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A CLASS OF ABSTRACT QUASI-LINEAR EVOLUTION EQUATIONS OF SECOND ORDER

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1. *Introduction*

In this paper we study the abstract quasi-linear evolution equation of second order

$$\begin{cases} u''(t) = A(t, u(t), u'(t))\, u(t) & \text{for } t \in [0, T] \\ u(0) = \phi,\ u'(0) = \psi \end{cases} \tag{1.1}$$

in a general Banach space Z. It is well-known that the abstract quasi-linear theory due to Kato [**10**, **11**] is widely applicable to quasi-linear partial differential equations of second order and that his theory is based on the theory of semigroups of class (C_0). (For example, see the work of Hughes *et al*. [**9**] and Heard [**8**].) However, even in the special case where $A(t, w, v) = A$ is independent of (t, w, v), it is found in [**2**] and [**14**] that there exist linear partial differential equations of second order for which Cauchy problems are not solvable by the theory of semigroups of class (C_0) but fit into the mould of well-posed problems where the solution and its derivative depend continuously on the initial data if the initial condition is measured in the graph norm of a suitable power of A. (See also work by Krein and Khazan [**13**] and Fattorini [**6**, Chapter 8].) This kind of Cauchy problem has recently been studied extensively, using the theory of integrated semigroups or regularized semigroups. The theory of integrated semigroups was studied intensively by Arendt [**1**] and that of regularized semigroups was initiated by Da Prato [**3**] and renewed by Davies and Pang [**4**]. For the theory of regularized semigroups we refer the reader to [**5**] and [**16**].

The second-order equation (1.1) is converted into the first-order system

$$(u(t), v(t))' = \tilde{A}^u(t)\,(u(t), v(t)) \quad \text{for } t \in [0, T] \quad \text{and} \quad (u(0), v(0)) = (\phi, \psi)$$

in a suitable Banach space $\tilde{X}$, where for each solution w of equation (1.1) the matrix operator $\tilde{A}^w(t)$ in $\tilde{X}$ is defined by $\tilde{A}^w(t)\,(u, v) = (v, A(t, w(t), w'(t))\, u)$. We are here interested in studying the case where each matrix operator $\tilde{A}^w(t)$ is the (complete infinitesimal) generator of a regularized semigroup on $\tilde{X}$. In Section 3 we set up basic hypotheses on the operators appearing in equation (1.1), and prove a fundamental existence and uniqueness theorem (Theorem 3.6) for the Cauchy problem (1.1). The proof is based on the theory of regularized evolution operators developed by the author [**15**], and a method of successive approximations proposed by Kobayasi and Sanekata [**12**] is applied to construct a unique twice continuously differentiable function u satisfying equation (1.1).

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Our formulation includes the abstract quasi-linear wave equation of Kirchhoff type

$$u''(t)+m(|A^{1/2}u(t)|^2)\,Au(t) = 0 \tag{1.2}$$

in a real Hilbert space H, where A is a nonnegative selfadjoint operator in H. Section 4 presents a regularized semigroup theoretical approach to the local solvability of equation (1.2) in the 'degenerate case' where the function $m(r)$ has zeros (Theorems 4.1 and 4.2), by using the result obtained in Section 3. In Section 2 we summarize some results on the generation of a regularized evolution operator associated with the linearized equation of (1.1), under the 'regularized stability' condition, and show that the family of matrix operators used to solve the linearized equation (1.2) satisfies the regularized stability condition. This fact will be useful for our arguments in Section 4.

2. *Regularized evolution operators*

In this section we consider a pair of real Banach spaces $(\tilde{X}, \tilde{Y})$ satisfying the following condition (2.1):

$$\tilde{Y} \text{ is densely and continuously embedded in } \tilde{X}. \tag{2.1}$$

The norms of $\tilde{X}$ and $\tilde{Y}$ are denoted by $\|\cdot\|_{\tilde{X}}$ and $\|\cdot\|_{\tilde{Y}}$ respectively. The symbol $B(\tilde{Y}, \tilde{X})$ denotes the set of all bounded linear operators on $\tilde{Y}$ to $\tilde{X}$. This section is devoted to an exposition on the generation of a regularized evolution operator associated with a given family $\{\tilde{A}(t) : t \in [0, T]\}$ of closed linear operators in $\tilde{X}$.

We begin by stating the following conditions (1)–(3) for the family $\{\tilde{A}(t) : t \in [0, T]\}$ of closed linear operators in $\tilde{X}$.

(1) There exist an injective operator $\tilde{C}$ in $B(\tilde{X})$ with dense range and two constants $M \geqslant 1$ and $\beta \geqslant 0$ such that if $\lambda > 0$ satisfies $\lambda\beta < 1$ then the following conditions are satisfied.

(i) $I-\lambda\tilde{A}(t)$ is injective for $t \in [0, T]$.

(ii) $D(\prod_{i=1}^{m}(I-\lambda\tilde{A}(t_i))^{-1}) \supset R(\tilde{C})$ and

$$\left\|\prod_{i=1}^{m}(I-\lambda\tilde{A}(t_i))^{-1}\,\tilde{C}\right\|_{\tilde{X}} \leqslant M$$

for every finite sequence $\{t_i\}_{i=1}^{m}$ with $0 \leqslant t_1 \leqslant \ldots \leqslant t_m \leqslant T$ and m with $0 \leqslant \lambda m \leqslant T$.

(iii) $\tilde{C}^{-1}\tilde{A}(t)\,\tilde{C} \supset \tilde{A}(t)$ for $t \in [0, T]$.

(2) There exists an isomorphism $\tilde{S}$ of $\tilde{Y}$ onto $\tilde{X}$ such that

$$\tilde{S}\tilde{A}(t)\,\tilde{S}^{-1} = \tilde{A}(t) \quad \text{for } t \in [0, T], \quad \text{and} \quad \tilde{S}\tilde{C}\tilde{S}^{-1} = \tilde{C}.$$

(3) $\tilde{Y} \subset D(\tilde{A}(t))$ for each $t \in [0, T]$, and $\tilde{A}(t)$ is norm continuous in $B(\tilde{Y}, \tilde{X})$ on $[0, T]$.

Condition (1) is called the *regularized stability condition*, and the set of all families $\{\tilde{A}(t) : t \in [0, T]\}$ of closed linear operators in $\tilde{X}$ satisfying the two conditions (1)(i) and (1)(ii) is denoted by $RS(\tilde{X}, M, \beta, \tilde{C})$. We are now in a position to state the generation theorem of regularized evolution operators.

THEOREM 2.1. *Under assumption* (2.1) *and conditions* (1)–(3), *there exists a unique family* $\{\tilde{U}(t, s)\}$ *in* $B(\tilde{X})$ *defined on the triangle*

$$\Delta = \{(t, s) : 0 \leqslant s \leqslant t \leqslant T\}$$

with the following properties.

(i) $\tilde{U}(t,t) = \tilde{C}$ *and* $\tilde{U}(t,r)\,\tilde{U}(r,s) = \tilde{U}(t,s)\,\tilde{C}$ *for* $(t,r),(r,s) \in \Delta$.

(ii) $\tilde{U}(t,s)$ *is strongly continuous in* $B(\tilde{X})$ *on* Δ, *and* $\|\tilde{U}(t,s)\|_{\tilde{X}} \leqslant M$ *for* $(t,s) \in \Delta$.

(iii) $\tilde{U}(t,s)(\tilde{Y}) \subset \tilde{Y}$, $\tilde{U}(t,s)$ *is strongly continuous in* $B(\tilde{Y})$ *on* Δ *and* $\|\tilde{U}(t,s)\|_{\tilde{Y}} \leqslant \overline{M}$ *for* $(t,s) \in \Delta$, *where* $\overline{M} = M\|\tilde{S}\|_{\tilde{Y},\tilde{X}}\,\|\tilde{S}^{-1}\|_{\tilde{X},\tilde{Y}}$.

(iv) $\partial\tilde{U}(t,s)/\partial t = \tilde{A}(t)\,\tilde{U}(t,s)$ *and* $\partial\tilde{U}(t,s)/\partial s = -\tilde{U}(t,s)\,\tilde{A}(s)$, *both of which exist in the strong sense in* $B(\tilde{Y},\tilde{X})$ *and are strongly continuous in* $B(\tilde{Y},\tilde{X})$ *on* Δ.

The family $\{\tilde{U}(t,s):(t,s)\in\Delta\}$ in $B(\tilde{X})$ obtained by Theorem 2.1 is called *the regularized evolution operator on* $\tilde{X}$ generated by $\{\tilde{A}(t):t\in[0,T]\}$. Theorem 2.1 was previously proved in [**15**, Theorem 2.1, Theorem 3.2], noting that the fact that $\tilde{C}(\tilde{Y}) \subset \tilde{Y}$ and $\tilde{C}(\tilde{Y})$ is dense in $\tilde{Y}$ is deduced from the assumption that $\tilde{S}\tilde{C}\tilde{S}^{-1} = \tilde{C}$ and $\tilde{C}$ has dense range. We state here the following relation, which will be used in Section 3.

$$\tilde{S}\tilde{U}(t,s) \supset \tilde{U}(t,s)\,\tilde{S} \quad \text{for } (t,s)\in\Delta. \tag{2.2}$$

This relation has already been obtained in [**15**, proof of Theorem 3.2].

We conclude this section by showing that the family of matrix operators used to solve the linearized equation of (1.2) satisfies the regularized stability condition. If A is a nonnegative selfadjoint operator in a real Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ then for each $k = 1,2,\ldots$, the Hilbert space $D(A^{k/2})$ equipped with the inner product

$$\langle u,v\rangle_{[D(A^{k/2})]} = \sum_{i=0}^{k} \langle A^{i/2}u, A^{i/2}v\rangle \quad \text{for } u,v\in D(A^{k/2})$$

is denoted by $[D(A^{k/2})]$.

THEOREM 2.2. *Assume that* A *is a nonnegative selfadjoint operator in a real Hilbert space* H *and that* $a(t)$ *is a nonnegative and nondecreasing continuous function on* $[0,T]$. *Let* $\tilde{X} = [D(A^{1/2})]\times H$ *and* $\tilde{Y} = [D(A)]\times[D(A^{1/2})]$. *Then the family* $\{\tilde{A}(t):t\in[0,T]\}$ *of linear operators in* $\tilde{X}$ *defined by*

$$\begin{cases} \tilde{A}(t)(u,v) = (v, -A(a(t)u)) & \textit{for } (u,v)\in D(\tilde{A}(t)) \\ D(\tilde{A}(t)) = \{(u,v): u\in D(A^{1/2}), v\in D(A^{1/2}) \text{ and } a(t)u\in D(A)\} \end{cases}$$

satisfies conditions (1)–(3) *with a positive constant* M, *depending only on* T *and* $\sup\{a(t):t\in[0,T]\}$, *which is nondecreasing with respect to* T, $\beta = 2$, *an isomorphism* $\tilde{S}$ *of* $\tilde{Y}$ *onto* $\tilde{X}$ *defined by*

$$\tilde{S}(u,v) = ((I+A)^{1/2}u, (I+A)^{1/2}v) \quad \textit{for } (u,v)\in\tilde{Y},$$

and $\tilde{C} := \tilde{S}^{-1}$.

Proof. It is easily seen that all the other conditions except conditions (1)(i) and (1)(ii) are satisfied. We begin by checking condition (1)(i). To this end, let $\lambda > 0$ and $t\in[0,T]$ and assume that $(u,v)\in D(A^{1/2})\times H$ satisfies $a(t)u\in D(A)$, $v\in D(A^{1/2})$ and $(I-\lambda\tilde{A}(t))(u,v) = (0,0)$ which is written as

$$u = \lambda v, \tag{2.3}$$

$$v + \lambda A(a(t)u) = 0. \tag{2.4}$$

By equation (2.4) we have

$$0 = |v|^2 + \langle \lambda v, A(a(t)\,u)\rangle = |v|^2 + a(t)\,\langle A^{1/2}(\lambda v), A^{1/2}u\rangle.$$

Since $a(t)$ is nonnegative we find $v = 0$ by substituting equation (2.3) into the equality above. It follows from (2.3) once again that $u = 0$, and so condition (1)(i) is satisfied.

To check condition (1)(ii), we use a family $\{a_\varepsilon(t) : \varepsilon \in (0,1]\}$ of auxiliary functions defined by $a_\varepsilon(t) = a(t) + \varepsilon$ for $t \in [0,T]$, and note that the operator $\tilde{A}_\varepsilon(t)$ in $\tilde{X}$ defined by $\tilde{A}_\varepsilon(t)\,(u,v) = (v, -a_\varepsilon(t)\,Au)$ for $(u,v) \in D(A) \times D(A^{1/2})$ has the property that for each $\lambda > 0$, $I - \lambda\tilde{A}_\varepsilon(t)$ is injective and $R(I - \lambda\tilde{A}_\varepsilon(t)) = \tilde{X}$. The proof of this property of $\tilde{A}_\varepsilon(t)$ is immediate, since $a_\varepsilon(t) \geqslant \varepsilon$ for $t \in [0,T]$ and A is a maximal monotone operator in H. Now, we show that condition (1)(ii) is satisfied. For this purpose, let $(u_0, v_0) \in D(A^{1/2}) \times D(A^{1/2})$ $(\supset R(\tilde{C}))$, let $\lambda \in (0, 1/2]$ and let $\{t_i\}_{i=1}^m$ be any finite sequence such that $0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_m \leqslant T$ and $0 \leqslant \lambda m \leqslant T$. If we set $(u_i^\varepsilon, v_i^\varepsilon) = \prod_{k=1}^{i} (I - \lambda\tilde{A}_\varepsilon(t_k))^{-1}(u_0, v_0)$ for $1 \leqslant i \leqslant m$, then we have $(u_i^\varepsilon, v_i^\varepsilon) \in D(A) \times D(A^{1/2})$ and

$$(u_i^\varepsilon - u_{i-1}^\varepsilon)/\lambda = v_i^\varepsilon, \tag{2.5}$$

$$(v_i^\varepsilon - v_{i-1}^\varepsilon)/\lambda + a_\varepsilon(t_i)\,Au_i^\varepsilon = 0 \tag{2.6}$$

for $1 \leqslant i \leqslant m$, where $(u_0^\varepsilon, v_0^\varepsilon) = (u_0, v_0)$. Let $1 \leqslant j \leqslant m$ and set $w_i^\varepsilon = \sum_{k=i}^{j} Au_k^\varepsilon$ for $1 \leqslant i \leqslant j$, and $w_{j+1}^\varepsilon = 0$. By (2.5) we have

$$(|A^{1/2}u_i^\varepsilon|^2 - |A^{1/2}u_{i-1}^\varepsilon|^2)/2\lambda \leqslant \langle A^{1/2}u_i^\varepsilon, A^{1/2}u_i^\varepsilon - A^{1/2}u_{i-1}^\varepsilon\rangle/\lambda = \langle w_i^\varepsilon - w_{i+1}^\varepsilon, v_i^\varepsilon\rangle \tag{2.7}$$

for $1 \leqslant i \leqslant j$. By (2.6) we have

$$\begin{aligned}\langle v_i^\varepsilon - v_{i-1}^\varepsilon, w_i^\varepsilon\rangle &= -\lambda a_\varepsilon(t_i)\,\langle w_i^\varepsilon - w_{i+1}^\varepsilon, w_i^\varepsilon\rangle \\ &\leqslant -\lambda a_\varepsilon(t_i)\,(|w_i^\varepsilon|^2 - |w_{i+1}^\varepsilon|^2)/2 \leqslant \lambda(a_\varepsilon(t_i)\,|w_{i+1}^\varepsilon|^2 - a_\varepsilon(t_{i-1})\,|w_i^\varepsilon|^2)/2\end{aligned} \tag{2.8}$$

for $1 \leqslant i \leqslant j$. To obtain the last inequality we have used the fact that $a(t)$ is nondecreasing. Addition of (2.7) and (2.8) gives

$$\begin{aligned}&(|A^{1/2}u_i^\varepsilon|^2 - |A^{1/2}u_{i-1}^\varepsilon|^2)/2\lambda + \langle v_i^\varepsilon, w_{i+1}^\varepsilon\rangle - \langle v_{i-1}^\varepsilon, w_i^\varepsilon\rangle \\ &\qquad - \lambda(a_\varepsilon(t_i)\,|w_{i+1}^\varepsilon|^2 - a_\varepsilon(t_{i-1})\,|w_i^\varepsilon|^2)/2 \leqslant 0\end{aligned}$$

for $1 \leqslant i \leqslant j$. We sum the inequalities above from $i = 1$ to $i = j$, and use the fact that $a(t)$ is nonnegative. This yields the estimate

$$\begin{aligned}|A^{1/2}u_j^\varepsilon|^2 &\leqslant |A^{1/2}u_0|^2 + 2\lambda \sum_{k=1}^{j} \langle A^{1/2}v_0, A^{1/2}u_k^\varepsilon\rangle \\ &\leqslant |A^{1/2}u_0|^2 + T|A^{1/2}v_0|^2 + \lambda \sum_{k=1}^{j} |A^{1/2}u_k^\varepsilon|^2\end{aligned}$$

for $0 \leqslant j \leqslant m$. Let α_j denote the right-hand side of the inequality above. Clearly, $|A^{1/2}u_j^\varepsilon|^2 \leqslant \alpha_j$ and $\alpha_j - \alpha_{j-1} = \lambda|A^{1/2}u_j^\varepsilon|^2 \leqslant \lambda\alpha_j$, which we solve to obtain

$$|A^{1/2}u_j^\varepsilon|^2 \leqslant \exp(2T)\,(|A^{1/2}u_0|^2 + T|A^{1/2}v_0|^2) \tag{2.9}$$

for $0 \leqslant j \leqslant m$. Here we have used the fact that $(1-t)^{-1} \leqslant \exp(2t)$ for $t \in [0, 1/2]$.

Next, we take the inner products of equations (2.5) and (2.6) with $a_\varepsilon(t_i)\,Au_i^\varepsilon$ and v_i^ε respectively, and add the two resulting equalities. This yields

$$a_\varepsilon(t_i)\,\langle A^{1/2}u_i^\varepsilon, A^{1/2}u_i^\varepsilon - A^{1/2}u_{i-1}^\varepsilon\rangle + \langle v_i^\varepsilon, v_i^\varepsilon - v_{i-1}^\varepsilon\rangle = 0,$$

from which it follows that

$$a_\varepsilon(t_i)\,|A^{1/2}u_i^\varepsilon|^2 - a_\varepsilon(t_{i-1})\,|A^{1/2}u_{i-1}^\varepsilon|^2 + |v_i^\varepsilon|^2 - |v_{i-1}^\varepsilon|^2 \leqslant (a_\varepsilon(t_i) - a_\varepsilon(t_{i-1}))\,|A^{1/2}u_{i-1}^\varepsilon|^2$$

for $1 \leqslant i \leqslant m$. Substituting (2.9) into the right-hand side of the inequality above and summing the resultant inequalities from $i = 1$ and $i = j$ $(\leqslant m)$, we find that

$$\begin{aligned} |v_j^\varepsilon|^2 &\leqslant |v_0|^2 + a_\varepsilon(t_0)\,|A^{1/2}u_0|^2 \\ &\quad + (a_\varepsilon(t_j) - a_\varepsilon(t_0))\exp(2T)\,(|A^{1/2}u_0|^2 + T|A^{1/2}v_0|^2) \\ &\leqslant |v_0|^2 + (a(t_j)+1)\exp(2T)\,(|A^{1/2}u_0|^2 + T|A^{1/2}v_0|^2) \end{aligned} \tag{2.10}$$

for $1 \leqslant j \leqslant m$. By equation (2.5) we have $|u_i^\varepsilon|^2 - |u_{i-1}^\varepsilon|^2 \leqslant 2\lambda\langle u_i^\varepsilon, v_i^\varepsilon\rangle \leqslant \lambda(|u_i^\varepsilon|^2 + |v_i^\varepsilon|^2)$, which implies that

$$|u_j^\varepsilon|^2 \leqslant (1-\lambda)^{-j}\left(|u_0|^2 + \lambda\sum_{i=1}^{j}|v_i^\varepsilon|^2\right)$$

for $1 \leqslant j \leqslant m$. Combining this inequality with (2.9) and (2.10), we obtain for $1 \leqslant j \leqslant m$,

$$|u_j^\varepsilon|^2_{[D(A^{1/2})]} + |v_j^\varepsilon|^2 \leqslant M_0(|u_0|^2_{[D(A^{1/2})]} + |v_0|^2_{[D(A^{1/2})]}), \tag{2.11}$$

where M_0 is a positive constant depending only on T and $\sup\{a(t): t\in[0,T]\}$, which is nondecreasing with respect to T. By the reflexivity of $[D(A^{1/2})]$ and H there exists a null sequence $\{\varepsilon(n)\}$ such that for $1 \leqslant j \leqslant m$, $\{u_j^{\varepsilon(n)}\}$ and $\{v_j^{\varepsilon(n)}\}$ converge weakly to u_j and v_j in $[D(A^{1/2})]$ and H respectively, as $n \to \infty$. From (2.11) it is easily seen that

$$|u_j|^2_{[D(A^{1/2})]} + |v_j|^2 \leqslant M_0(|u_0|^2_{[D(A^{1/2})]} + |v_0|^2_{[D(A^{1/2})]}) \tag{2.12}$$

for $1 \leqslant j \leqslant m$. Let $w \in D(A^{1/2})$. We take the inner product of (2.6) with w and the limit as $n \to \infty$. This yields

$$\langle w, (v_i - v_{i-1})/\lambda\rangle + a(t_i)\,\langle A^{1/2}w, A^{1/2}u_i\rangle = 0,$$

which implies that $a(t_i)\,u_i \in D(A)$ and $-A(a(t_i)\,u_i) = (v_i - v_{i-1})/\lambda$. Passing to the limit in (2.5) as $n \to \infty$ we obtain $v_i = (u_i - u_{i-1})/\lambda \in D(A^{1/2})$. Thus it has been shown that $(u_i, v_i) \in D(\tilde{A}(t_i))$ and $(I - \lambda\tilde{A}(t_i))(u_i, v_i) = (u_{i-1}, v_{i-1})$ for $1 \leqslant i \leqslant m$. Estimate (2.12) shows that condition (1)(ii) is satisfied with the operator $\tilde{C}$. □

3. *A class of quasi-linear evolution equations of second order*

In this section we consider a triplet of real Banach spaces $Y \subset X \subset Z$, with inclusions that are *continuous* and *dense*, and assume the following condition.

There exists an isomorphism S of X onto Z satisfying the property that

$$S(Y) = X,$$

and an injective operator C in $B(Z)$ satisfying

$$\begin{cases} C(Z) \text{ is dense in } Z, \\ C(X) \subset X \text{ and } C(X) \text{ is dense in } X, \end{cases}$$

such that

$$SCS^{-1} = C.$$

By the assumption of C, the linear operator $\tilde{C}$ in $\tilde{X}$ defined by

$$\tilde{C}(u,v) = (Cu, Cv) \quad \text{for } (u,v) \in \tilde{X} := X \times Z$$

is injective and bounded, and has dense range. This fact should be recalled in stating the regularized stability condition (Hypothesis 3.3 below).

We begin by setting up basic Hypotheses 3.1–3.5 on the operators $A(t,w,v)$ appearing in equation (1.1), with some comments.

HYPOTHESIS 3.1. There exist a nonempty open bounded subset W of $[C(Y)]$, a nonempty open bounded subset V of $[C(X)]$ and $T_0 > 0$ such that $A(t,w,v)$ is a closed linear operator in Z defined for each $(t,w,v) \in [0,T_0] \times W \times V$.

Here and subsequently, $[P(\mathfrak{X})]$ denotes the Banach space $P(\mathfrak{X})$ equipped with the norm $\|x\|_{[P(\mathfrak{X})]} = \|P^{-1}x\|_{\mathfrak{X}}$ for $x \in P(\mathfrak{X})$, if P is an injective operator in $B(\mathfrak{X})$ and $\mathfrak{X}$ is a real Banach space.

The aim of this section is to find a function u in the class

$$C([0,T];[C(Y)]) \cap C^1([0,T];[C(X)]) \cap C^2([0,T];[C(Z)])$$

satisfying equation (1.1) and the condition

$$u(t) \in W \quad \text{and} \quad u'(t) \in V \quad \text{for } t \in [0,T], \tag{3.1}$$

for a given $(\phi, \psi) \in W \times V$. Such a function u is called a *solution to* (1.1) on $[0,T]$.

For this purpose, let $(\phi,\psi) \in W \times V$ be fixed arbitrarily. To formulate the regularized stability condition, for each $\rho > 0$ and $\tau \in (0, T_0]$ we introduce the set $D(\rho,\tau)$ of all functions w in the class $C([0,\tau]; [C(Y)]) \cap C^1([0,\tau]; [C(X)]) \cap C^2([0,\tau]; [C(Z)])$ satisfying the property

$$\begin{cases} w(t) \in W \quad \text{and} \quad w'(t) \in V, \quad \text{for } t \in [0,\tau], \\ \|w''(t)\|_{[C(Z)]} \leqslant \rho \quad \text{for } t \in [0,\tau], \\ w(0) = \phi \quad \text{and} \quad w'(0) = \psi. \end{cases}$$

HYPOTHESIS 3.2. There exist $\rho_0 > 0$ and $\tau_0 \in (0, T_0]$ such that the set $D(\rho_0, \tau_0)$ is nonempty.

This condition is necessary for equation (1.1) to have a solution on $[0,T]$ where $T \in (0, T_0]$. Indeed, if $u(t)$ is a solution to (1.1) on $[0,T]$, then Hypothesis 3.2 is satisfied with $\tau_0 = T$ and $\rho_0 = \sup\{\|u''(t)\|_{[C(Z)]} : t \in [0,T]\}$.

By Hypothesis 3.2 the set $D(\rho,\tau)$ is nonempty for each $\rho \geqslant \rho_0$ and $\tau \in (0,\tau_0]$. Let $\rho \geqslant \rho_0$ and $\tau \in (0,\tau_0]$. For each $w \in D(\rho,\tau)$ we consider a family $\{\tilde{A}^w(t) : t \in [0,\tau]\}$ of closed linear operators in $\tilde{X}$ defined by

$$\begin{cases} \tilde{A}^w(t)(u,v) = (v, A(t,w(t),w'(t))u) \quad \text{for } (u,v) \in D(\tilde{A}^w(t)) \\ D(\tilde{A}^w(t)) = (D(A(t,w(t),w'(t))) \cap X) \times X. \end{cases}$$

HYPOTHESIS 3.3. For each $\rho \geqslant \rho_0$ there exist $M_\rho \geqslant 1$, $\beta_\rho \geqslant 0$ and $\tau_\rho \in (0,\tau_0]$ such that if $\tau \in (0,\tau_\rho]$ and $w \in D(\rho,\tau)$ then

$$\{\tilde{A}^w(t) : t \in [0,\tau]\} \in RS(\tilde{X}, M_\rho, \beta_\rho, \tilde{C}).$$

HYPOTHESIS 3.4. For each $(t, w, v) \in [0, T_0] \times W \times V$,

$$SA(t, w, v) S^{-1} = A(t, w, v) \quad \text{and} \quad C^{-1}A(t, w, v) C \supset A(t, w, v).$$

HYPOTHESIS 3.5. For each $(t, w, v) \in [0, T_0] \times W \times V$, $D(A(t, w, v)) \supset Y$. For each $(w, v) \in W \times V$, $A(\cdot, w, v)$ is norm continuous in $B(Y, Z)$ on $[0, T_0]$. There exists $L_A > 0$ such that

$$\|A(t, w, v) - A(t, \hat{w}, \hat{v})\|_{Y,Z} \leqslant L_A(\|w - \hat{w}\|_X + \|v - \hat{v}\|_Z)$$

for (t, w, v), $(t, \hat{w}, \hat{v}) \in [0, T_0] \times W \times V$.

This condition implies that there exists $M_A \geqslant 1$ such that

$$\|A(t, w, v)\|_{Y,Z} \leqslant M_A \quad \text{for } (t, w, v) \in [0, T_0] \times W \times V. \tag{3.2}$$

The main theorem in this paper can now be stated.

THEOREM 3.6. *If $(\phi, \psi) \in C^2(Y) \times C^2(X)$ then there exists $T \in (0, T_0]$ such that the evolution equation* (1.1) *has a unique solution on* $[0, T]$.

The proof will be divided into a sequence of lemmas. It should be noted that (ϕ, ψ) has been fixed in $W \times V$. Since W and V are open subsets of $[C(Y)]$ and $[C(X)]$ respectively, there exists $R_0 > 0$ such that

$$\tilde{W} := \{(w, v) \in C(Y) \times C(X) : \|w - \phi\|_{[C(Y)]} + \|v - \psi\|_{[C(X)]} \leqslant R_0\} \subset W \times V.$$

Put

$$\rho = M_A(\|\phi\|_{[C(Y)]} + R_0) \vee \rho_0. \tag{3.3}$$

To prove the theorem we use the method of iterations proposed by Kobayasi and Sanekata [**12**]. To do this, for each $\tau \in (0, \tau_\rho]$ we introduce the set $E(\tau)$ of all functions w in the class $C([0, \tau]; [C(Y)]) \cap C^1([0, \tau]; [C(X)]) \cap C^2([0, \tau]; [C(Z)])$ satisfying the property

$$\begin{cases} (w(t), w'(t)) \in \tilde{W} & \text{for } t \in [0, \tau], \\ \|w''(t)\|_{[C(Z)]} \leqslant \rho & \text{for } t \in [0, \tau], \\ w(0) = \phi \quad \text{and} \quad w'(0) = \psi. \end{cases}$$

Since $D(\rho, \tau_\rho) \neq \varnothing$ we choose $w_0 \in D(\rho, \tau_\rho)$ and then define

$$\bar{\tau}_0 = \sup\{\tau \in (0, \tau_\rho] : \|w_0(t) - \phi\|_{[C(Y)]} + \|w_0'(t) - \psi\|_{[C(X)]} \leqslant R_0 \text{ for } t \in [0, \tau]\}.$$

It is clear that $w_0|_{[0,\tau]} \in E(\tau)$ for each $\tau \in (0, \bar{\tau}_0]$. Thus we have the following lemma.

LEMMA 3.7. *There exists $\bar{\tau}_0 \in (0, \tau_\rho]$ such that the set $E(\tau)$ is nonempty for each $\tau \in (0, \bar{\tau}_0]$.*

Now, we set $\tilde{Y} = Y \times X$, and define an operator $\tilde{S}$ from $\tilde{Y}$ into $\tilde{X}$ by

$$\tilde{S}(u, v) = (Su, Sv) \quad \text{for } (u, v) \in \tilde{Y}.$$

Condition (2.1), stated in Section 2, is clearly satisfied.

LEMMA 3.8. *For all $w, z \in E(\tau)$ and $\tau \in (0, \bar{\tau}_0]$, the following assertions hold.*

(i) *There exist* $M \geqslant 1$ *and* $\beta \geqslant 0$, *independent of* w *and* τ, *such that*

$$\{\tilde{A}^w(t): t \in [0, \tau]\} \in RS(\tilde{X}, M, \beta, \tilde{C}).$$

(ii) $\tilde{C}^{-1}\tilde{A}^w(t)\,\tilde{C} \supset \tilde{A}^w(t)$ *for* $t \in [0, \tau]$.

(iii) *Operator* $\tilde{S}$ *is an isomorphism of* $\tilde{Y}$ *onto* $\tilde{X}$ *satisfying*

$$\tilde{S}\tilde{A}^w(t)\,\tilde{S}^{-1} = \tilde{A}^w(t) \quad \text{for } t \in [0, \tau], \quad \text{and} \quad \tilde{S}\tilde{C}\tilde{S}^{-1} = \tilde{C}.$$

(iv) $D(\tilde{A}^w(t)) \supset \tilde{Y}$ *for* $t \in [0, \tau]$, *and* $\tilde{A}^w(t)$ *is norm continuous in* $B(\tilde{Y}, \tilde{X})$ *on* $[0, \tau]$.

(v) $\|\tilde{A}^w(t) - \tilde{A}^z(t)\|_{\tilde{Y}, \tilde{X}} \leqslant L_A(\|w(t) - z(t)\|_X + \|w'(t) - z'(t)\|_Z)$ *for* $t \in [0, \tau]$.

Proof. Since $E(\tau) \subset D(\rho, \tau)$, assertion (i) follows immediately from Hypothesis 3.3. Assertion (ii) is easily checked. It is seen that all the other conditions of (iii) except the fact that $\tilde{S} \in B(\tilde{Y}, \tilde{X})$ are satisfied by a straightforward argument. The closed graph theorem implies that $\tilde{S} \in B(\tilde{Y}, \tilde{X})$, by showing that $\tilde{S}$ is a closed linear operator from $\tilde{Y}$ into $\tilde{X}$. The first half of assertion (iv) is obvious, and assertion (v) is proved as follows. Let $t \in [0, \tau]$ and $(u, v) \in \tilde{Y}$. Since

$$\tilde{A}^w(t)\,(u, v) - \tilde{A}^z(t)\,(u, v) = (0, A(t, w(t), w'(t))\,u - A(t, z(t), z'(t))\,u),$$

we have, by Hypothesis 3.5,

$$\|\tilde{A}^w(t)\,(u, v) - \tilde{A}^z(t)\,(u, v)\|_{\tilde{X}} \leqslant L_A(\|w(t) - z(t)\|_X + \|w'(t) - z'(t)\|_Z)\,\|u\|_Y.$$

This means that the desired inequality holds. Similarly to the preceding argument, we easily see that the second half of assertion (iv) is true. □

By Lemma 3.8 we can apply Theorem 2.1 to the family $\{\tilde{A}^w(t): t \in [0, \tau]\}$. This procedure, together with relation (2.2), yields the following lemma.

LEMMA 3.9. *For each* $w \in E(\tau)$ *and* $\tau \in (0, \bar{\tau}_0]$ *there exists a unique regularized evolution operator* $\{\tilde{U}^w(t, s)\}$ *defined on the triangle*

$$\Delta(\tau) = \{(t, s): 0 \leqslant s \leqslant t \leqslant \tau\}$$

generated in the sense of Theorem 2.1 *by* $\{\tilde{A}^w(t): t \in [0, \tau]\}$, *which satisfies the following properties.*

(i) $\|\tilde{U}^w(t, s)\|_{\tilde{X}} \leqslant M$ *and* $\|\tilde{U}^w(t, s)\|_{\tilde{Y}} \leqslant \bar{M}$ *for* $(t, s) \in \Delta(\tau)$, *where* $\bar{M} = M\|\tilde{S}\|_{\tilde{Y}, \tilde{X}}\,\|\tilde{S}^{-1}\|_{\tilde{X}, \tilde{Y}}$.

(ii) $\tilde{S}\tilde{U}^w(t, s) \supset \tilde{U}^w(t, s)\,\tilde{S}$ *for* $(t, s) \in \Delta(\tau)$.

It should be noted that M is the constant appearing in Lemma 3.8, which is independent of $w \in E(\tau)$ and $\tau \in (0, \bar{\tau}_0]$.

LEMMA 3.10. *For each* $w, z \in E(\tau)$ *and* $\tau \in (0, \bar{\tau}_0]$ *we have*

$$\|\tilde{U}^w(t, s)\,\tilde{C}\tilde{y} - \tilde{U}^z(t, s)\,\tilde{C}\tilde{y}\|_{\tilde{X}} \leqslant M\bar{M}L_A\|\tilde{y}\|_{\tilde{Y}} \int_s^t (\|w(\sigma) - z(\sigma)\|_X + \|w'(\sigma) - z'(\sigma)\|_Z)\,d\sigma$$

for $\tilde{y} \in \tilde{Y}$ *and* $(t, s) \in \Delta(\tau)$.

Proof. Let $\tilde{y} \in \tilde{Y}$ and $(t,s) \in \Delta(\tau)$. By Lemma 3.9 we have

$$\tilde{U}^z(t,s)\,\tilde{C}\tilde{y} - \tilde{U}^w(t,s)\,\tilde{C}\tilde{y} = \int_s^t \tilde{U}^w(t,\sigma)\,(\tilde{A}^z(\sigma) - \tilde{A}^w(\sigma))\,\tilde{U}^z(\sigma,s)\,\tilde{y}\,d\sigma \tag{3.4}$$

which is obtained by differentiating $\tilde{U}^w(t,\sigma)\,\tilde{U}^z(\sigma,s)\,\tilde{y}$ in σ and then integrating the resultant derivative over $\sigma \in [s,t]$. Property (i) of Lemma 3.9 and assertion (v) of Lemma 3.8 together imply that the integrand of equality (3.4) is estimated by

$$M\overline{M}L_A(\|w(\sigma) - z(\sigma)\|_X + \|w'(\sigma) - z'(\sigma)\|_Z)\,\|\tilde{y}\|_{\tilde{Y}}.$$

The desired inequality is thus obtained. □

Let $\tau \in (0, \bar{\tau}_0]$ and $w \in E(\tau)$. Since $(\phi, \psi) \in C^2(Y) \times C^2(X) = \tilde{C}^2(\tilde{Y})$, $\tilde{C}^{-1}(\phi,\psi) \in \tilde{C}(\tilde{Y}) \subset \tilde{Y}$. If we set $(u(t), v(t)) = \tilde{U}^w(t,0)\,\tilde{C}^{-1}(\phi,\psi)$ for $t \in [0,\tau]$, then we have, by Lemma 3.9, $(u,v) \in C([0,\tau];[\tilde{C}(\tilde{Y})]) \cap C^1([0,\tau];[\tilde{C}(\tilde{X})])$ and $(d/dt)\,(u(t),v(t)) = \tilde{A}^w(t)\,(u(t),v(t))$ for $t \in [0,\tau]$ and $(u(0),v(0)) = (\phi,\psi)$. Since $[\tilde{C}(\tilde{X})] = [C(X)] \times [C(Z)]$ it is seen that u belongs to the class

$$C([0,\tau];[C(Y)]) \cap C^1([0,\tau];[C(X)]) \cap C^2([0,\tau];[C(Z)])$$

and satisfies $u(0) = \phi$, $u'(0) = \psi$ and $u''(t) = A(t, w(t), w'(t))\,u(t)$ for $t \in [0,\tau]$. This fact enables us to consider a mapping Φ from $E(\tau)$ into $C([0,\tau];[C(Y)]) \cap C^1([0,\tau];[C(X)]) \cap C^2([0,\tau];[C(Z)])$ such that $(\Phi w)(t)$ is defined to be the first component of $\tilde{U}^w(t,0)\,\tilde{C}^{-1}(\phi,\psi)$ for $t \in [0,\tau]$. By the preceding argument we have $(\Phi w)(0) = \phi$, $(\Phi w)'(0) = \psi$ and

$$(\Phi w)''(t) = A(t, w(t), w'(t))\,(\Phi w)(t) \quad \text{for } t \in [0,\tau]. \tag{3.5}$$

Equation (3.5) is written as

$$(d/dt)\,((\Phi w)(t), (\Phi w)'(t)) = \tilde{A}^w(t)\,((\Phi w)(t), (\Phi w)'(t)) \quad \text{for } t \in [0,\tau]. \tag{3.6}$$

By the definition of Φ it is evident that

$$((\Phi w)(t), (\Phi w)'(t)) = \tilde{U}^w(t,0)\,\tilde{C}^{-1}(\phi,\psi) \quad \text{for } t \in [0,\tau]. \tag{3.7}$$

LEMMA 3.11. *There exists* $\tau \in (0, \bar{\tau}_0]$ *such that if* $T \in (0,\tau]$ *then* $\Phi w \in E(T)$ *for* $w \in E(T)$.

Proof. Let $\tilde{x} = \tilde{S}\tilde{C}^{-2}(\phi,\psi)$. This definition of $\tilde{x}$ makes sense, since $(\phi,\psi) \in \tilde{C}^2(\tilde{Y})$ and $\tilde{S}$ is an isomorphism of $\tilde{Y}$ onto $\tilde{X}$. Since $\tilde{Y}$ is dense in $\tilde{X}$ there exists $\tilde{y} \in \tilde{Y}$ such that

$$\|\tilde{S}^{-1}\|_{\tilde{X},\tilde{Y}}(M + \|\tilde{C}\|_{\tilde{X}})\,\|\tilde{x} - \tilde{y}\|_{\tilde{X}} \leqslant R_0/2. \tag{3.8}$$

Now, let us choose $\tau \in (0, \bar{\tau}_0]$ so that

$$\|\tilde{S}^{-1}\|_{\tilde{X},\tilde{Y}}\,MM_A\|\tilde{y}\|_{\tilde{Y}}\,\tau \leqslant R_0/2.$$

It will be shown that the lemma is true with this number τ. Let $T \in (0,\tau]$ and $w \in E(T)$, and then set $v = \Phi w$. By equation (3.7) we have $(v(t), v'(t)) = \tilde{U}^w(t,0)\,\tilde{C}^{-1}(\phi,\psi)$ for $t \in [0,T]$, and hence $\tilde{S}\tilde{C}^{-1}(v(t), v'(t)) = \tilde{U}^w(t,0)\,\tilde{x}$ for $t \in [0,T]$, by property (ii) of Lemma 3.9. Using the relation that $\tilde{S}\tilde{C}\tilde{S}^{-1} = \tilde{C}$ in assertion (iii) of Lemma 3.8 we obtain

$$\tilde{S}\tilde{C}^{-1}((v(t), v'(t)) - (\phi,\psi)) = \tilde{U}^w(t,0)\,\tilde{x} - \tilde{C}\tilde{x}$$

for $t \in [0,T]$, and the right-hand side is written as

$$\tilde{U}^w(t,0)\,(\tilde{x} - \tilde{y}) + (\tilde{U}^w(t,0)\,\tilde{y} - \tilde{C}\tilde{y}) + \tilde{C}(\tilde{y} - \tilde{x})$$

by means of $\tilde{y} \in \tilde{Y}$ satisfying inequality (3.8). Moreover, it is deduced from Lemma 3.9 that

$$\tilde{U}^w(t,0)\,\tilde{y} - \tilde{C}\tilde{y} = \int_0^t \tilde{U}^w(t,\sigma)\,\tilde{A}^w(\sigma)\,\tilde{y}\,d\sigma$$

for $t \in [0,T]$. Thus we have

$$\|\tilde{S}\tilde{C}^{-1}((v(t),v'(t)) - (\phi,\psi))\|_{\tilde{X}} \leqslant (M + \|\tilde{C}\|_{\tilde{X}})\,\|\tilde{x} - \tilde{y}\|_{\tilde{X}} + MM_A\|\tilde{y}\|_{\tilde{Y}}\,T$$

for $t \in [0,T]$. Here we have used the estimate that $\|\tilde{A}^w(t)\|_{\tilde{Y},\tilde{X}} \leqslant M_A$ for $t \in [0,T]$ which follows from (3.2). Since

$$\|\tilde{w}\|_{[\tilde{C}(\tilde{Y})]} = \|\tilde{C}^{-1}\tilde{w}\|_{\tilde{Y}} \leqslant \|\tilde{S}^{-1}\|_{\tilde{X},\tilde{Y}}\,\|\tilde{S}\tilde{C}^{-1}\tilde{w}\|_{\tilde{X}} \tag{3.9}$$

for $\tilde{w} \in \tilde{C}(\tilde{Y})$, it is seen from the choice of τ and $\tilde{y}$ that

$$\|(v(t),v'(t)) - (\phi,\psi)\|_{[\tilde{C}(\tilde{Y})]} \leqslant R_0$$

for $t \in [0,T]$, which implies that $(v(t),v'(t)) \in \tilde{W}$ for $t \in [0,T]$. Furthermore, we use inequality (3.2) to estimate (3.5) and find, by the second half of Hypothesis 3.4, that

$$\|v''(t)\|_{[C(Z)]} \leqslant M_A\|v(t)\|_{[C(Y)]}$$

for $t \in [0,T]$. The right-hand side is majorized by $M_A(\|\phi\|_{[C(Y)]} + R_0)$, and so it is concluded that v is an element of $E(T)$, by the choice of ρ (see expression (3.3)). □

Now, let us choose $T \in (0,\tau]$ so that

$$\alpha := M\bar{M}L_A\|\tilde{C}^{-2}(\phi,\psi)\|_{\tilde{Y}}\,T < 1. \tag{3.10}$$

By Lemma 3.7, the set $E(T)$ is nonempty. This fact enables us to choose $u_0 \in E(T)$. Since the set $E(T)$ is invariant under the mapping Φ by Lemma 3.11, we can define a sequence $\{u_n\}$ in $E(T)$ by $u_n = \Phi u_{n-1}$ for $n = 1, 2, \ldots$.

LEMMA 3.12. *The sequence $\{(u_n(t), u_n'(t))\}$ converges in $X \times Z$ uniformly on $[0,T]$, as $n \to \infty$.*

Proof. Since $(u_n(t), u_n'(t)) = \tilde{U}^{u_{n-1}}(t,0)\,\tilde{C}^{-1}(\phi,\psi)$ for $n = 1, 2, \ldots$ (by equation (3.7)), we have, by Lemma 3.10,

$$\begin{aligned}&\|(u_n(t),u_n'(t)) - (u_{n-1}(t),u_{n-1}'(t))\|_{\tilde{X}}\\ &\qquad\leqslant M\bar{M}L_A\|\tilde{C}^{-2}(\phi,\psi)\|_{\tilde{Y}}\int_0^t (\|u_{n-1}(\sigma) - u_{n-2}(\sigma)\|_X + \|u_{n-1}'(\sigma) - u_{n-2}'(\sigma)\|_Z)\,d\sigma,\end{aligned}$$

and hence, by (3.10),

$$d(u_n, u_{n-1}) \leqslant \alpha d(u_{n-1}, u_{n-2})$$

for $n \geqslant 2$, where d is the metric in $C([0,T];X) \cap C^1([0,T];Z)$ defined by

$$d(u,v) = \sup\{\|u(t) - v(t)\|_X + \|u'(t) - v'(t)\|_Z : t \in [0,T]\}.$$

If $n > m$, then $d(u_n, u_m) \leqslant (\sum_{k=m+1}^{n} \alpha^{k-1})\,d(u_1, u_0)$, and since $\alpha < 1$ the right-hand side tends to zero as $m \to \infty$, which proves the desired assertion. □

LEMMA 3.13. *The sequence $\{\tilde{U}^{u_n}(t,s) : (t,s) \in \Delta(T)\}$ of regularized evolution operators on $\tilde{X}$ is strongly convergent in $B(\tilde{X})$ uniformly on $\Delta(T)$, as $n \to \infty$.*

Proof. For $\tilde{x}\in\tilde{X}$, $\tilde{y}\in\tilde{Y}$ and $1\leqslant m\leqslant n$, we have

$$\|\tilde{U}^{u_n}(t,s)\tilde{x}-\tilde{U}^{u_m}(t,s)\tilde{x}\|_{\tilde{X}} \leqslant \|\tilde{U}^{u_n}(t,s)(\tilde{x}-\tilde{C}\tilde{y})\|_{\tilde{X}}+\|\tilde{U}^{u_n}(t,s)\tilde{C}\tilde{y}-\tilde{U}^{u_m}(t,s)\tilde{C}\tilde{y}\|_{\tilde{X}}+\|\tilde{U}^{u_m}(t,s)(\tilde{C}\tilde{y}-\tilde{x})\|_{\tilde{X}},$$

and we see from Lemma 3.10 that the right-hand side is bounded by

$$2M\|\tilde{C}\tilde{y}-\tilde{x}\|_{\tilde{X}}+M\overline{M}L_A\|\tilde{y}\|_{\tilde{Y}}(t-s)\,d(u_n,u_m).$$

The desired claim is obtained, since $\tilde{C}(\tilde{Y})$ is dense in $\tilde{X}$ and $\lim_{m,n\to\infty}d(u_n,u_m)=0$ by Lemma 3.12. □

Proof of Theorem 3.6. Let $\{u_n\}$ be the sequence defined in the paragraph before Lemma 3.12. We have, by the definition of u_n and equation (3.7),

$$\tilde{S}\tilde{C}^{-1}(u_n(t),u_n'(t))=\tilde{U}^{u_{n-1}}(t,0)\tilde{S}\tilde{C}^{-2}(\phi,\psi)$$

for $t\in[0,T]$ and $n\geqslant 1$, and the right-hand side converges in $\tilde{X}$ uniformly on $[0,T]$, by Lemma 3.13. This means that the sequence $\{(u_n(t),u_n'(t))\}$ converges in $[C(Y)]\times[C(X)]$ uniformly on $[0,T]$ as $n\to\infty$, by inequality (3.9) and the fact that $[\tilde{C}(\tilde{Y})]=[C(Y)]\times[C(X)]$.

It will be proved that the limit

$$u(t)=\lim_{n\to\infty}u_n(t)\quad\text{in }[C(Y)]$$

gives a unique solution to equation (1.1) on $[0,T]$. By the result in the preceding paragraph we have $u\in C([0,T];[C(Y)])\cap C^1([0,T];[C(X)])$ and $\lim_{n\to\infty}u_n'(t)=u'(t)$ in $[C(X)]$. Since $\tilde{W}$ is a closed subset of $[\tilde{C}(\tilde{Y})]$, $(u(t),u'(t))\in\tilde{W}\subset W\times V$ for $t\in[0,T]$. It is deduced from Hypothesis 3.5 that $A(t,u_{n-1}(t),u_{n-1}'(t))$ converges to $A(t,u(t),u'(t))$ in $B(Y,Z)$ uniformly on $[0,T]$ as $n\to\infty$. By equation (3.5) and Hypothesis 3.4 we have $C^{-1}u_n''(t)=A(t,u_{n-1}(t),u_{n-1}'(t))C^{-1}u_n(t)$, which converges to $A(t,u(t),u'(t))C^{-1}u(t)$ in Z uniformly on $[0,T]$ as $n\to\infty$. This implies that $u_n''(t)$ converges to $A(t,u(t),u'(t))u(t)$ in $[C(Z)]$ uniformly on $[0,T]$ as $n\to\infty$. It is therefore concluded that u belongs to the class $C([0,T];[C(Y)])\cap C^1([0,T];[C(X)])\cap C^2([0,T];[C(Z)])$ and satisfies condition (3.1) and equation (1.1), that is, u is a solution to equation (1.1) on $[0,T]$. Finally, to prove the uniqueness of the solution to (1.1) on $[0,T]$, let v be another solution to equation (1.1) on $[0,T]$. The preceding arguments imply that $u\in E(T)$, since $u_n\in E(T)$ for $n=1,2,\ldots$. It follows from Lemma 3.9 that the family $\{\tilde{A}^u(t):t\in[0,T]\}$ generates a regularized evolution operator $\{\tilde{U}^u(t,s):(t,s)\in\Delta(T)\}$ on $\tilde{X}$. Let $t\in[0,T]$. The equality

$$(u(t),u'(t))-(v(t),v'(t))=\int_0^t\tilde{U}^u(t,\sigma)(\tilde{A}^u(\sigma)-\tilde{A}^v(\sigma))\tilde{C}^{-1}(v(\sigma),v'(\sigma))\,d\sigma$$

is obtained by differentiating $\tilde{U}^u(t,\sigma)((u(\sigma),u'(\sigma))-(v(\sigma),v'(\sigma)))$ in σ, integrating the resultant derivative over $\sigma\in[0,t]$, and then using assertion (ii) of Lemma 3.8. We note that there exists $\tilde{M}>0$ such that $\|\tilde{C}^{-1}(v(t),v'(t))\|_{\tilde{Y}}\leqslant\tilde{M}$ for $t\in[0,T]$, and estimate the equality above by using assertion (v) of Lemma 3.8. This yields the inequality of Gronwall type

$$\|u(t)-v(t)\|_X+\|u'(t)-v'(t)\|_Z\leqslant M\tilde{M}L_A\int_0^t(\|u(\sigma)-v(\sigma)\|_X+\|u'(\sigma)-v'(\sigma)\|_Z)\,d\sigma$$

for $t\in[0,T]$, and so we conclude that $u=v$ on $[0,T]$. □

4. *Application to wave equations of Kirchhoff type*

Let us consider the abstract wave equation of Kirchhoff type

$$\begin{cases} u''(t)+m(|A^{1/2}u(t)|^2)\,Au(t)=0 \quad \text{for } t\in[0,\infty) \\ u(0)=\phi \quad \text{and} \quad u'(0)=\psi \end{cases} \tag{4.1}$$

in a real Hilbert space H with the inner product $\langle\cdot,\cdot\rangle$ and the associated norm $|\cdot|$. Here A is a nonnegative selfadjoint operator in H and $m\in C^1([0,\infty);\mathbb{R})$ satisfies the property that $m(r)\geqslant 0$ for $r\in[0,\infty)$.

The Cauchy problem (4.1) has been studied by Yamada [**17**], by a different method based on a certain kind of energy estimates. Our results (Theorems 4.1 and 4.2) give an improvement of [**17**, Theorem 2.1, Theorem 2.2].

THEOREM 4.1. *Assume that $m(|A^{1/2}\phi|^2)\neq 0$. If $\phi\in D(A)$ and $\psi\in D(A^{1/2})$, then there exists a positive number T such that the problem* (4.1) *has a unique solution u on* $[0,T]$ *in the class*

$$C([0,T];[D(A)])\cap C^1([0,T];[D(A^{1/2})])\cap C^2([0,T];H)$$

satisfying the property that $m(|A^{1/2}u(t)|^2)>0$ for $t\in[0,T]$.

Proof. We consider a triplet of real Banach spaces $[D(A)]\subset[D(A^{1/2})]\subset H$ as $Y\subset X\subset Z$, and take $(I+A)^{1/2}$ and the identity operator I as an isomorphism S of X onto Z and C in the previous section, respectively. Since $m(|A^{1/2}\phi|^2)\neq 0$ there exists $R>0$ such that $m(|A^{1/2}\phi|^2)>1/R$, $\|\phi\|_{[D(A)]}<R$ and $\|\psi\|_{[D(A^{1/2})]}<R$; we then define

$$W=\{w\in[D(A)]: m(|A^{1/2}w|^2)>1/R \text{ and } \|w\|_{[D(A)]}<R\}$$

and

$$V=\{v\in[D(A^{1/2})]: \|v\|_{[D(A^{1/2})]}<R\}.$$

It is obvious that W and V are open bounded subsets of $[D(A)]$ and $[D(A^{1/2})]$ respectively, and that $(\phi,\psi)\in W\times V$. Let $T_0>0$. We shall apply Theorem 3.6 to the family $\{A(t,w,v):(t,w,v)\in[0,T_0]\times W\times V\}$ defined by

$$A(t,w,v)\,u=-m(|A^{1/2}w|^2)\,Au \quad \text{for } u\in D(A).$$

Clearly, Hypotheses 3.1 and 3.4 are satisfied. It is well known that the Cauchy problem

$$\begin{cases} w''(t)+Aw(t)=0 \quad \text{for } t\in[0,\infty) \\ w(0)=\phi \quad \text{and} \quad w'(0)=\psi \end{cases}$$

has a unique solution $w\in C([0,\infty);[D(A)])\cap C^1([0,\infty);[D(A^{1/2})])\cap C^2([0,\infty);H)$. (See [**7**, Chapter II, Remark 7.5].) By the continuity of w there exists $\tau_0\in(0,T_0]$ such that $(w(t),w'(t))\in W\times V$ for $t\in[0,\tau_0]$. This implies that Hypothesis 3.2 is satisfied with such a number τ_0 and $\rho_0=\sup\{|w''(t)|:t\in[0,\tau_0]\}$. Since

$$\begin{aligned} &A(t,w,v)\,u-A(t,\hat{w},\hat{v})\,u \\ &\quad=-\left(\int_0^1 m'(\theta|A^{1/2}w|^2+(1-\theta)|A^{1/2}\hat{w}|^2)\,d\theta\right)(|A^{1/2}w|^2-|A^{1/2}\hat{w}|^2)\,Au \end{aligned}$$

we have

$$|A(t,w,v)\,u-A(t,\hat{w},\hat{v})\,u|\leqslant 2(\sup\{|m'(r)|:r\in[0,R^2]\})\,R\,\|w-\hat{w}\|_{[D(A^{1/2})]}|Au|$$

for $(t,w,v),(t,\hat{w},\hat{v})\in[0,T_0]\times W\times V$, from which Hypothesis 3.5 follows readily.

It remains to check Hypothesis 3.3. To do this, let $\rho \geqslant \rho_0$, $\tau \in (0, \tau_0]$ and $w \in D(\rho, \tau)$. For brevity in notation we write $a(t) = m(|A^{1/2}w(t)|^2)$ for $t \in [0, \tau]$. The definition of $D(\rho, \tau)$ in the previous section implies that $(w(t), w'(t)) \in W \times V$ for $t \in [0, \tau]$, and hence that $1/R < a(t) \leqslant \sup\{m(r) : r \in [0, R^2]\}$ and $|a'(t)| \leqslant 2 \sup\{|m'(r)| : r \in [0, R^2]\} R^2$ for $t \in [0, \tau]$. It follows that there exists $\omega \geqslant 0$, independent of ρ and τ, such that

$$|a(t) - a(s)|/a(s) \leqslant \omega(t-s) \tag{4.2}$$

for $0 \leqslant s \leqslant t \leqslant \tau$. Let $\lambda > 0$ and $\{t_i\}_{i=1}^n$ be any finite sequence with $0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n \leqslant \tau$ with $0 \leqslant \lambda n \leqslant \tau$. Since $a(t) \geqslant 1/R$ for $t \in [0, \tau]$ it is easily seen that the inverse of $I - \lambda \tilde{A}^w(t)$ exists in $B(\tilde{X})$. Let $(u_0, v_0) \in \tilde{X}$, and set $(u_i, v_i) = \prod_{k=1}^{i} (I - \lambda \tilde{A}^w(t_k))^{-1} (u_0, v_0)$ for $1 \leqslant i \leqslant n$. Let $1 \leqslant j \leqslant n$. Then we have $(u_i, v_i) \in [D(A)] \times [D(A^{1/2})]$ and

$$(u_i - u_{i-1})/\lambda = v_i, \tag{4.3}$$

$$(v_i - v_{i-1})/\lambda + a(t_i)\, A u_i = 0 \tag{4.4}$$

for $1 \leqslant i \leqslant j$. We use the inequality that $|u|^2 - |v|^2 \leqslant 2\langle u, u-v \rangle$ for $u, v \in H$, after taking inner products of (4.3) and (4.4) with $a(t_i)\, A u_i$ and v_i respectively and summing the two resulting equalities. This yields

$$a(t_i)\, |A^{1/2}u_i|^2 - a(t_{i-1})|\, A^{1/2}u_{i-1}|^2 + |v_i|^2 - |v_{i-1}|^2 \leqslant (a(t_i) - a(t_{i-1}))\, |A^{1/2}u_{i-1}|^2$$

for $i = 1, 2, \ldots, j$. Adding these inequalities, we obtain

$$a(t_j)\, |A^{1/2}u_j|^2 + |v_j|^2 \leqslant a(t_0)\, |A^{1/2}u_0|^2 + |v_0|^2 + \sum_{i=1}^{j} (a(t_i) - a(t_{i-1}))\, |A^{1/2}u_{i-1}|^2$$

for $j = 1, 2, \ldots, n$. Let α_j denote the right-hand side of the inequality above. Then we have $a(t_j)\, |A^{1/2}u_j|^2 + |v_j|^2 \leqslant \alpha_j$ and by (4.2)

$$\alpha_j - \alpha_{j-1} \leqslant |a(t_j) - a(t_{j-1})|\, a(t_{j-1})^{-1}\, \alpha_{j-1} \leqslant \omega(t_j - t_{j-1})\, \alpha_{j-1}$$

for $j = 1, 2, \ldots, n$. Solving this inequality we find that

$$a(t_j)\, |A^{1/2}u_j|^2 + |v_j|^2 \leqslant \exp(\omega(t_j - t_0))\, (a(t_0)|\, A^{1/2}u_0|^2 + |v_0|^2)$$

for $j = 1, 2, \ldots, n$. By (4.3) we have $|u_n| \leqslant |u_0| + \lambda \sum_{j=1}^n |v_j|$. These estimates imply that Hypothesis 3.3 is satisfied. Consequently, Theorem 4.1 follows from Theorem 3.6. □

THEOREM 4.2. *Assume that* $m(|A^{1/2}\phi|^2) = 0$. *If* $\phi \in D(A^2)$ *and* $\psi \in D(A^{3/2})$ *satisfy* $A^{1/2}\phi = 0$ *and* $A^{1/2}\psi \neq 0$ *respectively, then there exists a positive number T such that the problem* (4.1) *has a unique solution u on* $[0, T]$ *in the class*

$$C([0, T]; [D(A^{3/2})]) \cap C^1([0, T]; [D(A)]) \cap C^2([0, T]; [D(A^{1/2})]),$$

provided that the function $m(r)$ *also satisfies the property that* $m'(r) \geqslant 0$ *for* $r \in [0, \infty)$.

REMARK 4.3. If A is a positive selfadjoint operator in H and $m(r) = r^\alpha$ where $\alpha \geqslant 1$, then the following statements hold by Theorems 4.1 and 4.2 (cf. [**17**, Theorem 2.1, Theorem 2.2]).

(i) If $\phi \in D(A)$ satisfies $\phi \neq 0$ and $\psi \in D(A^{1/2})$, then there exists a positive number T such that the problem (4.1) has a unique solution u on $[0, T]$ in the class

$$C([0, T]; [D(A)]) \cap C^1([0, T]; [D(A^{1/2})]) \cap C^2([0, T]; H)$$

satisfying the property that $|A^{1/2}u(t)| > 0$ for $t \in [0, T]$.

(ii) If $\phi = 0$ and $\psi \in D(A^{3/2})$, then there exists a positive number T such that the problem (4.1) has a unique solution u on $[0, T]$ in the class

$$C([0,T];[D(A^{3/2})]) \cap C^1([0,T];[D(A)]) \cap C^2([0,T];[D(A^{1/2})]).$$

Proof of Theorem 4.2. We consider a triplet of real Banach spaces $[D(A)] \subset [D(A^{1/2})] \subset H$ as $Y \subset X \subset Z$, and take $(I+A)^{1/2}$ and $(I+A)^{-1/2}$ as an isomorphism S of X onto Z and an injective operator C in $B(Z)$, respectively. Choose $R > 0$ so that $\|\phi\|_{[D(A^{3/2})]} < R$ and $\|\psi\|_{[D(A)]} < R$, and define

$$W = \{w \in [D(A^{3/2})] : \|w\|_{[D(A^{3/2})]} < R\}$$

and

$$V = \{v \in [D(A)] : \|v\|_{[D(A)]} < R\}.$$

Clearly, W and V are open bounded subsets of $[D(A^{3/2})]$ and $[D(A)]$ respectively, and $(\phi, \psi) \in W \times V$.

Let $T_0 > 0$. We shall prove the theorem by applying Theorem 3.6 to the family $\{A(t,w,v) : (t,w,v) \in [0,T_0] \times W \times V\}$ defined by

$$\begin{cases} A(t,w,v)\,u = -A(m(|A^{1/2}w|^2)\,u) & \text{for } u \in D(A(t,w,v)) \\ D(A(t,w,v)) = \{u \in H : m(|A^{1/2}w|^2)\,u \in D(A)\}. \end{cases}$$

It is easily seen that all the other conditions of Theorem 3.6 except Hypotheses 3.2 and 3.3 are satisfied. Since $(\phi, \psi) \in D(A^{3/2}) \times D(A)$ one verifies that Hypothesis 3.2 is satisfied with some $\rho_0 > 0$ and $\tau_0 \in (0, T_0]$, similarly to the proof of Theorem 4.1. To check Hypothesis 3.3, let $\rho \geqslant \rho_0$ and choose $\tau_\rho \in (0, \tau_0]$ so that $\tau_\rho \leqslant (\sqrt{17}-3)\,|A^{1/2}\psi|/2\rho$. Here we have used the assumption that $A^{1/2}\psi \neq 0$. If $\tau \in (0, \tau_\rho]$ and $w \in D(\rho, \tau)$ then $w \in C([0,\tau];[D(A^{3/2})]) \cap C^1([0,\tau];[D(A)]) \cap C^2([0,\tau];[D(A^{1/2})])$ and

$$\|w''(t)\|_{[D(A^{1/2})]} \leqslant \rho \quad \text{for } t \in [0,\tau]. \tag{4.5}$$

Since $w(0) = \phi$ and $w'(0) = \psi$, the function w is written in the form

$$w(t) = \phi + t\psi + \int_0^t (t-s)\,w''(s)\,ds$$

for $t \in [0,\tau]$. By estimate (4.5) we have

$$\begin{aligned} &\langle A^{1/2}w(t), A^{1/2}w'(t)\rangle \\ &\quad \geqslant t|A^{1/2}\psi|^2 - |A^{1/2}\psi|\left(\int_0^t (t-s)\,|A^{1/2}w''(s)|\,ds + t\int_0^t |A^{1/2}w''(s)|\,ds\right) \\ &\qquad - \left(\int_0^t (t-s)\,|A^{1/2}w''(s)|\,ds\right)\left(\int_0^t |A^{1/2}w''(s)|\,ds\right) \\ &\quad \geqslant t|A^{1/2}\psi|^2 - |A^{1/2}\psi|(t^2\rho/2 + t^2\rho) - (t^2\rho/2)\,(t\rho) \end{aligned}$$

for $t \in [0,\tau]$. Here we have used the assumption that $A^{1/2}\phi = 0$. The right-hand side is greater than or equal to

$$t(|A^{1/2}\psi|^2 - 3\tau\rho|A^{1/2}\psi|/2 - \tau^2\rho^2/2),$$

which is nonnegative by the choice of τ_ρ. Hence

$$(d/dt)\,m(|A^{1/2}w(t)|^2) = 2m'(|A^{1/2}w(t)|^2)\,\langle A^{1/2}w(t), A^{1/2}w'(t)\rangle \geqslant 0$$

for $t \in [0,\tau]$. It follows from Theorem 2.2 that Hypothesis 3.3 is satisfied. □

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