l\langle n \rangle$ where l, m and n are positive integers with $m \neq n$. Since the set $\{j \in \mathbb{Z} | l < j < l + 1\}$ is empty, one sees $\mathcal{T}(1, l, m, n) = \{(t_1) | t_1 \in \mathbb{C}^\times\}$ and $\mathcal{T}(1, l, m, n)_1 = \{(1)\}$.

Proposition 3.1. *We have $\mathcal{T}(1, l, m, n)_1 = \{(1)\}$ and*

$$\beta_{1,l}(m, n; (1))(X) = (1 + X)^m \left(\sum_{j=0}^l c_j(-m/n, (1)) X^j \right)^n$$

where $c_j(-m/n, (1)) = (-m/n)(-m/n - 1) \cdots (-m/n - j + 1)/(j!)$ for every $j \in \mathbb{Z}$. In particular, the Belyi function $\beta_{1,l}(m, n; (1))(X)$ is defined over \mathbb{Q} .

We have the following proposition for the case of the flower trees with ramification $2\langle m \rangle + l\langle n \rangle$ where $m \neq n$.

Proposition 3.2. *If l is odd, then*

$$\mathcal{T}(2, l, m, n)_1 = \{(1, t_2) | c_{l+1}(-m/n, (1, t_2)) = 0\}.$$

When l is even, we have

$$\mathcal{T}(2, l, m, n)_1 = \{(1, t_2) | c_{l+1}(-m/n, (1, t_2)) = 0\} \cup \{(0, 1)\}.$$

For each $(1, t_2) \in \mathcal{T}(2, l, m, n)_1$, it holds that

$$\beta_{2,l}(m, n; (1, t_2))(X) = (1 + X + t_2 X^2)^m \left(\sum_{j=0}^l c_j(-m/n, (1, t_2)) X^j \right)^n$$

where

$$c_j(-m/n, (1, t_2)) = \sum_{i=0}^{\lfloor j/2 \rfloor} \frac{(-m/n)(-m/n - 1) \cdots (-m/n - (j - i) + 1)}{(j - 2i)! i!} t_2^i.$$

In particular, $\beta_{2,l}(m, n; (1, t_2))(X)$ is defined over the moduli field $\mathbb{Q}(t_2)$. For $(0, 1) \in \mathcal{T}(2, l, m, n)_1$ it satisfies $\beta_{2,l}(m, n; (0, 1))(X) = \beta_{1,l/2}(m, n; (1))(X^2)$.

Proof. Lemma 2.5 implies that $\text{pd}(\mathfrak{t}) = 1$ for every $\mathfrak{t} \in \mathcal{T}(2, l, m, n)$ if l is odd. This means $c_1(\mathfrak{t}) \neq 0$ and $\text{nom}(\mathfrak{t}) = (1, t_2)$ for some $t_2 \in \mathbb{C}$. When l is even, we have $(0, t_2) \in \mathcal{T}(2, l, m, n)$ since $c_{l+1}((-1)\mathfrak{s}) = -c_{l+1}(\mathfrak{s})$. Note that $\text{nom}((0, t_2)) = (0, 1) \in \mathcal{T}(2, l, m, n)_1$. Thus Theorem 1.4 shows the assertion. □

For the flower trees with ramification $2\langle m \rangle + 3\langle n \rangle = \langle m, m, n, n, n \rangle$ we have $\mathcal{T}(2, 3, m, n)_1 = \{(1, t_2^+), (1, t_2^-)\}$ where

$$t_2^\pm = \frac{3(m/n + 2) \pm \sqrt{3(m/n + 2)(2m/n + 3)}}{6},$$

respectively.

Corollary 3.3. *For every real quadratic field $\mathbb{Q}(\sqrt{d})$ there exist infinitely many flower tree dessins D with ramification $2\langle m \rangle + 3\langle n \rangle$ so that $\mathcal{M}(D) = \mathbb{Q}(\sqrt{d})$.*

Proof. Let $d \in \mathbb{Z}$ be a positive integer. Then there exist infinitely many rational numbers $r \in \mathbb{Q}$ such that $3/2 < r < 2$ and $r = du^2$ for some $u \in \mathbb{Q}$. For such an $r \in \mathbb{Q}$ not equal to $9/5$, let m and n be positive integers with $m/n = -3(r - 2)/(2r - 3)$. Then we have $\mathbb{Q}(t_2^+) = \mathbb{Q}(t_2^-) = \mathbb{Q}(\sqrt{d})$. \square

For the flower trees with ramification $2\langle 4 \rangle + 3\langle 1 \rangle = \langle 1, 1, 1, 4, 4 \rangle$, we have $\mathcal{T}(2, 3, 4, 1)_1 = \{(1, t_2^-), (1, t_2^+)\}$ where $t_2^\pm = (6 \pm \sqrt{22})/2$. The Belyi functions are equal to

$$\begin{aligned} \beta_{2,3}(4, 1; (1, t_2^\pm))(X) &= (1 + X + (6 \pm \sqrt{22})/2X^2)^4 \\ &\quad \times (1 - 4X - 2(1 \pm \sqrt{22})X^2 + 10(4 \pm \sqrt{22})X^3) \\ &\equiv 1 + 22(23 \pm 5\sqrt{22})X^5 \pmod{X^6\mathbb{Q}(\sqrt{22})[X]}, \end{aligned}$$

respectively. One can check that the dessin of $\beta_{2,3}(4, 1; (1, t_2^+))(X)$ is the left graph in Figure 3.4 and that of $\beta_{2,3}(4, 1; (1, t_2^-))(X)$ is the right one. The two graphs in Figure 3.4 are conjugate of each other under a Galois action $\sigma \in \Gamma_{\mathbb{Q}}$ such that $\sigma(\sqrt{22}) = -\sqrt{22}$.

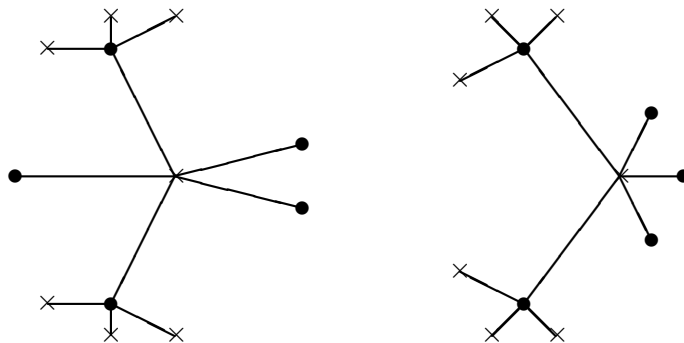


Figure 3.4 (two flower trees with ramification $2\langle 4 \rangle + 3\langle 1 \rangle$)

As a special case we can obtain the flower tree with one ramification index, that is, the flower tree with ramification $k\langle m \rangle$ where $k, m \in \mathbb{Z}$. In Theorems 1.3 and 1.4 let us take $l = 0$ and $n \geq 1$. Then $\mathbf{t} = (0, 0, \dots, 0, 1) \in \mathbb{C}^{k*}$

satisfies $c_j(-m/n, \mathbf{t}) = 0$ for every $0 < j < k$. Here $\beta_{k,0}(m, n; \mathbf{t}) = (1 + X^k)^m$ is a Belyi function whose dessin is the flower tree with ramification $k\langle m \rangle$. It is clear that there exists only one flower tree with ramification $k\langle m \rangle$.

Proposition 3.5. *The Belyi function for the flower tree with ramification $k\langle m \rangle$ is equal to $(1 + X^k)^m$, which is defined over \mathbb{Q} .*

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