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REAL HYPERSURFACES IN $P_n(C)$ WITH CONSTANT PRINCIPAL CURVATURES

Dedicated to Professor Hisao Nakagawa on his 60th birthday

U-HANG KI* and RYOICHI TAKAGI

Introduction. Let M be a real hypersurface in a complex projective space $P_n(C)$ of complex dimension n with the metric of constant holomorphic sectional curvature. We shall say M to be *d-isoparametric* if all principal curvatures of M are constant, and the number of different principal curvatures is equal to d. So far, T.E. Cecil and P.J. Ryan [1] proved that any real hypersurface in $P_n(C)$ with at most two principal curvatures must be a part of a geodesic hypersurface, and one of the present authors [3] and Q.M. Wang [4] classified 3-isoparametric hypersurfaces in $P_n(C)$. In this paper we consider a $d(\geq 4)$-isoparametric real hypersurface in $P_n(C)$. Our main result can be stated as follows:

Main Theorem. *Let M be a $d(\geq 4)$isoparametric real hypersurface in $P_n(C)$. Assume that* dim $M > 3\cdot 2^{d-2}$. *Then at each point of M there exists a principal subspace whose image by the complex structure of $P_n(C)$ is again tangent to M.*

Note that if M is a d-isoparametric real hypersurface in $P_n(C)$ and at each point of M there exist $d-1$ principal subspaces all of whose images by the complex structure of $P_n(C)$ are again tangent to M, then M is congruent to one of examples given in [3], which was proved by M. Kimura [2]. Thus our theorem gives a step to see whether a d-isoparametric real hypersurface in $P_n(C)$ exists except for known examples or not. Moreover, we observe that for $d \geq 4 (d \neq 5)$ we have no examples of a d-isoparametric real hypersurface in $P_n(C)$.

1. Preliminaries. Let M be a real hypersurface in $P_n(C)$. Choose a local field $\{\tilde{e}_1, \cdots\cdots, \tilde{e}_{2n}\}$ of orthonormal frame in such a way that, restricted to M, the vectors $\tilde{e}_1, \cdots\cdots, \tilde{e}_{2n-1}$ are tangent to M. Hereafter the indices $i, j, k, l, \alpha, \beta, \gamma$ run from 1 to $2n-1$, and the indices A, B, C, D run from 1 to $2n$. Let ω_A be the 1-forms dual to $\tilde{e}_A$. Then the connection forms ω_{AB} of $P_n(C)$ are the 1-forms uniquely determined by

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$$d\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0. \tag{1.1}$$

The curvature forms of $P_n(C)$ are given by

$$d\omega_{AB} + \sum_C \omega_{AC} \wedge \omega_{CB} = c\omega_A \wedge \omega_B + c\sum_{C,D}(I_{AC}I_{BD} + I_{AB}I_{CD})\omega_C \wedge \omega_D, \tag{1.2}$$

where $4c$ denotes the constant holomorphic sectional curvature of $P_n(C)$ and $I = (I_{AB})$ does the complex structure of $P_n(C)$. Moreover I satisfies

$$\sum_C I_{AC}I_{CB} = -\delta_{AB}, \qquad I_{AB} + I_{BA} = 0, \tag{1.3}$$

$$dI_{AB} = \sum_C I_{AC}\omega_{CB} - \sum_C I_{BC}\omega_{CA}. \tag{1.4}$$

In the sequel we denote by e_A, θ_A, θ_{AB} and J_{AB} the restriction of $\tilde{e}_A$, ω_A, ω_{AB} and I_{AB} to M respectively. Then we have

$$d\theta_i = -\sum_j \theta_{ij} \wedge \theta_j, \qquad \theta_{ij} + \theta_{ji} = 0, \tag{1.5}$$

which implies that θ_{ij} are the connection forms of M. By Cartan's lemma $\theta_{2n,i}$ can be written as

$$\phi_i := \theta_{2n,i} = \sum_j h_{ij}\theta_j, \qquad h_{ij} = h_{ji}. \tag{1.6}$$

The quadric form $\sum \phi_i\theta_i$ is called the second fundamental form of M with respect to the normal vector e_{2n}. The symmetric matrix (h_{ij}) of degree $2n-1$ is called the shape operator. An eigenvalue of the shape operator is called a principal curvature.

The curvature forms Θ_{ij} of M are defined by

$$\Theta_{ij} = d\theta_{ij} + \sum_k \theta_{ik} \wedge \theta_{kj}. \tag{1.7}$$

Then from (1.2) we have the equations of Gauss

$$\Theta_{ij} = \phi_i \wedge \phi_j + c\theta_i \wedge \theta_j + \sum_{k,l} c(J_{ik}J_{jl} + J_{ij}J_{kl})\theta_k \wedge \theta_l. \tag{1.8}$$

Put $f_i = J_{2n,i}$. Then from (1.2) and (1.6) we obtain the equations of Codazzi

$$d\phi_i + \sum_j \phi_j \wedge \theta_{ji} = c\sum_{j,k}(f_jJ_{ik} + f_iJ_{jk})\theta_j \wedge \theta_k. \tag{1.9}$$

Moreover it follows from (1.3) and (1.4) that

$$J_{ij} + J_{ji} = 0, \qquad \sum_k J_{ik}J_{kj} - f_if_j = -\delta_{ij},$$

$$\sum_j J_{ij}f_j = 0, \qquad \sum_i f_i^2 = 1, \tag{1.10}$$

$$dJ_{ij} = \sum_k J_{ik}\theta_{kj} - \sum_k J_{jk}\theta_{ki} - f_i\phi_j + f_j\phi_i, \tag{1.11}$$

$$df_i = \sum_j f_j\theta_{ji} - \sum_j J_{ji}\phi_j. \tag{1.12}$$

2. Formulas. In this section we assume that all principal curvatures $x_1, \cdots\cdots, x_{2n-1}$ of M are constant. Denote by $m(x_i)$ the multiplicity of x_i. We may set $\phi_i = x_i\theta_i$. Then by (1.9) we can write the connection forms θ_{ji} in the form

$$(x_i - x_j)\theta_{ij} = c\sum_k (A_{ijk} + f_iJ_{jk} + f_jJ_{ik})\theta_k, \tag{2.1}$$

where $A_{ijk} = A_{jik} = A_{ikj}$ (cf. [3]). In particular we have

$$A_{ijk} = -f_iJ_{jk} - f_jJ_{ik} \text{ if } x_i = x_j, \tag{2.2}$$

$$f_iJ_{jk} = 0 \text{ if } x_i = x_j = x_k. \tag{2.3}$$

For an index i, we donote by $[i]$ the set of indices j with $x_j = x_i$. Then it is obvious that the vector $F_i = \sum_{j\in[i]} f_je_j$ is independent of the choice of orthonormal frame $\{e_j \mid j \in [i]\}$ for the eigenspace belonging to x_i. Therefore for any index i we can indicate a special index i' so that the vector F_i linearly depends on $e_{i'}$. In other words, $f_j = 0$ for $j \in [i]\backslash\{i'\}$. Note that $f_{i'} = 0$ is possible.

Lemma 2.1. *Assume $J_{ij} = 0$ for $i, j \in [i']$. Then*

$$\sum_{\alpha\notin[i]} \frac{2c}{x_\alpha - x_i}(f_iJ_{j\alpha} - f_jJ_{i\alpha})J_{k\alpha} - f_ix_j\delta_{jk} + f_jx_i\delta_{ik} = 0 \tag{2.4}$$

for $i, j, k \in [i']$,

$$\sum_{\alpha\notin[i]}\left(\frac{J_{i\alpha}}{x_\alpha - x_i}(A_{\alpha j\beta} + f_\alpha J_{j\beta} + f_jJ_{\alpha\beta}) - \frac{J_{j\alpha}}{x_\alpha - x_i}(A_{\alpha i\beta} + f_\alpha J_{i\beta} + f_iJ_{\alpha\beta})\right) = 0 \tag{2.5}$$

for $\beta \notin [i']$ and $i, j \in [i']$.

Proof. From (1.11) we have

$$0 = dJ_{ij} = \sum_\alpha J_{i\alpha}\theta_{\alpha j} - \sum_\alpha J_{j\alpha}\theta_{\alpha i} - f_ix_j\theta_j + f_jx_i\theta_i.$$

It follows from this and (2.1) that

$$\sum_\beta \sum_{\alpha\notin[i]} \frac{cJ_{i\alpha}}{x_\alpha - x_j}(A_{\alpha j\beta} + f_\alpha J_{j\beta} + f_jJ_{\alpha\beta})\theta_\beta$$

$$-\sum_{\beta}\sum_{\alpha\notin[i]}\frac{cJ_{j\alpha}}{x_\alpha-x_i}(A_{\alpha i\beta}+f_\alpha J_{i\beta}+f_i J_{\alpha\beta})\theta_\beta-f_i x_j\theta_j+f_j x_i\theta_i=0.$$

Taking account of the coefficients of $\theta_k(k\in[i])$ and $\theta_\beta(\beta\notin[i])$, we have (2.4) and (2.5).

Lemma 2.2. *Assume $f_{i'}=0$. Then*

$$J_{ij}\Big(\sum_{\alpha\notin[i]}\frac{cf_\alpha^2}{x_\alpha-x_i}+x_i\Big)=0 \text{ for } i,\ j\in[i'], \tag{2.6}$$

$$\sum_{\beta\notin[i]}\frac{cf_\beta}{x_\beta-x_i}A_{\beta i\alpha}+J_{i\alpha}\Big(\sum_{\beta\notin[i]}\frac{cf_\beta^2}{x_\beta-x_i}+x_\alpha\Big)=0 \tag{2.7}$$

for $\alpha\notin[i']$ amd $i\in[i']$.

Proof. From $df_i=0$ and (1.12) we have

$$\sum_\alpha f_\alpha\theta_{\alpha i}-\sum_\alpha J_{\alpha i}x_\alpha\theta_\alpha=0,$$

which together with (2.1) implies that

$$\sum_\beta\sum_{\alpha\notin[i]}\frac{cf_\alpha}{x_\alpha-x_i}(A_{\alpha i\beta}+f_\alpha J_{i\beta}+f_i J_{\alpha\beta})\theta_\beta-\sum_\alpha J_{\alpha i}x_\alpha\theta_\alpha=0.$$

Taking account of the coefficients of $\theta_j(j\in[i'])$ and $\theta_\alpha(\alpha\notin[i'])$ and making use of (2.1), we have (2.6) and (2.7).

Lemma 2.3. *Assume $f_{i'}\neq 0$ and $m(x_i)\geq 2$. Then*

$$\sum_{\alpha\notin[i]}\frac{2c}{x_\alpha-x_i}J_{j\alpha}J_{k\alpha}-x_i\delta_{jk}=0 \text{ for } i,\ j,\ k\in[i'] \text{ and } j\neq i'.$$

Proof. Put $i=i'$ and take $j\in[i']\backslash\{i'\}$ in (2.4).

Lemma 2.4. *Let x_i be a positive maximal or negative minimum principal curvature. Then $f_{i'}=0$ or $m(x_i)=1$.*

Proof. For example, let x_i be a positive maximal principal curvature. If $f_{i'}\neq 0$ and $m(x_i)\geq 2$, we put $j=k\in[i']\backslash\{i'\}$ in Lemma 2.3. Then we have

$$\sum_{\alpha\notin[i]}\frac{2c}{x_\alpha-x_i}J_{j\alpha}J_{j\alpha}=x_i>0,$$

which contradicts the fact that $x_\alpha-x_i<0$ for $\alpha\notin[i']$.

Let $\lambda_1, \cdots\cdots, \lambda_d$ be all different principal curvatures with $\lambda_1 < \cdots\cdots < \lambda_d$, and put $m_r := m(\lambda_r)$.

We may assume $\lambda_1 < 0$ by reversing the unit normal vector e_{2n} if necessary.

Lemma 2.5. (1) *Let $r \in \{1, \cdots\cdots, d\}$ be an index such that $\lambda_r = x_i$ for some i with $f_{i'} \neq 0$. If $m_r \geq m_1 + \cdots\cdots + m_{r-1} + 2$, then $\lambda_r > 0$. If $m_r \geq m_{r+1} + \cdots\cdots + m_d + 2$, then $\lambda_r < 0$.*

(2) *Let $r, s \in \{1, \cdots\cdots, d\}$ be indices such that $r < s$ and $\lambda_r = x_i$, $\lambda_s = x_j$ for some i, j with $f_{i'} f_{j'} \neq 0$. Then $m_r \leq m_1 + \cdots\cdots + m_{r-1} + 1$ or $m_s \leq m_{s+1} + \cdots\cdots + m_d + 1$.*

Proof. We consider a matrix $A = (J_{ij})$, where the indices i and j take all values such that $x_i = \lambda_r$ and $x_j = \lambda_1$ or, $\cdots\cdots$, or λ_{r-1}. Thus A is an $m_r \times (m_1 + \cdots\cdots + m_{r-1})$-matrix, which can be considered as a linear mapping of the m_r-dimensional vector space $V(m_r)$ into the $(m_1 + \cdots\cdots + m_{r-1})$-dimensional vector space $V(m_1 + \cdots\cdots + m_{r-1})$. Denote by $V(m_r - 1)$ the orthogonal complement of $\boldsymbol{R}e_{i'}$ in $V(m_r)$. Putting $\tilde{A} := A \mid V(m_r - 1)$, we have $ker\ \tilde{A} \neq 0$ by the assumption. Then we retake an orthonormal frame $\{e_A\}$ so that $ker\ \tilde{A} \ni e_i$ for some $i \in [i']$. Now since $J_{ij} = 0$ for any j with $x_j = \lambda_1$ or, $\cdots\cdots$, or λ_{r-1}, by Lemma 2.3 we have $\lambda_r \geq 0$. But $\lambda_r = 0$ cannot occur by (1.10) and (2.3). Similarly we can prove the second statement. Now, (2) is an immediate consequence of (1).

Lemma 2.6. (1) *Let $r \in \{2, \cdots\cdots, d\}$ be an index such that $\lambda_r = x_i$ for some i with $f_{i'} \neq 0$. Assume $m_r \geq 2$. Then $\lambda_r(\lambda_{r-1} - \lambda_r) < 2c$ and*

$$m_1 + \cdots\cdots + m_{r-2} + m_{r+1} + \cdots\cdots + m_d \geq m_r - 1,$$

or $\lambda_r(\lambda_{r-1} - \lambda_r) = 2c$ and

$$m_{r-1} \geq m_r - 1.$$

(2) *Let $r \in \{1, \cdots\cdots, d-1\}$ be an index such that $\lambda_r = x_i$ for some i with $f_{i'} \neq 0$. Assume $m_r \geq 2$. Then $\lambda_r(\lambda_{r+1} - \lambda_r) < 2c$ and*

$$m_1 + \cdots\cdots + m_{r-1} + m_{r+2} + \cdots\cdots + m_d \geq m_r - 1,$$

or $\lambda_r(\lambda_{r+1} - \lambda_r) = 2c$ and

$$m_{r-1} \geq m_r - 1.$$

Proof. (1) According to Lemma 2.3., we have

(2-8) $$\sum_{\alpha \notin [i]} \frac{2c}{x_\alpha - x_i} J_{j\alpha} J_{k\alpha} = x_i \delta_{jk} \quad \text{for } i, j, k \in [i'] \text{ and } j \neq i'.$$

On the other hand, it follows from (1.10) that

$$\frac{2c}{\lambda_{r-1} - \lambda_r} \sum_\alpha J_{j\alpha} J_{k\alpha} = \frac{2c}{\lambda_{r-1} - \lambda_r} \delta_{jk}.$$

Subtracting this from (2.8), we have

$$2c \sum_{\alpha \notin [i]} \frac{\lambda_{r-1} - x_\alpha}{(x_\alpha - \lambda_r)(\lambda_{r-1} - \lambda_r)} J_{j\alpha} J_{k\alpha}$$
$$= \left(\lambda_r - \frac{2c}{\lambda_{r-1} - \lambda_r}\right) \delta_{jk} \text{ for } j, k \in [i'], j \neq i'.$$

This implies that $\lambda_r(\lambda_{r-1} - \lambda_r) \leq 2c$ and there are $m_r - 1$ linearly independent $(m_1 + \cdots\cdots + m_{r-2} + m_{r+1} + \cdots\cdots + m_d)$-dimensional vectors.

If $\lambda_r(\lambda_{r-1} - \lambda_r) = 2c$, then we have $J_{j\alpha} = 0$ for $j \in [i'] \setminus \{i'\}$ and any α with $x_\alpha \neq \lambda_{r-1}, \lambda_r$. Thus we have

$$\sum_\alpha J_{j\alpha} J_{k\alpha} = \delta_{jk} \text{ for } j, k \in [i'] \setminus \{i'\},$$

where the summation is taken over all α's with $x_\alpha = x_{r-1}$. This implies $m_{r-1} \geq m_r - 1$.

Similarly, by considering λ_{r+1} instead of λ_{r-1}, we have (2).

3. Proof of Theorem. As for special indices i', confer the beginning of section 2. It is sufficient to prove $f_{i'} = 0$ for some index i. For this, assume the contrary. Then we see $m_1 = 1$ by Lemma 2.4. We need to consider two cases.

Case I: $m_d \geq 2$. Then from Lemma 2.5 (2) we have

$$m_r \leq 1 + m_1 + \cdots\cdots + m_{r-1} \quad \text{for } r = 1, \cdots\cdots, d-1.$$

This implies $m_r \leq 2^{r-1}$ for $r = 1, \cdots\cdots, d-1$. And hence

(3.1) $$m_d = \dim M - m_1 - \cdots\cdots - m_{d-1} \geq \dim M - 2^{d-1} + 1.$$

On the other hand, it follows from Lemma 2.6 (1) that

(3.2) $$m_d \leq 2^{d-2} + 1,$$

which together with (3.1) implies $\dim M \leq 2^{d-1} + 2^{d-2} = 3 \cdot 2^{d-2}$.

Case II: $m_d = 1$. Define two indices r and s by

$$r := \max\{t \mid m_{t-1} \leq 1 + m_1 + \cdots\cdots + m_{t-2}\}$$

and

$$s := \min\{t \mid m_t \leq 1+m_{t+1}+ \cdots\cdots +m_d\}.$$

Then $2 \leq r,\ s \leq d$. If $r < s$, then owing to Lemma 2.5 (2), we have $s = r+1$. Then, as in the Case I we have

$$m_t \leq 2^{t-1} \quad \text{for } t = 1, \cdots\cdots, r-1,$$
$$m_t \leq 2^{d-1} \quad \text{for } t = r+1, \cdots\cdots, d.$$

If $r \geq s$, then above two inequalities hold obviously. Consequently we see that

$$\begin{aligned} m_r &= \dim M-(m_1+ \cdots\cdots +\widehat{m}_r+ \cdots\cdots +m_d) \\ &\geq 2n-1-2^{r-1}-2^{d-r}+2 > n \end{aligned}$$

since $n \geq 3\cdot 2^{d-3}+1$ by the assumption. Thus from (2.3) we have $J_{ij} = 0$ for all i and j with $x_i = x_j = \lambda_r$, which contradicts the fact rank $J = 2n-2$.

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