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Note on Almost Krull Semigroup Rings

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NOTE ON ALMOST KRULL SEMIGROUP RINGS

To the memory of Professor Hisao Tominaga

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Let S be a torsion-free cancellative commutative (additive) semigroup $\supsetneq \{0\}$. Torsion-free commutative (additive) group $\supsetneq \{0\}$ is denoted by G throughout the paper. Let D be an integral domain. The semigroup ring of S over D is denoted by $D[X;S]$. If D_M is a Noetherian domain for each maximal ideal M of D, then D is called a locally Noetherian domain. If D_M is a Krull domain for each maximal ideal M of D, then D is called an almost Krull domain. In this paper we get a necessary and sufficient condition for $D[X;S]$ to be an almost Krull domain for a locally Noetherian domain D of characteristic $p > 0$.

Lemma 1 (A part of [1, Theorem] and [4, Proposition (2.2)]). *Let F be a free subgroup of G such that G/F is torsion. Then $D[X;G]$ is a locally Noetherian domain if and only if D is a locally Noetherian domain, torsion-free rank of G is finite and $(G/F)_p$ is a finite group for each nonunit (of D) prime number p; where $(G/F)_p$ denotes the p-primary component of G/F (that is $(G/F)_p = \{\bar{\alpha} \in G/F \mid p^n\bar{\alpha} = \bar{0}$ for some $n \in \mathbf{N}\}$).*

Lemma 2. *Let D be a Noetherian Krull domain of characteristic $p > 0$. Let G be of the form $G = \mathbf{Z}_{(p)}e_1 + \cdots + \mathbf{Z}_{(p)}e_n$, where the set $e_1, \cdots, e_n$ of elements is linearly independent over $\mathbf{Z}$. Then $D[X;G]$ is an almost Krull domain.*

Proof. We set $F = \mathbf{Z}e_1 + \cdots + \mathbf{Z}e_n$. Then $(G/F)_p = 0$. Lemma 1 implies that $D[X;G]$ is a locally Noetherian domain. It follows that $D[X;G]_M$ is a Noetherian integrally closed domain for each maximal ideal M of $D[X;G]$. Hence $D[X;G]$ is an almost Krull domain.

The quotient field of a domain D is denoted by $\mathrm{q}(D)$. Let $\{v_i \mid i \in I\}$ be a set of valuations on a field k. If the set $\{v_i \mid v_i(a) \neq 0\}$ is finite for each nonzero $a \in k$, then $\{v_i \mid i \in I\}$ is said to be of finite character.

Proposition 3. *Let D be a Noetherian Krull domain of characteristic $p > 0$. Let $G = \sum_{\lambda\in\Lambda} \mathbf{Z}_{(p)}e_\lambda$, where the subset $\{e_\lambda \mid \lambda \in \Lambda\}$ of G is linearly independent over $\mathbf{Z}$. Then $D[X;G]$ is an almost Krull domain.*

Proof. The proof is similar to that of [3, Lemma 26]. We may assume that Λ is an infinite set by Lemma 2. Let $\{\Lambda(\sigma) | \sigma \in \Sigma\}$ be the set of non-empty finite subsets of Λ, and set $G_\sigma = \sum_{\lambda \in \Lambda(\sigma)} \mathbf{Z}_{(p)} e_\lambda$ for each $\sigma \in \Sigma$. Let $\{P_\tau^\sigma | \tau \in T_\sigma\}$ be the set of prime ideals of $D[X; G_\sigma]$ of height one, and let v_τ^σ be the valuation on $q(D[X; G_\sigma])$ which has center P_τ^σ on $D[X; G_\sigma]$. Let w_τ^σ be the canonical extension of v_τ^σ to $q(D[X; G])$, and let $\overline{P_\tau^\sigma}$ be the center of w_τ^σ on $D[X; G]$. Then w_τ^σ is essential for $D[X; G]$. It follows that $\{\overline{P_\tau^\sigma} | \sigma \in \Sigma, \tau \in T_\sigma\}$ is the set of prime ideals of $D[X; G]$ of height one.

Let M be a nonzero prime ideal of $D[X; G]$, and let $\{\overline{P_i} | i \in I\}$ be the set of height one prime ideals of $D[X; G]$ contained in M. Let W_i be the valuation ring $D[X; G]_{\overline{P_i}}$, and let w_i be the valuation of W_i. If $\varphi \in \bigcap_i W_i$, there exists $\sigma \in \Sigma$ such that $\varphi \in q(D[X; G_\sigma])$ and the restriction N of M to $D[X; G_\sigma]$ is nonzero. Let $\{P_j | j \in J\}$ be the set of height one prime ideals of $D[X; G_\sigma]$ contained in N, and let v_j be the valuation on $q(D[X; G_\sigma])$ which has center P_j on $D[X; G_\sigma]$ for each $j \in J$. Since $v_j(\varphi) \geq 0$ for each j, we have $\varphi \in D[X; G_\sigma]_N$. It follows that $D[X; G]_M = \bigcap_i W_i$.

Next let $0 \neq f \in M$. We have $f \in D[X; G_\sigma]$ for some σ. We set $L = \{i \in I | \overline{P_i} \cap D[X; G_\sigma] \neq (0)\}$. If $i \in I - L$, then $w_i(f) = 0$. Let $P_l = \overline{P_l} \cap D[X; G_\sigma]$, and let v_l be the valuation on $q(D[X; G_\sigma])$ which has center P_l on $D[X; G_\sigma]$ for $l \in L$. We have $P_l \neq P_{l'}$ if $\overline{P_l} \neq \overline{P_{l'}}$. Since $D[X; G_\sigma]_N$ is a Krull domain, $\{v_l | l \in L\}$ has finite character.

It follows that $\{w_i | i \in I\}$ has finite character and that $D[X; G]_M$ is a Krull domain.

Lemma 4 ([3, Lemma 27,(2)]). *Let k be a field of characteristic $p > 0$. Let F be a free subgroup of G such that G/F is torsion, and let H be a subgroup of G such that $(G/F)_p = H/F$. Then $k[X; G]$ is an almost Krull domain if and only if H satisfies the ascending chain condition on cyclic subgroups.*

Lemma 5 ([3, Lemma 17,(2)]). *Let H be a subgroup of G. If $D[X; G]$ is an almost Krull domain, then $D[X; H]$ is also an almost Krull domain.*

Lemma 6 ([5, Theorem 2.13]). *Let D be an almost Krull domain with quotient field K, and let L be a finite algebraic extension field of K. Then the integral closure of D in L is an almost Krull domain.*

Let H be a subgroup of G. Assume that for each $\alpha \in G$ there exists $n \in \mathbf{N}$ such that $n\alpha \in H$. Then G is said to be integral over H.

Lemma 7. *Let D be an integrally closed domain. Let H be a subgroup of G such that G is integral over H. Then the properties (LO), (GU) and (GD) hold for the pair of rings $D[X;G]$ and $D[X;H]$.*

Proof. Because $D[X;G]$ is an integral extension of $D[X;H]$, and $D[X;H]$ is an integrally closed domain.

Proposition 8. *Let D be a Noetherian Krull domain of characteristic $p > 0$. Let F be a free subgroup of G such that G/F is torsion, and let H be a subgroup of G such that $(G/F)_p = H/F$. Then $D[X;G]$ is an almost Krull domain if and only if H satisfies the ascending chain condition on cyclic subgroups.*

Proof. We set $k = \mathrm{q}(D)$.

The necessity.

$k[X;G]$ is an almost Krull domain. Lemma 4 implies that H satisfies the ascending chain condition on cyclic subgroups.

The sufficiency.

We set $K = \{\alpha \in G \mid l\alpha \in F$ for some $l \in \mathbf{N}$ with $(l,p) = 1\}$. Then K is a subgroup of a group of the form $\overline{K} = \sum_{\lambda\in\Lambda} \mathbf{Z}_{(p)} e_\lambda$, where the subset $\{e_\lambda \mid \lambda \in \Lambda\}$ is linearly independent over $\mathbf{Z}$. Proposition 3 shows that $D[X;\overline{K}]$ is an almost Krull domain. Then $D[X;K]$ is an almost Krull domain by Lemma 5. Also $k[X;G]$ is an almost Krull domain by Lemma 4. It follows that $D[X;G]_{\overline{P}}$ is a discrete valuation ring for each height one prime ideal $\overline{P}$ of $D[X;G]$.

Let M be a nonzero prime ideal of $D[X;G]$, and let $\{\overline{P_i} \mid i \in I\}$ be the set of height one prime ideal of $D[X;G]$ contained in M. Let $0 \neq \varphi \in \bigcap_i D[X;G]_{\overline{P_i}}$. We have $\varphi = f/g$ for $f,g \in D[X;G]$. There exist $\alpha_1,\cdots,\alpha_l \in G$ such that $f,g \in D[X;\sum_i \mathbf{Z}\alpha_i]$. Set $F + \sum_i \mathbf{Z}\alpha_i = G_0$. Lemma 6 implies that $D[X;G_0]$ is an almost Krull domain. We set $M \cap D[X;G_0] = N$, and set $\overline{P_i} \cap D[X;G_0] = P_i$ for each i. The properties (LO), (GU) and (GD) hold for the pair of rings $D[X;G]$ and $D[X;G_0]$ (Lemma 7). We see that $\{P_i \mid i \in I\}$ is the set of height one prime ideals of $D[X;G_0]$ contained in N. Let w_i be the valuation of $D[X;G]_{\overline{P_i}}$, and let v_i be the valuation of $D[X;G_0]_{P_i}$ for each i. Then the restriction of w_i to $\mathrm{q}(D[X;G_0])$ is (equivalent to) v_i for each i. Since $w_i(\varphi) \geq 0$, we have $v_i(\varphi) \geq 0$. It follows that $\varphi \in D[X;G_0]_N$, and hence $\varphi \in D[X;G]_M$. Therefore $D[X;G]_M = \bigcap_i D[X;G]_{\overline{P_i}}$.

Next let $0 \neq f \in M$. We have $f^{p^n} \in D[X;K]$ for some $n \in \mathbf{N}$. We set $M \cap D[X;K] = N$. Height one prime ideals of $D[X;K]$ containing f^{p^n} and contained in N are of finite number. Let $P_1, \cdots, P_m$ be the set of such prime ideals. We note that the properties (LO),(GU) and (GD) hold for the pair $D[X;G]$ and $D[X;K]$. Let $\overline{P}$, $\overline{P'}$ be distinct height one prime ideals of $D[X;G]$ which contain f and is contained in M. Set $\overline{P} \cap D[X;K] = P$, and $\overline{P'} \cap D[X;K] = P'$. Then we see that P, P' are distinct height one prime ideals of $D[X;K]$.

It follows that height one prime ideals of $D[X;G]$ which contain f and is contained in M are only of finite number. Now the proof of Proposition 8 is complete.

Lemma 9 ([2, Proposition 14]). *$D[X;S]$ is an almost Krull domain if and only if S is a Krull semigroup, and D and $D[X;G_0]$ are almost Krull domains, where G_0 is the maximal subgroup of S.*

Theorem 10. *Let D be a locally Noetherian domain of characteristic $p > 0$. Let G_0, F_0 and H_0 be subgroups of S such that G_0 is the maximal subgroup of S, F_0 is a free subgroup of G_0 such that G_0/F_0 is torsion and $(G_0/F_0)_p = H_0/F_0$. Then $D[X;S]$ is an almost Krull domain if and only if D is an almost Krull domain, S is a Krull semigroup and H_0 satisfies the ascending chain condition on cyclic subgroups.*

Proof. This follows from Proposition 8 and Lemma 9.

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