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Throughout, A will represent a ring with the center C. For $x, y \in A$ and a positive integer k, we define inductively $[x, y]_0 = x$, $[x, y]_1 = [x, y]$ $(= xy - yx)$, $[x, y]_k = [[x, y]_{k-1}, y]$.

M. S. Putcha and A. Yaqub have made a remark in [8] that any semisimple ring A is commutative if $xy^2x = yx^2y$ for every pair of elements x and y in A, and they gave an example of a non-commutative, non-semisimple ring satisfying the above identity. We have extended the result to semiprime rings in [7]. On the other hand, H. E. Bell [2] has proved that if for each $x, y \in A$ there exist positive integers m, n such that $xy = y^m x^n$, then A is commutative. Now, let D be a division ring, and $A_k = \{(a_{ij}) \in (D)_k \mid a_{ij} = 0\ (i \leqslant j)\}$. If $k > 2$ then A_k is a non-commutative nilpotent ring of index k. For any positive integers m, n, A_3 does not satisfy the identity $xy - y^m x^n = 0$, but $[xy - y^m x^n, z] = 0$. Or, more generally, A_5 does not satisfy $xy^2x - (yx)^m (xy)^n = 0$, but $[xy^2x - (yx)^m (xy)^n, z] = 0$. Therefore, it is natural to investigate the structure of rings satisfying the last identity. The purpose of this note is prove the following

Theorem. *Suppose A satisfies one of the following polynomial identities:*

(P$_1$) $[(xy)^n - x^n y^n, x] = [(xy)^n - x^n y^n, y] = 0$, *where* $n > 1$.

(P$_2$) $[(xy)^n - y^n x^n, x] = [(xy)^n - y^n x^n, y] = 0$, *where* $n > 1$.

(P$_3$) $[(x+y)^n - x^n - y^n, x] = 0$, *where* $n > 1$.

(P$_4$) $[xy^2x - (yx)^m (xy)^n, z] = 0$, *where* $m, n \geq 1$.

(P$_5$) $[xy^2 - y^m x^n, z] = 0$, *where* $m, n \geq 1$.

(P$_6$) $[[x, y]z - z^m [x, y]^n, w] = 0$, *where* $m, n \geq 1$.

Then the prime radical of A coincides with the set of all nilpotent elements and includes the commutator ideal of A.

We begin with

Lemma 1. (1) *Let A be a prime ring, and c in C. If $ac = 0$ then either $a = 0$ or $c = 0$.*

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(2) *Any reduced prime ring is an integral domain.*

(3) *Let A be a semiprime ring. If $a^2=0$ and $ax^2a=0$ for any $x \in A$, then $a=0$.*

Proof. It is enough to prove (3) only. For any $x, y \in A$ we have $0=a(x+ay)^2a=axaya$, whence it follows that $(aA)^3=0$. Thus $a=0$.

Lemma 2. *Let A be a division ring, and k a non-negative integer. If there exist integers $n>m\geq 1$ such that*

$$[x, y]_k^n - [x, y]_k^m \in C \quad \textit{for all } x, y \in A,$$

then $[x, y]_k^m \in C$. If furthermore $(m, n)=1$, then A is commutative.

Proof. Assume there exist some $a, b \in A$ such that $[a, b]_k^m \notin C$. Let $d=[a, b]_k$, and c an arbitrary element of C. Then $cd=[ca, b]_k$ and $(c^n-c^m)d^m=(cd)^n-(cd)^m-c^n(d^n-d^m) \in C$. Hence, c^n-c^m must be 0, which means that C is a finite field. On the other hand, A is finite dimensional over C by Kaplansky's theorem [6]. This amounts to saying that A is a finite field, a contradiction. In case $(m, n)=1$, we readily obtain $[x, y]_k \in C$. Hence, the latter assertion is a consequence of [4, Theorem].

Lemma 3. *If a division ring A satisfies one of the polynomial identities $(P_1)-(P_6)$, then A is commutative.*

Proof. According to [3, Lemma 1], in order to prove the lemma for $(P_1)-(P_3)$, it is enough to show that $[x^s, y^t]=0$ for any non-zero $x, y \in A$ with some positive integers s, t.

(P_1) Setting $c_1=x[x^{n-1}, y^n]$, we have

$$c_1=x^ny^n-(xy)^n+x\{(yx)^n-y^nx^n\}x^{-1},$$

so that $[c_1, x]=[c_1, y]=0$, and similarly for $c_2=x^2[x^{2(n-1)}, y^n]$ there holds $[c_2, y]=0$. By a brief computation, one obtains $2c_1-c_2x^{-n}=x^nc_1x^{-n}+xc_1x^{-1}-c_2x^{-n}=0$. If $[x^{2(n-1)}, y^n]=x^{-2}c_2$ is non-zero, then $x^{-n}=c_2^{-1}(2c_1)$ commutes with y, that is, $[x^n, y]=0$.

(P_2) Setting $c_1=x^{-1}[x^{n+1}, y^n]$, we have

$$c_1=x^ny^n-(yx)^n+x^{-1}\{(xy)^n-y^nx^n\}x,$$

so that $[c_1, x]=[c_1, y]=0$, and similarly for $c_2=x^{-2}[x^{2(n+1)}, y^n]$ there holds $[c_2, y]=0$. One obtains also $2c_1-c_2x^{-n}=x^nc_1x^{-n}+x^{-1}c_1x-c_2x^{-n}=0$. If $[x^{2(n+1)}, y^n]=x^2c_2$ is non-zero, then $x^{-n}=c_2^{-1}(2c_1)$ commutes with y, namely, $[x^n, y]=0$.

(P_3) By Kaplansky's theorem [6], A is finite dimensional over C. Since $[x^n, y]-[x, y^n]=[x^n+y^n-(x+y)^n, x+y]=0$, for any $c \in C$ we have

$(c^n-c)[x^n, y]=[c^n x^n, y]-c[x, y^n]=[(cx)^n, y]-[cx, y^n]=0$. If $[x^n, y]\neq 0$, then $c^n-c=0$ for all $c\in C$. Obviously, C is then finite, and A is also finite. Hence A is commutative, which is a contradiction.

(P_4) If $m=n=1$, then $[xy, yx]=xy^2x-yx^2y\in C$. Since $[x, y]_2=[(1+x)y, y(1+x)]-[xy, yx]\in C$, A is commutative by [4, Theorem]. In below, we assume $m+n>2$. In general, $[x, y]^2-[x, y]^{m+n}=[x, y]\cdot 1^2\cdot [x, y]-(1\cdot[x, y])^m([x, y]\cdot 1)^n\in C$, and hence by Lemma 2 we have $[x, y]^2$, $[x, y]^{m+n}\in C$. Especially, in case $m+n$ is odd, A is commutative by Lemma 2. Henceforth, we may restrict our attention to the case $m+n$ is even. According to [3, Lemma 1], it suffices to show that $x^2\in C$ for all $x\in A$. Assume there exists an a such that $a^2\notin C$. Then $d=[a, b]\neq 0$ for some b. Recalling that $(da)^2=[a, ba]^2\in C$, one obtains $(ad)^2=(ad)^3(ad)^{-1}=a(da)^2d(ad)^{-1}=(da)^2\in C$. If both m and n are even, then $a^2=[\{ad^2a-(da)^m(ad)^n\}+(da)^m(ad)^n]d^{-2}\in C$, a contradiction. Finally, we consider the case $m=2h+1$ and $n=2k+1$, where $s=h+k>0$. Since $ad^2a-(da)^m(ad)^n\in C$, we get $a^2d^2+(ad)^{2s}a^2d^2=\{1+(ad)^{2s}\}a^2d^2\in C$. It follows then $1+(ad)^{2s}=0$, and hence $(ad)^{4s}=1$. Similarly, noting that $(ca)^2\notin C$ and $cd=[ca, b]$ for any non-zero $c\in C$, we obtain $(c^2ad)^{4s}=1$. Hence, $c^{8s}=c^{8s}(ad)^{4s}=(c^2ad)^{4s}=1$. This implies that C is finite. Again by Kaplansky's theorem [6], A will be seen to be commutative. This is a contradiction.

(P_5) Since $x-x^n$, y^2-y^m, $y-y^{m-n}\in C$, if either $n>1$ or $n=1$ and $m\neq 2$ then A is commutative by Lemma 2. If $n=1$ and $m=2$ then $xy^2-y^2x\in C$, and A is commutative by [4, Theorem].

(P_6) Since $[x, y]-[x, y]^n\in C$, if $n>1$ then A is commutative by Lemma 2. Next, if $m=n=1$ then $[x, y]_2=[x, y]y-y[x, y]\in C$, and A is commutative by [4, Theorem]. Finally, if $m>1$ and $n=1$ then $(c-c^m)[x, y]\in C$ for all $c\in C$. If A is not commutative, Kaplansky's theorem [6] will yield a contradiction.

Proof of Theorem. To our end, it suffices to prove that if A is prime and satisfies one of the polynomial identities (P_1)−(P_6) then A is commutative. According to Amitsur's theorem [1], any integral domain satisfying a polynomial identity (with coefficients ± 1) has the division ring of quotients satisfying the same polynomial identity. Thus, by the validity of Lemma 3, our proof will be completed by showing that a prime ring A satisfying one of the identities (P_1)−(P_6) is an integral domain. To this end, we assume that there exists a non-zero element a with $a^2=0$ (see Lemma 1 (2)).

(P_1) and (P_2) To be easily seen, $(ax)^n$ commutes with a for any $x\in$

A. Thus, $0=a(ax)^n=(ax)^n a=(ax)^{n+1}$, whence it follows a contradiction $a=0$ ([5, Lemma 1. 1]).

(P_3) Obviously, $(ax)^{n-1}a=(a+ax)^n-(ax)^n=(a+ax)^n-a^n-(ax)^n$ commutes with ax for all $x\in A$, that is, $0=(ax)^{n-1}a(ax)x=(ax)(ax)^{n-1}ax=(ax)^{n+1}$. Again by [5, Lemma 1. 1], we have a contradiction $a=0$.

(P_4) For any $x\in A$ we have $ax^2a=ax^2a-(xa)^m(ax)^n\in C$. Since $a(ax^2a)=0$, it follows that $a=0$ by Lemma 1 (1) and (3).

(P_5) For any $x\in A$, $a(xa)^2=a(xa)^2-(xa)^m a^n\in C$. By Lemma 1 (1), $a\cdot a(xa)^2=0$ implies that $a(xa)^2=0$. Hence $(ax)^3=a(xa)^2x=0$, and $a=0$ by [5, Lemma 1. 1].

(P_6) Obviously, $(ax)^2a=[ax, a]xa-(xa)^m[ax, a]^n\in C$. By Lemma 1 (1), $a(ax)^2a=0$ implies that $(ax)^2a=0$, so that $(ax)^3=0$. Hence, $a=0$ by [5, Lemma 1. 1].

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REFERENCES

[1] S. A. AMITSUR: On rings with identities, J. London Math. Soc. **30** (1955), 464—470.

[2] H. E. BELL: A commutativity condition for rings, Canad. J. Math. **28** (1976), 986—991.

[3] L. P. BELLUCE, I. N. HERSTEIN and S. K. JAIN: Generalized commutative rings, Nagoya Math. J. **27** (1966), 1—5.

[4] A. HARMANCI: On the commutativity of some class of rings, J. Austral. Math. Soc. **21** (1976), 376—380.

[5] I. N. HERSTEIN: Topics in Ring Theory, Univ. of Chicago Press, 1969.

[6] I. KAPLANSKY: Rings with a polynomial identity, Bull. Amer. Math. Soc. **54** (1948), 575—580.

[7] A. KAYA: Notes on the commutativity of rings, METU J. Pure Appl. Sci. **9** (1976), 261—265.

[8] M. S. PUTCHA and A. YAQUB: Multiplicative commutators in division rings, Math. Japonica **19** (1974), 111—115.

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