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Note on a mean ergodic theorem

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NOTE ON A MEAN ERGODIC THEOREM

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S.-Y. Shaw [2] showed an interesting mean ergodic theorem, and recently R. Sato [1] improved his result. In this paper we shall show that an analogous result holds for k-parameter semigroups of operators. To do this we apply the method of Sato.

The result is the following

Theorem. *Let* $\{T(t_1,\cdots,t_k)\ ;\ t_1,\cdots,t_k \geqq 0\}$ *be a strongly measurable k-parameter semigroup of uniformly bounded linear operators on a Banach space X. Suppose there exist* $\delta_i > 0$ $(i=1,\cdots,k)$ *such that* $\|T(0,\cdots,0,t_i,0,\cdots,0)-I\| < 2$ *for all* $0 < t_i < \delta_i$. *Then for each* $0 < t_i < \delta_i$ $(i=1,\cdots,k)$ *we have*

$$\lim_{a_1,\cdots,a_k\to\infty}\frac{1}{a_1\cdots a_k}\int_0^{a_k}\cdots\int_0^{a_1} T(s_1,\cdots,s_k)x\ ds_1\cdots ds_k$$
$$=\lim_{n\to\infty}\frac{1}{n^k}\sum_{i_1=0}^{n-1}\cdots\sum_{i_k=0}^{n-1} T(i_1t_1,\cdots,i_kt_k)x$$

whenever one of these limits exists.

Proof Let X_0 (resp. $X_{t_1,\cdots,t_k}$) be the set of x for which

$$P_0x \equiv \lim_{a_1,\cdots,a_k\to\infty}\frac{1}{a_1\cdots a_k}\int_0^{a_k}\cdots\int_0^{a_1} T(s_1,\cdots,s_k)x\ ds_1\cdots ds_k$$
$$(\text{resp. } P_{t_1,\cdots,t_k}x \equiv \lim_{n\to\infty}\frac{1}{n^k}\sum_{i_1=0}^{n-1}\cdots\sum_{i_k=0}^{n-1} T(i_1t_1,\cdots,i_kt_k)x)$$

exists. Then by the uniform boundedness of the semigroup we get

$$X_0 = \bigcap_{s_1,\cdots,s_k>0} N[T(s_1,\cdots,s_k)-I]\oplus \overline{\mathrm{sp}}\bigcup_{s_1,\cdots,s_k>0} R[T(s_1,\cdots,s_k)-I]$$

and

$$X_{t_1,\cdots,t_k} = \bigcap_{i_1,\cdots,i_k=1}^{\infty} N[T(i_1t_1,\cdots,i_kt_k)-I]$$
$$\oplus\ \overline{\mathrm{sp}}\bigcup_{i_1,\cdots,i_k=1}^{\infty} R[T(i_1t_1,\cdots,i_kt_k)-I],$$

where $\overline{\mathrm{sp}}\ U$ denotes the closed linear space spanned by U. It is clear that if $x\in \bigcap_{i_1,\cdots,i_k=1}^{\infty} N[T(i_1t_1,\cdots,i_kt_k)-I]$ then P_0x exists, thus $X_{t_1,\cdots,t_k}\subset X_0$. On the other hand, we have

$$\frac{1}{n^k}\sum_{i_1=0}^{2n-1}\sum_{i_2=0}^{n-1}\cdots\sum_{i_k=0}^{n-1} T(\frac{i_1t_1}{2}, i_2t_2, \cdots, i_kt_k)x$$
$$= [2I+(T(\frac{t_1}{2},0,\cdots,0)-I)]\,[\frac{1}{n^k}\sum_{i_1=0}^{n-1}\cdots\sum_{i_k=0}^{n-1} T(i_1t_1,\cdots,i_kt_k)x],$$

so that for any x in $\overline{\mathrm{sp}}\bigcup_{i_1,\cdots,i_k=1}^{\infty} R[T(\frac{i_1t_1}{2}, i_2t_2,\cdots,i_kt_k)-I]$, namely for any x with $P_{t_1/2,t_2,\cdots,t_k}x=0$,

$$\left\|\frac{1}{n^k}\sum_{i_1=0}^{n-1}\cdots\sum_{i_k=0}^{n-1} T(i_1t_1,\cdots,i_kt_k)x\right\|$$
$$\leqq (2-\left\|T(\frac{t_1}{2},0,\cdots,0)-I\right\|)^{-1}\left\|\frac{1}{n^k}\sum_{i_1=0}^{2n-1}\sum_{i_2=0}^{n-1}\cdots\sum_{i_k=0}^{n-1} T(\frac{i_1t_1}{2}, i_2t_2,\cdots,i_kt_k)x\right\| \to 0$$

as $n\to\infty$. Thus $P_{t_1,\cdots,t_k}x=0$, that is,

$$\overline{\mathrm{sp}}\bigcup_{i_1,\cdots,i_k=1}^{\infty} R[T(\frac{i_1t_1}{2}, i_2t_2,\cdots,i_kt_k)-I]$$
$$\subset \overline{\mathrm{sp}}\bigcup_{i_1,\cdots,i_k=1}^{\infty} R[T(i_1t_1,\cdots,i_kt_k)-I].$$

Doing this process successively and applying an approximation argument, we finally observe that

$$\overline{\mathrm{sp}}\bigcup_{s_1,\cdots,s_k>0} R[T(s_1,\cdots,s_k)-I]$$
$$\subset \overline{\mathrm{sp}}\bigcup_{i_1,\cdots,i_k=1}^{\infty} R[T(i_1t_1,\cdots,i_kt_k)-I].$$

Since P_0 (resp. $P_{t_1,\cdots,t_k}$) is a projection onto $\bigcap_{s_1,\cdots,s_k>0} N[T(s_1,\cdots,s_k)-I]$ (resp. $\bigcap_{i_1,\cdots,i_k=1}^{\infty} N[T(i_1t_1,\cdots,i_kt_k)-I]$), the proof is completed.

References

[1] R. Sato: On a mean ergodic theorem, Proc. Amer. Math. Soc. (to appear).

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