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HARMONIC AND ISOMETRIC ROTATIONS AROUND A SUBMANIFOLD

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1. Introduction. *Reflections* with respect to points, curves and submanifolds in a Riemannian manifold have been studied extensively in the framework of Riemannian, Hermitian, symplectic and contact geometry. We refer to [2], [18] for a survey and for further references. This study shows that the properties of the reflections influence strongly the curvature of the ambient space and also the extrinsic and intrinsic geometry of the submanifold. It also shows that these properties may be used to characterize some special classes of Riemannian manifolds and submanifolds. (See also [6].)

In [15], [16] the authors extended this theory and initiated the study of the more general notion of *rotation around points and curves* in a Riemannian manifold. Their aim was to treat similar problems as those treated for reflections.

In this paper we continue this study by introducing the notion of *rotation around a submanifold.* More specifically, we will deal with *harmonic* rotations and direct our attention towards the relation between harmonic and isometric rotations. This type of problem is similar to that for reflections with respect to points, curves and submanifolds treated in [1], [7], [8], [19] and for rotations around points and curves considered in [3], [15], [16], [17].

In Section 2 and Section 3 we introduce the basic material and discuss the notion of rotation around a submanifold. In particular we derive, for analytic data, necessary and sufficient conditions for an isometric rotation. In Section 4 we concentrate on harmonic rotations and in Section 5 we prove, among others, that harmonic and isometric rotations around a totally geodesic submanifold with flat normal connection coincide when the ambient space is a locally symmetric Einstein space. This result shows that the study of harmonic rotations is much more complicated than that of harmonic reflections. This may be seen by noting that in [8] it is proved that, in the analytic case, a reflection with respect to a submanifold in a

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general Riemannian manifold is harmonic if and only if it is isometric. Up to now we do not know if the results proved in the present paper may be extended to more general cases.

2. Preliminaries. Let (M,g) be a (connected) n-dimensional Riemannian manifold and let B be a (connected) topologically embedded submanifold of dimension q. Further, let ν be the normal bundle of B an denote by $\exp_\nu$ the exponential map of ν. It is defined on some open domain containing the zero section of ν. We shall always suppose that this domain is sufficiently small in order to have a diffeomorphic $\exp_\nu$.

Next, let $\{E_1,...,E_n\}$ be a local orthonormal frame field of (M,g) along B in a neighborhood $U \subset B$ of a point $m \in B$. We choose $E_1,...,E_q$ to be tangent vector fields and $E_{q+1},...,E_n$ orthonormal sections of ν. If $(y^1,...,y^q)$ is a system of coordinates for B on U such that $\partial/\partial y^i(m) = E_i(m), i = 1,...,q$, then the *Fermi coordinates* $x^1,...,x^n$ relative to $m,(y^1,...,y^q)$ and the frame field $\{E_{q+1},...,E_n\}$ are given by (see [11], [13], [18], for example)

$$x^i(\exp_\nu \sum_{\alpha=q+1}^{n} t^\alpha E_\alpha(b)) = y^i(b), \quad i = 1,...,q,$$

$$x^a(\exp_\nu \sum_{\alpha=q+1}^{n} t^\alpha E_\alpha(b)) = t^\alpha, \quad a = q+1,...,n$$

where $b \in U$ and the $t^\alpha, \alpha = q+1,...,n$ are small enough, in accordance with the hypothesis made above for $\exp_\nu$.

Now, fix a normal unit vector u at m and consider the geodesic normal to B given by $\gamma(t) = \exp_m(tu)$. Here we have $\gamma(0) = m, \gamma'(0) = u$ and we will specialize the frame field $\{E_1,..., E_m\}$ in such a way that $E_n(m) = u$. Next, let $\{e_1(t),...,e_n(t)\}$ be the frame field along $\gamma(t)$ obtained by parallel translation of $\{E_1(m),...,E_n(m)\}$ with respect to the Levi Civita connection ∇ of (M,g).

Further, it is easily seen that the vector fields

$$Y_i(t) = \frac{\partial}{\partial x^i}|_{\gamma(t)}, \quad Y_a(t) = t\frac{\partial}{\partial x^a}|_{\gamma(t)}, \tag{1}$$

$i = 1,...,q; a = q+1,...,n-1$ are Jacobi vector fields along γ with initial conditions

$$\begin{cases} Y_i(0) = E_i(m), & Y_i'(0) = \nabla_u \frac{\partial}{\partial x^i} \\ Y_a(0) = 0, & Y_a'(0) = E_a(m) \end{cases} \tag{2}$$

where the prime denotes covariant differentiation along γ. Then the endomorphism-valued function $t \to D_u(t)$ defined by

(3) $$Y_\alpha(t) = D_u(t)e_\alpha(t), \quad \alpha = 1, ..., n-1,$$

satisfies the Jacobi equation

(4) $$D_u'' + R \circ D_u = 0$$

where $R(t)X = R_{\gamma'(t)X}\gamma'(t), X \in \{\gamma'(t)\}^\perp$ and

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y],$$

for all vector fields X, Y of M, is the Riemann curvature tensor of (M, g). Each $D_u(t)$ is an endomorphism of the space $\{\gamma'(t)\}^\perp$ and these spaces may be identified via the parallel translation along γ by using the parallel basis $\{e_A(t)\}$.

The initial conditions for $D_u(t)$ are obtained through the Gauss and Weingarten formulae for a submanifold given by [4], [14]

(5) $$\begin{cases} \nabla_X Y &= \tilde{\nabla}_X Y + T_X Y, \\ \nabla_X N &= T(N)X + \nabla^\perp_X N \end{cases}$$

for all X, Y tangent to B and all N normal to B. Here $\tilde{\nabla}$ denotes the Levi Civita connection of (B, g), $T_X Y$ is the second fundamental form of B, $T(N)$ is the shape operator of B corresponding to the normal vector N and $\nabla^\perp$ denotes the normal connection along B. Note that $g(T(N)X, Y) = -g(T_X Y, N)$. Now, using (5), we then have in matrix form with respect to the basis $\{E_1(m), ..., E_{n-1}(m)\}$:

(6) $$D_u(0) = \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix}, \quad D_u'(0) = \begin{pmatrix} T(u) & 0 \\ -{}^t\!\perp(u) & I_{n-q-1} \end{pmatrix}$$

where

$$T(u)_{ij} = g(T(u)E_i, E_j)(m), \quad \perp(u)_{ia} = g(\nabla^\perp_{E_i} E_a, E_n)(m)$$

and I_q, I_{n-q-1} are the identity matrices of order q and $n-q-1$, respectively.

Note that the local orthonormal frame $\{E_{q+1}, ..., E_n\}$ defined above can always be chosen to be parallel with respect to the normal connection at a single point in U. Moreover, it can be chosen to be parallel on U

if and only if the normal connection is flat, that is, the curvature of the normal connection vanishes identically [4].

We finish this section with the computation of the components of the metric tensor g and its inverse g^{-1} with respect to the Fermi coordinates introduced above. We have

$$g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}), \quad g_{ia} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^a}), \quad g_{ab} = g(\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}),$$

$i,j = 1,\dots,q; a,b = q+1,\dots,n-1$ and further, from the generalized Gauss lemma (see, for example [11]) we also have

$$g_{in} = g_{an} = 0, \qquad g_{nn} = 1.$$

Next, using (1) and (3) we have for $p = \exp_m(tu), u \in T_m^{\perp}B, \| u \| = 1$:

$$\begin{aligned} g_{ij}(p) &= g(D_u(t)e_i, D_u(t)e_j), \\ g_{ia}(p) &= \frac{1}{t} g(D_u(t)e_i, D_u(t)e_a), \\ g_{ab}(p) &= \frac{1}{t^2} g(D_u(t)e_a, D_u(t)e_b). \end{aligned}$$

Further, using the Jacobi equation (4) and the initial conditions (6) we get the following power series expansions for the components of g and g^{-1}. (By abuse of notation we will also denote the linear map corresponding to the matrix ${}^t\perp(u)$ by the same symbol.) We have

$$\begin{aligned} g_{ij}(p) = {} & g(E_i, E_j)(m) + 2tg(T(u)E_i, E_j)(m) \\ & + t^2\{-g(R(u)E_i, E_j) + g(T(u)^2E_i, E_j) + g({}^t\perp(u)E_i, {}^t\perp(u)E_j)\}(m) \\ & - \frac{1}{3}t^2\{g(R'(u)E_i, E_j) + 2g(R(u)E_i, T(u)E_j) + 2g(R(u)E_j, T(u)E_i) \\ & - 2g(R(u)E_i, {}^t\perp(u)E_j) - 2g(R(u)E_j, {}^t\perp(u)E_i\}(m) \\ & - \frac{1}{12}t^4\{g(R''(u)E_i, E_j) - 4g(R(u)^2E_i, E_j) + 3g(R'(u)E_i, T(u)E_j) \\ & + 3g(R'(u)E_j, T(u)E_i) + 4g(R(u)T(u)E_i, T(u)E_j) \\ & - 3g(R'(u)E_i, {}^t\perp(u)E_j) - 3g(R'(u)E_j, {}^t\perp(u)E_i) \\ & + 4g(R(u){}^t\perp(u)E_i, {}^t\perp(u)E_j) - 4g(R(u){}^t\perp(u)E_j, T(u)E_i) \\ & - 4g(R(u)T(u)E_j, {}^t\perp(u)E_i)\}(m) + O(t^5), \end{aligned}$$

$$g_{ia}(p) = -tg({}^t\perp(u)E_i, E_a)(m) - \frac{2}{3}t^2 g(R(u)E_i, E_a)(m)$$

$$-\frac{1}{12}t^3\{3g(R'(u)E_i, E_a) + 4g(R(u)T(u)E_i, E_a)$$
$$-4g(R(u)^t\perp(u)E_i, E_a)\}(m)$$
$$-\frac{1}{30}t^4\{2g(R''(u)E_i, E_a) - 4g(R(u)^2E_i, E_a)$$
$$+5g(T(u)E_i, R'(u)E_a) - 5g({}^t\perp(u)E_i, R'(u)E_a)\}(m) + O(t^5),$$

$$g_{ab}(p) = g(E_a, E_b)(m) - \frac{1}{3}t^2g(R(u)E_a, E_b)(m) - \frac{1}{6}t^3g(R'(u)E_a, E_b)(m)$$
$$+\frac{1}{180}t^4\{8g(R(u)^2E_a, E_b) - 9g(R''(u)E_a, E_b)\}(m) + O(t^5)$$

and

$$g^{ij}(p) = g(E_i, E_j)(m) - 2tg(T(u)E_i, E_j)(m)$$
$$+t^2\{3g(T(u)^2E_i, E_j) + g(R(u)E_i, E_j)\}(m)$$
$$+\frac{1}{3}t^3\{g(R'(u)E_i, E_j) - 4g(T(u)E_i, R(u)E_j)$$
$$-4g(R(u)E_i, T(u)E_j) - 12g(T(u)^3E_i, E_j)\}(m) + O(t^4),$$

$$g^{ia}(p) = tg({}^t\perp(u)E_i, E_a)(m) + \frac{2}{3}t^2\{g(R(u)E_i, E_a)$$
$$-3\sum_k g(T(u)E_i, E_k)g({}^t\perp(u)E_k, E_a)\}(m)$$
$$\frac{1}{4}t^3\{g(R'(u)E_i, E_a) - 4g(R(u)T(u)E_i, E_a)$$
$$+4\sum_k g(R(u)E_i, E_k)g({}^t\perp(u)E_k, E_a)$$
$$+12\sum_k g(T(u)^2E_i, E_k)g({}^t\perp(u)E_k, E_a)\}(m) + O(t^4),$$

$$g^{ab}(p) = g(E_a, E_b)(m) + \frac{1}{3}t^2\{g(R(u)E_a, E_b)$$
$$+3\sum_k g({}^t\perp(u)E_k, E_a)g({}^t\perp(u)E_k, E_b)\}(m) + \frac{1}{6}t^3\{g(R'(u)E_a, E_b)$$
$$-12\sum_{k,\ell} g(T(u)E_k, E_\ell)g({}^t\perp(u)E_k, E_a)g({}^t\perp(u)E_\ell, E_b)$$
$$+4\sum_k g(R(u)E_k, E_a)g({}^t\perp(u)E_k, E_b)$$
$$+4\sum_k g(R(u)E_k, E_b)g({}^t\perp(u)E_k, E_a)\}(m) + O(t^4),$$

where $R(u) = R_u.u, R'(u) = (\nabla_u R)_u.u, R''(u) = (\nabla^2_{uu} R)_u.u$.

3. Rotations around a submanifold. Let f be an isometry of (M, g) and suppose it has a fixed point set of positive dimension. Let B be a (totally geodesic) connected component of this fixed point set. Then, on a sufficiently small tubular neighborhood of B, f can be represented as

$$f = \exp_\nu \circ df|_B \circ \exp_\nu^{-1}$$

where $df|_B$ is the differential map of f calculated along B. So, $df|_B$ is a $(1, 1)$-tensor field along B which is a linear isometry on each fiber of the normal bundle and which is the identity map on the vectors tangent to B.

With this example in mind we will now introduce the notion of rotation around a submanifold. Therefore, let B be a q-dimensional submanifold of (M, g) as specified in the preceding section. Denote by $\mathfrak{X}(B)$ the C^∞ tangent vector fields of B and by $\bar{\mathfrak{X}}(B)$ the C^∞ tangent vector fields of M along B. Then $\bar{\mathfrak{X}}(B) = \mathfrak{X}(B) \oplus \mathfrak{X}^\perp(B)$ where $\mathfrak{X}^\perp(B)$ consists of all C^∞ vector fields normal to B. Further, $\mathcal{F}(\mathcal{B})$ will denote the algebra of real-valued C^∞ functions on B.

Definitions. A $(1, 1)$-tensor field S along B, that is, an $\mathcal{F}(\mathcal{B})$-linear map

$$S : \bar{\mathfrak{X}}(B) \to \bar{\mathfrak{X}}(B),$$

is said to be a *rotation field along* B if

$$S(\mathfrak{X}^\perp(B)) \subseteq \mathfrak{X}^\perp(B), \quad S|_{\mathfrak{X}(B)} = id_{\mathfrak{X}(B)}$$

and

$$g(SU, SV) = g(U, V)$$

for all $U, V \in \mathfrak{X}^\perp(B)$.

On a sufficiently small tubular neighborhood of B the local diffeomorphism defined by

$$s_B = \exp_\nu \circ S \circ \exp_\nu^{-1}$$

is said to be a (*local*) *S-rotation around the submanifold B*. If $S - I$ is non-singular on the normal bundle, s_B is said to be a *free S-rotation.*

Note that for $S = -I$, s_B defines the (local) reflection with respect to B. Further, we have

$$s_B : \exp_\nu(m, v) \mapsto \exp_\nu(m, Sv).$$

Moreover, B is contained in the fixed point set of s_B. Finally, the analytic expression of s_B in terms of Fermi coordinates is

$$\begin{cases} x^i \circ s_B = x^i, & i = 1, ..., q; \\ x^a \circ s_B = S^a_b x^b, & a, b = q+1, ..., n \end{cases}$$

where S^a_b are the components of S with respect to the basis $\{E_{q+1}, ..., E_n\}$ introduced above.

The covariant differential of S along B is the $\mathcal{F}(B)$-linear function $\nabla S : \mathfrak{X}(B) \times \bar{\mathfrak{X}}(B) \to \bar{\mathfrak{X}}(B)$ defined by

$$(\nabla S)(X, V) = (\nabla_X S)V = \nabla_X(SV) - S\nabla_X V.$$

Then it is easy to see that $\nabla S = 0$ is equivalent with the two following statements :

i) S preserves the second fundamental form of B, that is, $ST_XY = T_XY$;

ii) S preserves the normal connection of B, that is, $S\nabla^{\perp}_X U = \nabla^{\perp}_X(SU)$ (or equivalently $(\nabla^{\perp}_X S)U = 0$),

for $X, Y \in \mathfrak{X}(B)$ and $U \in \mathfrak{X}^{\perp}(B)$. Hence, if S *is free,* $\nabla S = 0$ *implies that B is totally geodesic.* In particular, when S determines the reflection with respect to B, then $\nabla S = 0$ is equivalent to the fact that B is totally geodesic since $\nabla^{\perp}_X S = 0$ is automatically satisfied.

From the remarks made at the beginning of this section it follows that an isometry f of (M, g) is a rotation around the (totally geodesic) connected components of its fixed point set (which was supposed to have a positive dimension). Its rotation field is the differential map of f along B. It is easy to see that this field is parallel along B. Now we will derive a criterion for a rotation to be isometric. It will be used in Section 5.

Proposition 3.1. *Let B be a submanifold of (M, g) as specified above and let s_B be an S-rotation around B. Then we have:*

A. *If s_B is an isometry, then*

i) $\nabla S = 0$ *along B;*

ii) $(\nabla^k_{u...u} R)_{uxuy} = (\nabla^k_{Su...Su} R)_{SuSxSuSy}$

for all normal vectors u, all tangent vectors x, y to M and all $k \in \mathbb{N}$.

B. *The converse also holds for given analytic data.*

Proof. If $s_B = \exp_\nu \circ S \circ \exp_\nu^{-1}$ is an isometry, then $s_B = \exp_\nu \circ ds_{B|B} \circ \exp_\nu^{-1}$ and hence $S = ds_{B|B}$ is parallel along B. Moreover, since any isometry preserves the curvature tensor and its covariant derivatives, we have ii).

Conversely, given i) and ii), we have to prove that $s_B^* g = g$. Using i) this reduces to

$$g_{ij}(p) = g_{ij}(s_B(p)), \quad g_{ia}(p) = g_{i\alpha}(s_B(p))S_a^\alpha(m),$$
$$g_{ab}(p) = g_{\beta\gamma}(s_B(p))S_a^\beta(m)S_b^\gamma(m),$$

for $i, j = 1, ..., q$ and $a, b, \alpha, \beta, \gamma = q+1, ..., n$ where $p = \exp_m(tu)$. Now, as we explained in Section 2, the components of the metric tensor are given in terms of the operator D_u. The Jacobi equation (4) yields

$$D_u^{\ell+2}(0) = -\sum_{k=0}^{\ell} \binom{\ell}{k} R^{(\ell-k)}(0) D_u^{(k)}(0), \quad \ell \in \mathbb{N}.$$

Then, the Taylor expansion of $D_u(t)$ together with the initial conditions,

$$T(Su) = ST(u) = T(u), \quad S\nabla^\perp u = \nabla^\perp(Su),$$

and ii) yield the required result.

4. Harmonic rotations. Let $\varphi : M \to N$ be a smooth map between two Riemannian manifolds with metrics g and h, respectively. The covariant derivative $\nabla d\varphi$ of the differential $d\varphi : TM \to TN$ is a symmetric bilinear form on TM with values in $\varphi^{-1}(TN)$ and is called the *second fundamental form* of φ. The trace of $\nabla d\varphi$ taken with respect to the metric g is called the *tension field* of φ and denoted by $\tau(\varphi)$. The map φ is said to be *harmonic* if $\tau(\varphi) = 0$ (see [9], [10]).

Let $U \subset M$ be a domain with coordinates $(x^1, ..., x^m)$ and $V \subset N$ a domain with coordinates $(y^1, ..., y^n)$ such that $\varphi(U) \subset V$ and suppose φ is locally represented by $y^\alpha = \varphi^\alpha(x^1, ..., x^m), \alpha = 1, ..., n$. Then we have

$$(7) \qquad (\nabla d\varphi)_{ij}^\gamma = \frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j} - {}^M\Gamma_{ij}^k \frac{\partial \varphi^\gamma}{\partial x^k} + {}^N\Gamma_{\alpha\beta}^\gamma(\varphi) \frac{\partial \varphi^\alpha}{\partial x^i} \frac{\partial \varphi^\beta}{\partial x^j},$$

$i, j = 1, ..., m$ and $\gamma = 1, ..., n$. Here ${}^M\Gamma_{ij}^k$ and ${}^N\Gamma_{\alpha\beta}^\gamma$ denote the Christoffel symbols of (M, g) and (N, h), respectively. Hence, φ is harmonic if and only if

$$(8) \qquad \tau(\varphi)^\gamma = g^{ij}(\nabla d\varphi)_{ij}^\gamma = 0.$$

Using Fermi coordinates and with the notations just introduced we have

Proposition 4.1. *The S-rotation s_B with respect to the submanifold B is harmonic if and only if for all $m \in B$*

$$(9)\quad \tau(s_B)^k(p) = \{g^{ij}(\nabla ds_B)^k_{ij} + 2g^{ia}(\nabla ds)^k_{ia} + g^{ab}(\nabla ds_B)^k_{ab}\}(p) = 0,$$

$$(10)\quad \tau(s_B)^c(p) = \{g^{ij}(\nabla ds_B)^c_{ij} + 2g^{ia}(\nabla ds_B)^c_{ia} + g^{ab}(\nabla ds_B)^c_{ab}\}(p) = 0$$

for $i,\ j,\ k = 1,\dots,q$ and $a,\ b,\ c = q+1,\ \dots,\ n$, where $p = \exp_m(tu)$, $u \in T^\perp_m B, \| u \| = 1$ and

$$\begin{aligned}
(11)\quad (\nabla ds_B)^k_{ij}(p) &= -\Gamma^k_{ij}(p) + \Gamma^k_{ij}(s_B(p)) + \Gamma^k_{i\beta}(s_B(p))\frac{\partial S^\beta_\mu}{\partial x^j}x^\mu \\
&\quad + \Gamma^k_{\alpha j}(s_B(p))\frac{\partial S^\alpha_\delta}{\partial x^i}x^\delta + \Gamma^k_{\alpha\beta}(s_B(p))\frac{\partial S^\alpha_\delta}{\partial x^i}\frac{\partial S^\beta_\mu}{\partial x^j}x^\delta x^\mu, \\
(\nabla ds_B)^k_{ia}(p) &= -\Gamma^k_{ia}(p) + \Gamma^k_{i\beta}(s_B(p))S^\beta_a + \Gamma^k_{\alpha\beta}(s_B(p))\frac{\partial S^\alpha_\delta}{\partial x^i}S^\beta_a x^\delta, \\
(\nabla ds_B)^k_{ab}(p) &= -\Gamma^k_{ab}(p) + \Gamma^k_{\alpha\beta}(s_B(p))S^\alpha_a S^\beta_b, \\
(\nabla ds_B)^c_{ij}(p) &= \frac{\partial^2 S^c_\alpha}{\partial x^i \partial x^j}x^\alpha - \Gamma^\ell_{ij}(p)\frac{\partial S^c_\delta}{\partial x^\ell}x^\delta - \Gamma^\alpha_{ij}(p)S^c_\alpha \\
&\quad + \Gamma^c_{ij}(s_B(p)) + \Gamma^c_{i\beta}(s_B(p))\frac{\partial S^\beta_\delta}{\partial x^j}x^\delta \\
&\quad + \Gamma^c_{\alpha j}(s_B(p))\frac{\partial S^\alpha_\mu}{\partial x^i}x^\mu + \Gamma^c_{\alpha\beta}(s_B(p))\frac{\partial S^\alpha_\delta}{\partial x^i}\frac{\partial S^\beta_\mu}{\partial x^j}x^\delta x^\mu, \\
(\nabla ds_B)^c_{ia}(p) &= \frac{\partial S^c_a}{\partial x^i} - \Gamma^\ell_{ia}(p)\frac{\partial S^c_\delta}{\partial x^\ell}x^\delta - \Gamma^\alpha_{ia}(p)S^c_\alpha \\
&\quad + \Gamma^c_{i\beta}(s_B(p))S^\beta_a + \Gamma^c_{\alpha\beta}(s_B(p))\frac{\partial S^\alpha_\delta}{\partial x^i}S^\beta_a x^\delta, \\
(\nabla ds_B)^c_{ab}(p) &= -\Gamma^\ell_{ab}(p)\frac{\partial S^c_\delta}{\partial x^\ell}x^\delta - \Gamma^\alpha_{ab}(p)S^c_\alpha + \Gamma^c_{\alpha\beta}(s_B(p))S^\alpha_a S^\beta_b
\end{aligned}$$

where S^α_β and its partial derivatives are evaluated at m.

To express the harmonicity of s_B we need to know the Christoffel symbols with respect to a system of Fermi coordinates. For this we will use the well-known formula

$$(12)\qquad \Gamma^A_{BC} = \frac{1}{2}\sum_{D=1}^{n} g^{AD}\{\frac{\partial g_{BD}}{\partial x^C} + \frac{\partial g_{CD}}{\partial x^B} - \frac{\partial g_{BC}}{\partial x^D}\},$$

$A, B, C = 1, ..., n$, giving the Christoffel symbols in terms of the metric tensor. Then we will use the expressions for g and g^{-1} given in Section 2.

5. Harmonic and isometric rotations. In this section we shall concentrate on harmonic S-rotations in relation with isometric S-rotations around a submanifold of a Riemannian manifold. More specifically, we shall prove

Theorem 5.1. *Let B be a submanifold of (M, g) as specified above and let s_B be an S-rotation around B. If s_B is harmonic, then S preserves the mean curvature vector field of B and if s_B is a free rotation, then B is a minimal submanifold. Moreover, if B is totally geodesic with flat normal connection and s_B a harmonic rotation, then $\nabla S = 0$ along B.*

Theorem 5.2. *Let (M, g) be a locally symmetric space with S-invariant Ricci tensor and let B be a totally geodesic submanifold with flat normal connection and s_B an S-rotation around B. Then s_B is harmonic if and only if it is isometric.*

This gives at once

Corollary 5.3. *Let (M, g) be a locally symmetric Einstein space and let B be a totally geodesic submanifold with flat normal connection and s_B a rotation around B. Then s_B is harmonic if and only if s_B is isometric.*

To prove these results we put

$$\tau(s_B)^c = \sum_{k=0}^{3} A_k^c t^k + O(t^4), \qquad c = q+1, ..., n,$$

$$\tau(s_B)^i = \sum_{k=0}^{3} A_k^i t^k + O(t^4), \qquad i = 1, ..., q.$$

From (9) and (10) we then get the following necessary conditions for s_B to be a harmonic rotation :

$$A_k^c = 0, \qquad A_k^i = 0$$

for $k = 0, 1, 2, 3; c = q+1, ..., n$ and $i = 1, ..., q$. To compute A_k^c and A_k^i we use (9), (10), (11), the expressions for g_{AB}, g^{AB} obtained in Section 2 and (12). We delete the lengthy but straightforward computations.

Proof of Theorem 5.1. First, from $A_0^c = 0, c = q+1, ..., n$ we get

$$S \sum_{i=1}^{q} T_{E_i} E_i = \sum_{i=1}^{q} T_{E_i} E_i$$

which is equivalent with $SH = H$ where H is the mean curvature vector. So, if s_B is a free rotation, we get $H = 0$ and hence, B is a minimal submanifold.

Next, suppose that B is totally geodesic and that $\nabla^\perp$ is flat. Then the frame field $\{E_{q+1}, ..., E_n\}$ can be taken parallel with respect to the normal connection in a neighborhood of $m \in B$. Under these hypotheses, the conditions $A_1^c = 0$ yield

$$(13) \qquad \sum_{i=1}^{q} g((\nabla^{\perp 2}_{E_i E_i} S)u, SE_c) - \sum_{i=1}^{q} (R_{uici} - R_{SuiSci}) \\ -\frac{2}{3} \sum_{a=q+1}^{n} (R_{caua} - R_{ScSaSuSa}) = 0$$

where $R_{ABCD} = R_{E_A E_B E_C E_D}, A, B, C, D = 1, ..., n$. Now, for a normal unit vector field u we have $g((\nabla^\perp_{E_i} S)u, Su) = 0$ and then, differentiating again and using ${}^t SS = I$, we get

$$g((\nabla^{\perp 2}_{E_i E_i} S)u, Su) = -g((\nabla^\perp_{E_i} S)u, (\nabla^\perp_{E_i} S)u).$$

Next, we substitute this in (13), replace u by E_c and sum with respect to $c = q+1, ..., n$. This gives

$$\sum_{i=1}^{q} \sum_{c=q+1}^{n} g((\nabla^\perp_{E_i} S)E_c, (\nabla^\perp_{E_i} S)E_c) + \sum_{i=1}^{q} \sum_{c=q+1}^{n} (R_{cici} - R_{SciSci}) \\ + \frac{2}{3} \sum_{a,c=q+1}^{n} (R_{caca} - R_{ScSaScSa}) = 0,$$

and this yields

$$\sum_{i=1}^{q} \sum_{c=q+1}^{n} g((\nabla^\perp_{E_i} S)E_c, (\nabla^\perp_{E_i} S)E_c) = 0.$$

Since $\nabla^{\perp}S : \mathfrak{X}(B) \times \mathfrak{X}^{\perp}(B) \to \mathfrak{X}^{\perp}(B)$ is $\mathcal{F}(B)$-linear, we obtain $\nabla^{\perp}S = 0$ along B. So, since B is totally geodesic, $\nabla S = 0$ along B.

Proof of Theorem 5.2. Since each isometry is harmonic we have only to prove the converse. So, let s_B be a harmonic rotation around B. According to Proposition 3.1 and Theorem 5.1 we have to prove that

$$R_{iuju} = R_{iSujSu}, \quad R_{iuau} = R_{iSuSaSu}, \quad R_{aubu} = R_{SaSuSbSu}$$

for $i, j = 1, ..., q$ and $a, b = q+1, ..., n$.

First, since $\nabla S = 0$ and since the Ricci tensor is S-invariant, (13) yields, by taking $E_c = u$,

$$\sum_{i=1}^{q}(R_{uiui} - R_{SuiSui}) = 0 \tag{14}$$

for all $u \in T_m^{\perp}B$.

Next, we consider the conditions $A_3^c = 0, c = q+1, ..., n$. Then, after a lengthy computation, we obtain

$$\begin{aligned} 30\sum_{i,j=1}^{q}(R_{uiuj}^2 - R_{SuiSuj}^2) &- 45\sum_{i,j=1}^{q} R_{uiuj}(R_{uiuj} - R_{SuiSuj}) \\ &-60\sum_{i=1}^{q}\sum_{a=q+1}^{n} R_{uiua}(R_{uiua} - R_{SuiSuSa}) \\ &+36\sum_{i=1}^{q}\sum_{a=q+1}^{n}(R_{iuau}^2 - R_{iSuSaSu}^2) \\ &+6\sum_{a,b=1}^{n}(R_{uaub}^2 - R_{SuaSub}^2) \\ &-10\sum_{a,b=q+1}^{n} R_{uaub}(R_{uaub} - R_{SuSaSuSb}) = 0. \end{aligned} \tag{15}$$

The left hand side number of (15) may be considered as a function on the unit sphere $S^{n-q-1}(1)$ in $T_m^{\perp}B$. We shall integrate this function over this sphere. Therefore we use the next lemma (see, for example, [5], [11], [12]). We have

Lemma 5.4. *Let $u = \sum_{a=q+1}^{n} u_a E_a$ be an orthonormal decomposition of the unit vector $u \in T_m^{\perp}B$ with respect to an orthonormal basis*

$\{E_{q+1},...,E_n\}$ of $T_m^{\perp}B$. Then we have

$$\int_{S^{n-q-1}(1)} u_a d\mu = 0, \quad \int_{S^{n-q-1}(1)} u_a u_b u_c d\mu = 0,$$

$$\int_{S^{n-q-1}(1)} u_a u_b d\mu = c_{n-q-1}\frac{1}{n-q}\delta_{ab},$$

$$\int_{S^{n-q-1}(1)} u_a^2 u_b^2 d\mu = \frac{1}{3}\int_{S^{n-q-1}(1)} u_a^4 d\mu = c_{n-q-1}\frac{1}{(n-q)(n-q+2)}, \quad a \neq b,$$

$$\int_{S^{n-q-1}(1)} u_a u_b u_c u_d d\mu = 0 \text{ whenever at least three indices are different},$$

$$\int_{S^{n-q-1}(1)} u_a^3 u_b d\mu = 0, \quad a \neq b,$$

where $a,b,c,d = q+1,...,n$; $d\mu$ denotes the volume element of $S^{n-q-1}(1)$ and

$$c_{n-q-1} = \frac{(n-q)\pi^{\frac{n-q}{2}}}{(\frac{n-q}{2})!}$$

is the volume of a unit sphere in the Euclidean space $\mathbb{E}^{n-q}$.

As a consequence of this lemma one gets at once that the integrals of

$$\sum_{i,j=1}^{q}(R^2_{uiuj} - R^2_{SuiSuj}), \quad \sum_{i=1}^{q}\sum_{a=q+1}^{n}(R^2_{uiua} - R^2_{SuiSuSa})$$

and

$$\sum_{a,b=q+1}^{n}(R^2_{uaub} - R^2_{SuaSub})$$

vanish. Next, put

$$A = \sum_{i,j=1}^{q} R_{uiuj}(R_{uiuj} - R_{SuiSuj}),$$

$$B = \sum_{i=1}^{q}\sum_{a=q+1}^{n} R_{uiua}(R_{uiua} - R_{SuiSuSa}),$$

$$C = \sum_{a,b=q+1}^{n} R_{uaub}(R_{uaub} - R_{SuSaSuSb}).$$

Then, by integration, we get first

$$\int_{S^{n-q-1}(1)} A d\mu = \frac{c_{n-q-1}}{(n-q)(n-q+2)}\sum_{i,j=1}^{q}\sum_{a,b=q+1}^{n}\{R_{aibj}R_{aibj} + R_{aibj}R_{biaj}$$

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$$-R_{aibj}R_{SaiSbj}-R_{aibj}R_{SbiSaj}\}.$$

Now, we use the first Bianchi identity and the Ricci equation for B given by (see [4])

$$g(R^{\perp}_{XY}U,V)=g(R_{XY}U,V)+g(T(U)X,T(V)Y)-g(T(V)X,T(U)Y)$$

for tangent vectors X,Y of B and normal vectors U,V of B. Then we get

$$\int_{S^{n-q-1}(1)}Ad\mu=\frac{c_{n-q-1}}{(n-q)(n-q+2)}\sum_{i,j=1}^{q}\sum_{a,b=q+1}^{n}(R_{aibj}-R_{SaiSbj})^2.$$

Next, we have

$$\int_{S^{n-q-1}(1)}Bd\mu=\frac{c_{n-q-1}}{(n-q)(n-q+2)}\sum_{i=1}^{q}\sum_{a,b,c=q+1}^{n}\{R_{biba}R_{cica}+R_{bica}R_{bica}$$
$$+R_{bica}R_{ciba}-R_{biba}R_{SciScSa}-R_{bica}R_{SbiScSa}-R_{bica}R_{SciSbSa}\}.$$

We observe that

$$\sum_{c=q+1}^{n}(R_{biba}R_{cica}-R_{biba}R_{SciScSa})=R_{biba}(\rho_{ia}-\sum_{j=1}^{q}R_{jija}$$
$$-\rho_{iSa}+\sum_{j=1}^{q}R_{jijSa})=\sum_{j=1}^{q}R_{biba}(R_{jijSa}-R_{jija})$$

since ρ is S-invariant. This last expression vanishes since $R_{E_jE_i}E_j$ is tangent to B because the submanifold is totally geodesic. Furthermore, because of the curvature identities

$$\sum_{a,b,c}R_{ibca}R_{icba}=\frac{1}{2}\sum_{a,b,c}R_{ibca}R_{ibca},$$
$$\sum_{a,b,c}R_{ibca}R_{iScSbSa}=\frac{1}{2}\sum_{a,b,c}R_{ibca}R_{iSbScSa},$$

we finally obtain

$$\int_{S^{n-q-1}(1)}Bd\mu=\frac{3c_{n-q-1}}{2(n-q)(n-q+2)}\sum_{i=1}^{q}\sum_{a,b,c=q+1}^{n}R_{ibca}\{R_{ibca}-R_{iSbScSa}\}.$$

For the integral of C we get

$$\int_{S^{n-q-1}(1)} C d\mu = \frac{c_{n-q-1}}{(n-q)(n-q+2)} \sum_{a,b,\alpha,\beta=q+1}^{n} \{R_{\alpha a\alpha b}R_{\beta a\beta b} + R_{\alpha a\beta b}R_{\alpha a\beta b} + R_{\alpha a\beta b}R_{\beta a\alpha b} - R_{\alpha a\alpha b}R_{S\beta Sa S\beta Sb} - R_{\alpha a\beta b}R_{S\alpha Sa S\beta Sb} - R_{\alpha a\beta b}R_{S\beta Sa S\alpha Sb}\}.$$

Proceeding in the same way as for B, we obtain

$$\int_{S^{n-q-1}(1)} C d\mu = \frac{c_{n-q-1}}{(n-1)(n-q+2)} \sum_{a,b,\alpha,\beta=q+1}^{n} \{R_{\alpha a\alpha b}R_{\beta a\beta b} - R_{\alpha a\alpha b}R_{S\beta Sa S\beta Sb} + \frac{3}{2}R_{\alpha a\beta b}(R_{\alpha a\beta b} - R_{S\alpha Sa S\beta Sb})\}.$$

Now, the S-invariance of ρ yields

$$\sum_{a,b,\alpha,\beta=q+1}^{n} R_{\alpha a\alpha b}(R_{\beta a\beta b}-R_{S\beta Sa S\beta Sb}) = -\sum_{i=1}^{q}\sum_{a,b,\alpha=q+1}^{n} R_{\alpha a\alpha b}(R_{iaib}-R_{iSaiSb})$$

and this vanishes because of (14). So, we get

$$\int_{S^{n-q-1}(1)} C d\mu = \frac{3c_{n-q-1}}{2(n-q)(n-q+2)} \sum_{a,b,\alpha,\beta=q+1}^{n} R_{\alpha a\beta b}(R_{\alpha a\beta b} - R_{S\alpha Sa S\beta Sb})$$
$$= \frac{3c_{n-q-1}}{4(n-q)(n-q+2)} \sum_{a,b,\alpha,\beta=q+1}^{n} (R_{\alpha a\beta b} - R_{S\alpha Sa S\beta Sb})^2.$$

From all these computations we see that the integration of (15) over $S^{n-q-1}(1)$ yields

$$6\sum_{i,j=1}^{q}\sum_{a,b=q+1}^{n}(R_{aibj} - R_{SaiSbj})^2 + 6\sum_{i=1}^{q}\sum_{a,b,c=q+1}^{n}(R_{iabc} - R_{iSaSbSc})^2 + \sum_{a,b,\alpha,\beta=q+1}^{n}(R_{\alpha a\beta b} - R_{S\alpha Sa S\beta Sb})^2 = 0,$$

and this is equivalent to

$$R_{aibj} = R_{SaiSbj}, \quad R_{iabc} = R_{iSaSbSc}, \quad R_{\alpha a\beta b} = R_{S\alpha Sa S\beta Sb}$$

for $i,j = 1,\dots,q$ and $a,b,c = q+1,\dots,n$, from which the required result follows

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