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The Role of Commutators in a Non-Cancellation Phenomenon

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Abstract

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1. Introduction

The term non-cancellation phenomenon is used here in the following sense:

Differentiable manifolds M_1, M_2 and N are given to satisfy

(i) $M_1 \times N$ is diffeomorphic to $M_2 \times N$;
(ii) $M_1 \not\simeq M_2$, where $\simeq$ means same homotopy type.

Charlap in 1965, furnished one of the early examples of such phenomenon in [Ch], where M_1, M_2 are flat riemannian manifolds and $N = S^1$. Such an example is obtained as a consequence of his classification for $\mathbb{Z}_p$-manifolds (riemannian manifolds with holonomy group equal to $\mathbb{Z}_p$) with p prime.

In 1969, Hilton and Roitberg [HR2] considered the case where M_1 and M_2 are total spaces of principal bundles and N the corresponding structural group, more precisely, they consider principal S^3-bundles over spheres S^n. The principal S^3-bundles over S^n are classified by $\pi_n(BS^3) \cong \pi_{n-1}(S^3)$ (cf. [St]). So, we have for each $\alpha \in \pi_{n-1}(S^3)$ the corresponding S^3-bundle $S^3 \cdots E_\alpha \xrightarrow{p_\alpha} S^n$ classified by the adjoint $\alpha_0 \in \pi_n(BS^3)$ of α. Given $\alpha, \beta \in \pi_{n-1}(S^3)$ let $E_{\alpha\beta}$ be the principal S^3-bundle over E_α induced from the bundle E_β by the projection $p_\alpha : E_\alpha \longrightarrow S^n$. We have in this way the following commutative diagram:

$$\begin{array}{ccccc}
S^3 & & S^3 & & \\
\vdots & & \vdots & & \\
E_{\alpha\beta} & \longrightarrow & E_\beta & & \\
\downarrow & & \downarrow{\scriptstyle p_\beta} & & \\
E_\alpha & \xrightarrow{p_\alpha} & S^n & \xrightarrow{\beta_0} & BS^3.
\end{array}$$

diagram 1

Theorem 1 (Hilton-Roitberg [HR2])**.**

i) *$E_\alpha \simeq E_\beta \Longleftrightarrow \alpha = \pm\beta$;*
ii) *Let $\alpha \in \pi_{n-1}(S^3)$ be an element of order k and $\beta = l\alpha$, $l \in \mathbb{Z}$. If there exists l', $l' \equiv l \bmod k$, such that*

$$\frac{l'(l'-1)}{2}\omega \circ \Sigma^3\alpha = 0 \in \pi_{n+2}(S^3),$$

where $\omega \in \pi_6(S^3)$ is the generator, Σ^3 is the 3-fold suspension of α, then the bundle $S^3 \cdots E_{\alpha\beta} \to E_\alpha$ is trivial.

We observe that by the construction of the induced bundle it follows easily that $E_{\alpha\beta} = E_{\beta\alpha}$. This observation together with the above theorem give the following example of non-cancellation phenomena:

Consider S^3-bundles over S^7, $M_1 = E_\alpha$, where $\alpha = \omega \in \pi_6(S^3) \cong \mathbb{Z}_{12}$ is a generator, and $M_2 = E_\beta$ with $\beta = 7\alpha$. Since $\frac{7(7-1)}{2}\omega \circ \Sigma^3\alpha = 21\omega \circ \Sigma^3\alpha = 0$ by $\pi_9(S^3) \cong \mathbb{Z}_3$, similarly as $\alpha = 7\beta$ we also have $\frac{7(7-1)}{2}\omega \circ \Sigma^3\beta = 0$. As E_ω is the canonical S^3-bundle $Sp(2)$ over S^7 we have now

$$Sp(2) \times S^3 = E_{7\omega} \times S^3 \quad \text{and} \quad Sp(2) \not\simeq E_{7\omega}. \tag{1}$$

This example is also relevant in a different context:

$Sp(2)$ and $E_{7\omega}$ are the only total spaces of principal S^3-bundles over S^7, up to orientation, that admit a loop-space structure (cf. [HMR2], [CM] or [Z]). This, together with the second part of (1), tells us that $Sp(2)$ and $E_{7\omega}$ are H-spaces with distinct H-structures.

In the examples above M_1, M_2 and N are at most 2-connected. In 1972 Hilton, Mislin and Roitberg [HMR1] provided examples of M_1, M_2 and N all arbitrarily highly connected.

These examples of non-cancellation phenomena are obtained by indirect ways, that is, the diffeomorphism (1) is not explicitated.

Hilton and Roitberg ([HR1], [HR2]) consider the cellular decomposition of the spaces E_α ($\alpha \in \pi_{n-1}(S^3)$):

$$E_\alpha = (S^3 \cup_\alpha e^n) \cup e^{n+3} = C_\alpha \cup e^{n+3}.$$

It is shown that under the same conditions in which we have $E_\alpha \times S^3 = E_\beta \times S^3$ we also have $C_\alpha \vee S^3 \simeq C_\beta \vee S^3$ (where $\vee$ denotes the one point union) although, in general $E_\alpha \not\simeq E_\beta$ and $C_\alpha \not\simeq C_\beta$. They suggest then a more careful analysis of the diffeomorphism between $E_\alpha \times S^3$ and $E_\beta \times S^3$.

The subject of this paper is to analyse the example of Hilton-Roitberg above, trying to give an idea of the complexity of the diffeomorphism between $Sp(2) \times S^3$ and $E_{7\omega} \times S^3$.

To do this, we worked with the models for principal S^3-bundles over S^7 denoted in [R] by $\tilde{P}_n$. Such bundles are represented by 10-dimensional submanifolds of $Sp(n)$ and have transition functions $g^n_{VU} : U \cap V \longrightarrow S^3$ relative to an open covering of S^7 by just two sets $U = \left\{ \binom{a}{b} \in S^7; a \neq 0 \right\}$ and $V = \left\{ \binom{a}{b} \in S^7; b \neq 0 \right\}$, given by $g^n_{VU}\binom{a}{b} = \frac{b^{n-1}(a\bar{b})^{n-1}\bar{a}^{n-1}}{(|a||b|)^{2(n-1)}}$ and the method used in [R] shows that

$$(2) \qquad g^n_{VU} \overset{H}{\simeq} 1 \Longrightarrow \begin{array}{l}\text{The bundle } \tilde{P}_n \text{ is trivial and a global section} \\ \text{can be constructed explicitly by means of } H.\end{array}$$

There exists a diffeomorphism $\beta : S^3 \times S^3 \times (0, \frac{\pi}{2}) \longrightarrow U \cap V$ such that $g^n_{VU} \circ \beta(A, B, \theta) = B^{n-1}(A\bar{B})^{n-1}\bar{A}^{n-1}$. Thus, working in the group of homotopy classes of maps $[S^3 \times S^3, S^3]$ we obtained that $B^8(A\bar{B})^8\bar{A}^8 \simeq 1$ and the implication (2) can be realized for $n = 9$, which coincides with the classification of the bundles $\tilde{P}_n$ given in [B2]. We use the same idea to construct a diffeomorphism between $Sp(2) \times S^3$ and $E_{7\omega} \times S^3$. In this case the bundles $E_{\alpha\beta}$ are modeled through the $\tilde{P}_n$'s via the pull-back construction providing the principal S^3-bundles $\tilde{P}_{n,m}$ over $\tilde{P}_n$, in such a way that the homotopy classes of transition functions of these bundles are in one to one correspondence with $[S^3 \times S^3 \times S^3, S^3]$, so the method applied above works here but the calculations are much more complicated.

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2. The bundles $\tilde{P}_n$ and $\tilde{P}_{n,m}$

Let $M_n = M_n(a, b, x_1, x_2, \ldots, x_n) \in Sp(n)$ $(n \geq 3)$ be given by

$$M_3 = \begin{pmatrix} a & -b|b|^2 & x_1 \\ b & b\bar{a}b & x_2 \\ 0 & a\sqrt{1+|b|^2} & x_3 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} a & -b|b|^2L(4)^{-1} & 0 & x_1 \\ b & b\bar{a}bL(4)^{-1} & 0 & x_2 \\ 0 & a|a|^2L(4)^{-1} & -b & x_3 \\ 0 & a\bar{b}aL(4)^{-1} & a & x_4 \end{pmatrix},$$

where $L(4) = \sqrt{|a|^4 + |b|^4}$, and for $n \geq 5$

$$M_n = \begin{pmatrix} a & \frac{-b|b|^2}{L(n)} & 0 & \dots & 0 & 0 & 0 & 0 & x_1 \\ b & \frac{b\bar{a}b}{L(n)} & 0 & \dots & 0 & 0 & 0 & 0 & x_2 \\ 0 & \frac{af_{n-5}}{L(n)} & \frac{-b}{L_{n-4}} & \dots & 0 & 0 & 0 & 0 & x_3 \\ 0 & \frac{(a\bar{b})af_{n-6}}{L(n)} & \frac{af_{n-6}}{L_{n-4}} & \dots & 0 & 0 & 0 & 0 & x_4 \\ 0 & \frac{(a\bar{b})^2af_{n-7}}{L(n)} & \frac{(a\bar{b})af_{n-7}}{L_{n-4}} & \dots & 0 & 0 & 0 & 0 & x_5 \\ 0 & \frac{(a\bar{b})^3af_{n-8}}{L(n)} & \frac{(a\bar{b})^2af_{n-8}}{L_{n-4}} & \dots & 0 & 0 & 0 & 0 & x_6 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \frac{(a\bar{b})^{n-7}af_2}{L(n)} & \frac{(a\bar{b})^{n-8}af_2}{L_{n-4}} & \dots & \frac{-b}{L_3} & 0 & 0 & 0 & x_{n-4} \\ 0 & \frac{(a\bar{b})^{n-6}af_1}{L(n)} & \frac{(a\bar{b})^{n-7}af_1}{L_{n-4}} & \dots & \frac{af_1}{L_3} & \frac{-b}{L_2} & 0 & 0 & x_{n-3} \\ 0 & \frac{(a\bar{b})^{n-5}af_0}{L(n)} & \frac{(a\bar{b})^{n-6}af_0}{L_{n-4}} & \dots & \frac{(a\bar{b})af_0}{L_3} & \frac{af_0}{L_2} & \frac{-b}{L_1} & 0 & x_{n-2} \\ 0 & \frac{(a\bar{b})^{n-4}a|a|^2}{L(n)} & \frac{(a\bar{b})^{n-5}a|a|^2}{L_{n-4}} & \dots & \frac{(a\bar{b})^2a|a|^2}{L_3} & \frac{(a\bar{b})a|a|^2}{L_2} & \frac{a|a|^2}{L_1} & -b & x_{n-1} \\ 0 & \frac{(a\bar{b})^{n-3}a}{L(n)} & \frac{(a\bar{b})^{n-4}a}{L_{n-4}} & \dots & \frac{(a\bar{b})^3a}{L_3} & \frac{(a\bar{b})^2a}{L_2} & \frac{(a\bar{b})a}{L_1} & a & x_n \end{pmatrix},$$

where

$$\begin{aligned} L_1^2 &= |a|^4 + |b|^2, \\ L_k^2 &= |a|^{2(k+1)}(L_1L_2L_3\dots L_{k-1})^2 + |b|^2,\ k = 2,3,4,\dots,n-4, \\ L(n)^2 &= |a|^{2(n-2)}(L_1L_2L_3\dots L_{n-4})^2 + |b|^4, \\ f_0 &= |a|^4, \\ f_k &= |a|^{2(k+2)}(L_1L_2L_3\dots L_{k-1})^2,\ k = 2,3,4,\dots,n-4. \end{aligned}$$

The bundles $S^3 \cdots \tilde{P}_n \xrightarrow{\tilde{p}_n} S^7$ are such that $\tilde{P}_2$ is the canonical S^3-bundle $Sp(2)$ over S^7 and for $n > 2$, $\tilde{P}_n = \{M \in Sp(n); M = M_n(a,b,x_1,\dots,x_n)$, for certain a, b, x_i in $\mathbb{H}\}$, $\tilde{p}_n(M_n(a,b,x_1,\dots,x_n)) = \binom{a}{b} \in S^7$ and S^3 acts on $\tilde{P}_n$ by multiplication on the right over the last column. We have the classification theorem.

Theorem 2 ([B2]). *If $\omega \in \pi_6(S^3)$ is the preferred generator then*

$$\tilde{P}_n \cong E_{\varphi(n-1)\omega},$$

where $\varphi(n) = \binom{n+1}{2} = \frac{(n+1)n}{2}$.

We calculate here the transition functions of the bundles $\tilde{P}_n$ over S^7 and of the bundles $E_{\alpha\beta}$ over E_α with respect to a certain open covering of S^7 and E_α respectively containing just 2 open sets.

Let $S^7 = \left\{\binom{a}{b} \in \mathbb{H}^2; a\bar{a} + b\bar{b} = 1\right\}$ be the seven sphere and U, V the open subsets defined by $U = \left\{\binom{a}{b} \in S^7; a \neq 0\right\}$, $V = \left\{\binom{a}{b} \in S^7; b \neq 0\right\}$. Thus for $n \geq 5$, $\tilde{p}_n^{-1}(U) = \{M_n(a, b, y_1, y_2, \dots, y_n) \in Sp(n); a \neq 0\}$ and $\tilde{p}_n^{-1}(V) = \{M_n(a, b, z_1, z_2, \dots, z_n) \in Sp(n); b \neq 0\}$, where

$$
\begin{aligned}
y_1 &= -(a\bar{b})|a|^{-2}y_2,\\
|y_2| &= |y_2(a,b)| = |a|^{n-1}(L_1L_2\dots L_{n-4})L(n)^{-1},\\
y_k &= -(a\bar{b})^{k-1}|a|^{-2(k-1)}(L_{n-(k+1)}L_{n-k}\dots L_{n-4})^{-2}y_2,\ 3 \leq k \leq n-2,\\
y_{n-1} &= -(a\bar{b})^{n-2}|a|^{-2(n-2)}(L_1L_2\dots L_{n-4})^{-2}y_2,\\
y_n &= -(a\bar{b})^{n-1}|a|^{-2(n-1)}(L_1L_2\dots L_{n-4})^{-2}y_2,\\
z_1 &= (b\bar{a})^{n-2}|b|^{-2(n-2)}(L_1L_2\dots L_{n-4})^2z_n,\\
z_2 &= -(b\bar{a})^{n-1}|b|^{-2(n-1)}(L_1L_2\dots L_{n-4})^2z_n,\\
z_k &= (b\bar{a})^{n-k}|b|^{-2(n-k)}(L_1L_2\dots L_{n-(k+2)})^2x_n,\ 3 \leq k \leq n-3,\\
z_{n-2} &= (b\bar{a})^2|b|^{-4}z_n,\\
z_{n-1} &= (b\bar{a})|b|^{-2}z_n,\\
|z_n| &= |z_n(a,b)| = |b|^{n-1}(L_1L_2\dots L_{n-4}L(n))^{-1}
\end{aligned}
$$

are obtained solving the equations $\overline{(\operatorname{col} n)} \cdot (\operatorname{col} i) = 0$ $(i = 1, 2, \dots, n-1)$ and using the fact that $a \neq 0$ and $b \neq 0$ in U and V respectively (cf. [R]).

We note that an element of $\tilde{p}_n^{-1}(U)$ depends only on the values of a, b and y_2, thus we can write $M_n(a, b, y_1, y_2, \dots, y_n) = M_n(a, b, y_2) \in \tilde{p}_n^{-1}(U)$. Similarly, $\tilde{p}_n^{-1}(V)$ depends only on the values of a, b and z_n, then $M_n(a, b, z_1, z_2, \dots, z_n) = M_n(a, b, z_n) \in \tilde{p}_n^{-1}(V)$.

We define the partial sections $S_U^n : U \longrightarrow \tilde{p}_n^{-1}(U)$ over U and $S_V^n : V \longrightarrow \tilde{p}_n^{-1}(V)$ over V by

$$
\begin{aligned}
S_U^n\begin{pmatrix} a\\ b\end{pmatrix} &= M_n(a, b, y_2), \qquad y_2 = \bar{a}^{n-1}(L_1L_2\dots L_{n-4})L(n)^{-1},\\
S_V^n\begin{pmatrix} a\\ b\end{pmatrix} &= M_n(a, b, z_n), \qquad z_n = -\bar{b}^{n-1}(L_1L_2\dots L_{n-4}L(n))^{-1}.
\end{aligned}
$$

Note that y_2 and z_n are restricted only by their modules, i.e., they belong to certain S^3's. Their values were chosen so that the transition function g_{VU}^n can be factored through the $S^3 \wedge S^3 = S^6$ (cf. §3).

If $g_{VU}^n : U \cap V \longrightarrow S^3$ is a transition function of the bundle $\tilde{P}_n$ with respect to the open sets U and V, then

$$
S_V^n\begin{pmatrix} a\\ b\end{pmatrix} g_{VU}^n\begin{pmatrix} a\\ b\end{pmatrix} = S_U^n\begin{pmatrix} a\\ b\end{pmatrix}, \quad \forall\begin{pmatrix} a\\ b\end{pmatrix} \in U \cap V. \tag{3}
$$

As the action of S^3 on $\tilde{P}_n$ is by multiplication from the right in the last column, we have

$$z_i g^n_{VU}\begin{pmatrix} a \\ b \end{pmatrix} = y_i, \;\; i = 1, 2, \ldots, n, \tag{4}$$

so, for example

$$\begin{aligned} & z_n g^n_{VU}\begin{pmatrix} a \\ b \end{pmatrix} \\ & = y_n = -(a\bar{b})^{n-1}|a|^{-2(n-1)}(L_1L_2\ldots L_{n-4})^{-2}\bar{a}^{n-1}(L_1L_2\ldots L_{n-4})L(n)^{-1}, \end{aligned}$$

hence

$$-\bar{b}^{n-1}(L_1L_2\ldots L_{n-4}L(n))^{-1}g^n_{VU}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{-(a\bar{b})^{n-1}\bar{a}^{n-1}}{(L_1L_2\ldots L_{n-4})|a|^{2(n-1)}L(n)},$$

thus

$$-b^{n-1}(-\bar{b}^{n-1})g^n_{VU}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{b^{n-1}(a\bar{b})^{n-1}\bar{a}^{n-1}}{|a|^{2(n-1)}},$$

therefore

$$g^n_{VU}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{b^{n-1}(a\bar{b})^{n-1}\bar{a}^{n-1}}{(|a||b|)^{2(n-1)}}. \tag{5}$$

Analogously we can define sections $S^k_U : U \longrightarrow \tilde{p}_k^{-1}(U)$ and $S^k_V : V \longrightarrow \tilde{p}_k^{-1}(V)$ $(k = 2, 3, 4)$ in such a way that $g^k_{VU}\binom{a}{b} = \frac{b^{k-1}(a\bar{b})^{k-1}\bar{a}^{k-1}}{(|a||b|)^{2(k-1)}}$ is a transition function of $\tilde{P}_k$ with respect to the open sets U and V. Thus we have

Lemma 1. *If $S^7 = \left\{\binom{a}{b} \in I\!\!H; a\bar{a} + b\bar{b} = 1\right\}$ is the 7-sphere and U, V are the open subsets $U = \left\{\binom{a}{b} \in S^7; a \neq 0\right\}$ and $V = \left\{\binom{a}{b} \in S^7; b \neq 0\right\}$ respectively, then a transition function of the bundle $S^3 \cdots \tilde{P}_n \longrightarrow S^7$ with respect to the open sets U and V is the function $g^n_{VU} : U \cap V \longrightarrow S^3$ given by (5) above.*

We now apply an analogous procedure to obtain transition functions of $E_{\alpha\beta}$.

Let $\tilde{P}_{n,m}$ be the principal S^3-bundle over $\tilde{P}_n$ induced from $\tilde{P}_m$ by the projection $\tilde{p}_n : \tilde{P}_n \longrightarrow S^7$. So, we have the commutative diagram:

$$\begin{array}{ccccc} S^3 & & S^3 & & S^3 \\ \vdots & & \vdots & & \vdots \\ \tilde{P}_{n,m} & \longrightarrow & \tilde{P}_m & \longrightarrow & Sp(2) \\ \downarrow & & \downarrow{\scriptstyle \tilde{p}_m} & & \downarrow \\ \tilde{P}_n & \xrightarrow{\tilde{p}_n} & S^7 & \xrightarrow{\varphi(m-1)\iota_7} & S^7. \end{array}$$

diagram 2

By the definition of the induced bundle we have $\tilde{P}_{n,m} = \{(M_n, M_m) \in \tilde{P}_n \times \tilde{P}_m; \tilde{p}_n(M_n) = \tilde{p}_m(M_m)\}$. This provides a model for $E_{\varphi(n-1)\omega,\varphi(m-1)\omega}$ where $\omega \in \pi_6(S^3)$ is the preferred generator.

Let us consider the open sets $\tilde{U}_n = \tilde{p}_n^{-1}(U) = \{M_n(a,b,y_2) \in \tilde{P}_n; a \neq 0\}$ and $\tilde{V}_n = \tilde{p}_n^{-1}(V) = \{M_n(a,b,z_n) \in \tilde{P}_n; b \neq 0\}$. For each $k, t \in \mathbb{Z}$ such that $\frac{m-tkn+tk-1}{t}$ is an integer, let us define partial sections $s^{kt}_{\tilde{U}_n} : \tilde{U}_n \longrightarrow \tilde{P}_{n,m}$ over $\tilde{U}_n$ and $s^{kt}_{\tilde{V}_n} : \tilde{V}_n \longrightarrow \tilde{P}_{n,m}$ over $\tilde{V}_n$ by

$$s^{kt}_{\tilde{U}_n}(M_n(a,b,y_2)) = (M_n(a,b,y_2), M_m(a,b,Y_2(k,t))),$$
$$s^{kt}_{\tilde{V}_n}(M_n(a,b,z_n)) = (M_n(a,b,z_n), M_m(a,b,Z_m(k,t))),$$

where

$$Y_2(k,t) = (y_2^k \bar{a}^{\frac{m-tkn+tk-1}{t}})^t \frac{L_1L_2\ldots L_{m-4}L(n)^{tk}}{(L_1L_2\ldots L_{n-4})^{tk}L(m)},$$
$$Z_m(k,t) = (-1)^{tk+1}(z_n^k \bar{b}^{\frac{m-tkn+tk-1}{t}})^t \frac{(L_1L_2\ldots L_{n-4}L(n))^{kt}}{L_1L_2\ldots L_{m-4}L(m)}.$$

By using the expression of y_n on page 77, if $M_n(a,b,y_2) = M_n(a,b,z_n)$ over $\tilde{U}_n \cap \tilde{V}_n$ then

$$(6) \quad z_n = \frac{-(a\bar{b})^{n-1}y_2}{|a|^{2(n-1)}(L_1L_2\ldots L_{n-4})^2} \Longrightarrow z_n^k = \frac{(-1)^k((a\bar{b})^{n-1}y_2)^k}{|a|^{2k(n-1)}(L_1L_2\ldots L_{n-4})^{2k}}.$$

A transition function $g_{n,m,k,t} : \tilde{U}_n \cap \tilde{V}_n \longrightarrow S^3$ of $\tilde{P}_{n,m}$ with respect to the open sets $\tilde{U}_n$ and $\tilde{V}_n$ can be given solving the equation

$$(7) \quad s^{kt}_{\tilde{V}_n}(M).g_{n,m,k,t}(M) = s^{kt}_{\tilde{U}_n}(M), \quad \forall M \in \tilde{U}_n \cap \tilde{V}_n.$$

It follows from the expressions on page 77 that

$$(8) \quad Y_m(k,t) = \frac{-(a\bar{b})^{m-1}(y_2^k \bar{a}^{\frac{m-tkn+tk-1}{t}})^t L(n)^{tk}}{|a|^{2(m-1)}(L_1L_2\ldots L_{m-4})(L_1L_2\ldots L_{n-4})^{tk}L(m)}.$$

Setting $M_n = M_n(a,b,y_2)$ it follows from (7) that

$$(9) \quad Z_m(k,t)g_{n,m,k,t}(M_n) = Y_m(k,t),$$

hence

$$(-1)^{tk+1}(z_n^k \bar{b}^{\frac{m-tkn+tk-1}{t}})^t \frac{(L_1L_2\ldots L_{n-4}L(n))^{kt}}{L_1L_2\ldots L_{m-4}L(m)}.g_{n,m,k,t}(M_n)$$
$$= \frac{-(a\bar{b})^{m-1}(y_2^k \bar{a}^{\frac{m-tkn+tk-1}{t}})^t L(n)^{tk}}{|a|^{2(m-1)}(L_1L_2\ldots L_{m-4})(L_1L_2\ldots L_{n-4})^{tk}L(m)},$$

using (6) we obtain

$$\frac{(-1)^{tk+1}((-1)^k((a\bar{b})^{n-1}y_2)^k\bar{b}^{\frac{m-tkn+tk-1}{t}})^t}{(|a|^{2k(m-1)}(L_1L_2\dots L_{n-4})^{2k})^t}g_{n,m,k,t}(M_n)$$
$$=\frac{-(a\bar{b})^{m-1}(y_2^k\bar{a}^{\frac{m-tkn+tk-1}{t}})^t}{|a|^{2(m-1)}(L_1L_2\dots L_{n-4})^{2tk}},$$

thus, setting $p=\frac{m-ktn+tk-1}{t}$ we have

$$\frac{(-1)^{2kt+1}(|a||b|)^{2kt(n-1)}|y_2|^{2kt}|b|^{2pt}}{|a|^{2kt(n-1)}}g_{n,m,k,t}(M_n)$$
$$=\frac{-(b^p(\bar{y}_2(b\bar{a})^{n-1})^k)^t(a\bar{b})^{m-1}(y_2^k\bar{a}^p)^t}{|a|^{2(m-1)}},$$

therefore

$$g_{n,m,k,t}(M_n)=\frac{(b^{\frac{m-tkn+tk-1}{t}}(\bar{y}_2(b\bar{a})^{n-1})^k)^t(a\bar{b})^{m-1}(y_2^k\bar{a}^{\frac{m-tkn+tk-1}{t}})^t}{|ab|^{2(m-1)}|y_2|^{2tk}}. \tag{10}$$

Evidently, from the definition of the induced bundle we have that

$$\tilde{g}_{n,m}(M_n(a,b,y_2))=g^m_{VU}\circ\tilde{p}_n(M_n(a,b,y_2))=\frac{b^{m-1}(a\bar{b})^{m-1}\bar{a}^{m-1}}{(|a||b|)^{2(m-1)}} \tag{11}$$

is also a transition function of $\tilde{P}_{n,m}$, which coincides with $g_{n,m,0,1}$.

We have in this way the following:

Lemma 2. *Let n, m be integers greater than 1. If $\tilde{U}_n=\{M_n(a,b,y_2)\in\tilde{P}_n; a\neq 0\}$, $\tilde{V}_n=\{M_n(a,b,z_n)\in\tilde{P}_n; b\neq 0\}$ then for all $k,t\in\mathbb{Z}$ such that $\frac{m-tkn+tk-1}{t}\in\mathbb{Z}$ the functions $g_{n,m,k,t}:\tilde{U}_n\cap\tilde{V}_n\longrightarrow S^3$ given by*

$$g_{n,m,k,t}(M_n(a,b,y_2))=\frac{(b^{\frac{m-tkn+tk-1}{t}}(\bar{y}_2(b\bar{a})^{n-1})^k)^t(a\bar{b})^{m-1}(y_2^k\bar{a}^{\frac{m-tkn+tk-1}{t}})^t}{|ab|^{2(m-1)}|y_2|^{2tk}}$$

are equivalent transition functions (in the sense of equivalent coordinate transformation, Lemma 2.10 from Steenrod [St]*) of the bundle $S^3\cdots\tilde{P}_{n,m}\longrightarrow\tilde{P}_n$.*

3. Trivialization of $\tilde{P}_9$

Here we construct a global section of $\tilde{P}_9$ up to a homotopy. To do this we use some elementary results from the theory of nilpotent groups, which we list below, and finally we show how this method can be applied to construct a diffeomorphism between $Sp(2)\times S^3$ and $E_{7\omega}\times S^3$ up to homotopies of powers of commutators.

Let Γ be a multiplicative group, given elements x and y in Γ we recall that (cf. [Ha]) the **commutator** of x and y is defined by $[x,y]=x^{-1}y^{-1}xy$ and for x,y and $z\in\Gamma$ we have the following properties:

p$_1$: $[x,y]^{-1}=[y,x]$,
p$_2$: $[xy,z]=[x,z][[x,z],y][y,z]$,
p$_3$: $[x,yz]=[x,z][x,y][[x,y],z]$,
p$_4$: $xy=yx[x,y]$,
p$_5$: $xy=[x^{-1},y^{-1}]yx$.

Given subsets X and Y of a group Γ, we define $[X,Y]$ as the subgroup of Γ generated by every $[x,y]\in\Gamma$ such that $x\in X$ and $y\in Y$.

A group Γ is called **nilpotent of class $\leq$ *r*** if there are subgroups $\Gamma_0,\Gamma_1,\ldots,\Gamma_r$ of Γ such that

(12) $$\Gamma=\Gamma_0\supseteq\Gamma_1\supseteq\Gamma_2\supseteq\cdots\supseteq\Gamma_r=\{1\},$$

(13) $$\Gamma_i \text{ is a normal subgroup of } \Gamma,\ i=1,2,\ldots,r,$$

(14) $$[\Gamma_i,\Gamma]\subseteq\Gamma_{i+1},\ i=0,1,2,\ldots,r-1.$$

The series of subgroups (12) satisfying (13) and (14) is called **central series** or **central chain**. Observe that (14) is equivalent to $\frac{\Gamma_i}{\Gamma_{i+1}}\subseteq\text{center}(\frac{G}{\Gamma_{i+1}})$, in particular

(15) $$\Gamma \text{ is nilpotent of class } \leq r \Longrightarrow \Gamma_{r-1}\subseteq\text{center}(\Gamma).$$

If $G2$ and $G3$ are nilpotent groups of classes ≤ 2 and ≤ 3 respectively with central chains $G2=\Omega_0\supseteq\Omega_1\supseteq\Omega_2=\{1\}$ and $G3=\Gamma_0\supseteq\Gamma_1\supseteq\Gamma_2\supseteq\Gamma_3=\{1\}$, then the following formulas are easily obtained from properties p$_1$, …,p$_5$:

N1: $[x,y]=[y^{-1},x]=[y,x^{-1}]=[x^{-1},y^{-1}]\quad\forall x,y\in G2$,
N2: $[x,y]^n=[x,y^n]=[x^n,y]\quad\forall x,y\in G2$ and $\forall n\in\mathbb{Z}$,
N3: $(xy)^n=y^n[x,y]^{\varphi(n)}x^n\quad\forall x,y\in G2$, where $\varphi(n)=\binom{n+1}{2}=\frac{n(n+1)}{2}$,
N4: $[x,y^n]=[x,y]^n[[x,y],y]^{\varphi(n-1)}\quad\forall x,y\in G3$ and $n\in\mathbb{N}$,
N5: $[x^n,y]=[x,y]^n[x,[y,x]]^{\varphi(n-1)}\quad\forall x,y\in G3$ and $n\in\mathbb{N}$,
N6: Given $x,y\in G3$ and $n\in\mathbb{Z}$ we have
i) if $n\geq 0$
$(xy)^n=y^n[x^{-1},[y,x]]^{a(n)}[[x,y],y]^{a(n-1)}[x,y]^{\varphi(n)}x^n$,
ii) if $n<0$
$(xy)^n=y^n[y^{-1},[x^{-1},y^{-1}]]^{a(n)}[[x^{-1},y^{-1}],x^{-1}]^{a(n-1)}[x^{-1},y^{-1}]^{\varphi(n)}x^n$,
where $a(n)=\binom{n+2}{3}=\frac{(n+2)(n+1)n}{6}$,
N7: $[[x,y],z][[y,z],x][[z,x],y]=1\quad\forall x,y,z\in G3$.

Remark 1. The function φ in N4, N5 and N6 is the same as that of N3.

To trivialize $\tilde{P}_9$ let us consider the diffeomorphisms $\alpha : U\cap V \longrightarrow S^3\times S^3\times (0,\frac{\pi}{2})$ and $\beta : S^3\times S^3\times(0,\frac{\pi}{2}) \longrightarrow U\cap V$ given by $\alpha\binom{a}{b} = (\frac{a}{|a|},\frac{b}{|b|},\cos^{-1}|a|)$ and $\beta(A,B,\theta) = \binom{\cos\theta A}{\sin\theta B}$, which are mutual inverses (cf. [R]).

Let $S^9_U : U \longrightarrow \tilde{p}_9^{-1}(U)$ and $S^9_V : V \longrightarrow \tilde{p}_9^{-1}(V)$ be the sections given in page 77 above (setting $n = 9$), that is,

$$S^9_U\binom{a}{b} = M_9(a,b,\bar{a}^8(L_1L_2L_3L_4L_5)L(9)^{-1}),$$
$$S^9_V\binom{a}{b} = M_9(a,b,-\bar{b}^8(L_1L_2L_3L_4L_5L(9))^{-1}).$$

$\tilde{P}_9$ has as transition function $g^9_{VU} : U\cap V \longrightarrow S^3$, $g^9_{VU}\binom{a}{b} = \frac{b^8(a\bar{b})^8\bar{a}^8}{(|a||b|)^{16}}$.

We have that $g^9_{VU}\circ\beta(A,B,\theta) = B^8(A\bar{B})^8\bar{A}^8$. As $S^3\times S^3\times(0,\frac{\pi}{2})\simeq S^3\times S^3$ it follows that $[S^3\times S^3\times(0,\frac{\pi}{2}),S^3]\cong[S^3\times S^3,S^3]$, where $[X,Y]$ denotes the set of homotopy classes of maps from X to Y.

For arbitrary positive integers n_i $(i = 1,2,\ldots,k)$ consider the group $\Gamma = [S^{n_1}\times S^{n_2}\times\cdots\times S^{n_k},G]$ where S^{n_i} is the n_i-dimensional sphere and G is a topological group. For $i = 1,2,\ldots,k$ let P_i be the set of all points in $S^{n_1}\times\cdots\times S^{n_k}$ with at least $k-i$ coordinates equal to the base point and Γ_i be the group of homotopy classes of maps $f\in\Gamma$ such that $f|_{P_i}$ is nullhomotopic, then we have

Theorem 3 (G. W. Whitehead [W1]). *The group $[S^{n_1}\times S^{n_2}\times\cdots\times S^{n_k},G]$ has the central chain $\Gamma = \Gamma_0\supseteq\Gamma_1\supseteq\Gamma_2\supseteq\cdots\supseteq\Gamma_k = \{1\}$ and*

$$\frac{\Gamma_{i-1}}{\Gamma_i}\cong\prod_{|I|=i}\pi_{n(I)}(G),$$

where $I\subseteq\{1,2,3,\ldots,k\}$, $|I|$ = cardinality of I and $n(I) = \sum_{i\in I}n_i$.

If $\Gamma = [S^3\times S^3,S^3]$, then $G = S^3$, $k = 2$, $n_1 = n_2 = 3$ and by the result above Γ is nilpotent of class ≤ 2 and has a central chain $\Gamma = \Gamma_0\supseteq\Gamma_1\supseteq\Gamma_2 = \{1\}$ such that

$$\frac{\Gamma_0}{\Gamma_1}\cong\mathbb{Z}\oplus\mathbb{Z}\quad\text{and}\quad\frac{\Gamma_1}{\Gamma_2}\cong\Gamma_1\cong\mathbb{Z}_{12}.$$

Remark 2. Recently M. Mimura and H. Ōshima (cf. [MO]) described the group structure of $[S^n\times S^m,S^k,\mu_r]$ for $m,n,k\in\{1,3,7\}$ and of $[E_\alpha,E_\beta,\mu_0^{(r)}]$ for $\alpha,\beta\in\pi_6(S^3)$, where $\mu_r(x,y) = xy[x,y]^r$ and $\mu_0(x,y) = xy$ are the usual complex, quaternionic or Cayley multiplications and $\mu_0^{(r)}$ is defined similarly for convenient multiplications $\mu_0^{(0)}$. Thus, for example $[S^3\times S^3,S^3,\mu_0]$ is the group generated by the projections $p_1,p_2 : S^3\times S^3\longrightarrow S^3$ and with relations $p_1[p_1,p_2] = [p_1,p_2]p_1$, $p_2[p_1,p_2] = [p_1,p_2]p_2$, $[p_1,p_2]^{12} = 1$.

Given $f : S^3 \times S^3 \longrightarrow S^3$ we denote by $\bar{f} : S^3 \times S^3 \longrightarrow S^3$ the map $\bar{f}(x,y) = \overline{f(x,y)}$ (quaternionic conjugation of f) and as in the remark above $p_i : S^3 \times S^3 \longrightarrow S^3$ $(i = 1,2)$ are the projections.

We know that, if $f, g \in \Gamma = [S^3 \times S^3, S^3]$, then $f.g$ is the homotopy class of the product of f by g in S^3 with this we observe that $\bar{p}_i = p_i^{-1}$ in Γ $(i = 1,2)$ and from N3 we have that $(p_1\bar{p}_2)^n = \bar{p}_2^n[\bar{p}_2, \bar{p}_1]^{\varphi(n)}p_1^n$ so $p_2^n(p_1\bar{p}_2)^n\bar{p}_1^n = [\bar{p}_2, \bar{p}_1]^{\varphi(n)}$ in Γ.

We observe that $p_2^{n-1}(p_1\bar{p}_2)^{n-1}\bar{p}_1^{n-1} = g_{VU}^n \circ \beta$, and so we conclude

$$g_{VU}^n \circ \beta \simeq [\bar{p}_2, \bar{p}_1]^{\varphi(n-1)}. \tag{16}$$

Thus, $g_{VU}^9 \circ \beta \simeq [\bar{p}_2, \bar{p}_1]^{36}$, as $[\bar{p}_2, \bar{p}_1] \in \Gamma_1 \cong \mathbb{Z}_{12}$ it follows that $g_{VU}^9 \circ \beta \simeq 1$.

Let $F : S^3 \times S^3 \times [0, \frac{\pi}{2}] \longrightarrow S^3$ be a smooth homotopy such that

$$F(A,B,\theta) = \begin{cases} B^8(A\bar{B})^8\bar{A}^8 & \text{if } \theta \in [0, \frac{\pi}{6}] \\ 1 & \text{if } \theta \in [\frac{\pi}{3}, \frac{\pi}{2}], \end{cases}$$

then $S : S^7 \longrightarrow \tilde{P}_9$ given by

$$S\begin{pmatrix} a \\ b \end{pmatrix} = \begin{cases} S_V^9\binom{a}{b} & \text{if } \frac{5\pi}{12} \le \cos^{-1}|a| \le \frac{\pi}{2} \\ S_V^9\binom{a}{b}(F\circ\alpha)\binom{a}{b} & \text{if } \frac{\pi}{12} \le \cos^{-1}|a| \le \frac{5\pi}{12} \\ S_U^9\binom{a}{b} & \text{if } 0 \le \cos^{-1}|a| \le \frac{\pi}{12} \end{cases}$$

is a global section of $\tilde{P}_9$.

A diffeomorphism $\Phi : S^7 \times S^3 \longrightarrow \tilde{P}_9$ is given by

$$\Phi\left(\begin{pmatrix} a \\ b \end{pmatrix}, \nu\right) = M_9(a, b, w_1\nu, w_2\nu, w_3\nu, w_4\nu, w_5\nu, w_6\nu, w_7\nu, w_8\nu, w_9\nu)$$

where

$$w_1 = \begin{cases} -(b\bar{a})^7\bar{b}^8|b|^{-14}(L_1L_2L_3L_4L_5)L(9)^{-1} & \text{if } \frac{5\pi}{12} \le \cos^{-1}|a| \le \frac{\pi}{2} \\ -(b\bar{a})^7\bar{b}^8|b|^{-14}(L_1L_2L_3L_4L_5)L(9)^{-1}(F\circ\alpha)\binom{a}{b} & \text{if } \frac{\pi}{12} \le \cos^{-1}|a| \le \frac{5\pi}{12} \\ -(a\bar{b})\bar{a}^8|a|^{-2}(L_1L_2L_3L_4L_5)L(9)^{-1} & \text{if } 0 \le \cos^{-1}|a| \le \frac{\pi}{12}, \end{cases}$$

$$w_2 = \begin{cases} (b\bar{a})^8\bar{b}^8|b|^{-16}(L_1L_2L_3L_4L_5)L(9)^{-1} & \text{if } \frac{5\pi}{12} \le \cos^{-1}|a| \le \frac{\pi}{2} \\ (b\bar{a})^8\bar{b}^8|b|^{-16}(L_1L_2L_3L_4L_5)L(9)^{-1}(F\circ\alpha)\binom{a}{b} & \text{if } \frac{\pi}{12} \le \cos^{-1}|a| \le \frac{5\pi}{12} \\ \bar{a}^8(L_1L_2L_3L_4L_5)L(9)^{-1} & \text{if } 0 \le \cos^{-1}|a| \le \frac{\pi}{12}, \end{cases}$$

$$w_k = \begin{cases} -(b\bar{a})^{9-k}\bar{b}^8|b|^{-2(9-k)}\frac{(L_1\ldots L_{9-(k+2)})}{L_{9-(k+1)}\ldots L_5L(9)} & \text{if } \frac{5\pi}{12} \le \cos^{-1}|a| \le \frac{\pi}{2} \\ -(b\bar{a})^{9-k}\bar{b}^8|b|^{-2(9-k)}\frac{(L_1\ldots L_{9-(k+2)})}{L_{9-(k+1)}\ldots L_5L(9)}(F\circ\alpha)\binom{a}{b} & \text{if } \frac{\pi}{12} \le \cos^{-1}|a| \le \frac{5\pi}{12} \\ -(a\bar{b})^{k-1}\bar{a}^8|a|^{-2(k-1)}\frac{(L_1\ldots L_{9-(k+2)})}{L_{9-(k+1)}\ldots L_5L(9)} & \text{if } 0 \le \cos^{-1}|a| \le \frac{\pi}{12} \end{cases}$$

for $3 \leq k \leq 6$,

$$w_7 = \begin{cases} -(b\bar{a})^2\bar{b}^8|b|^{-4}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ -(b\bar{a})^2\bar{b}^8|b|^{-4}(L_1L_2L_3L_4L_5L(9))^{-1}(F\circ\alpha)\binom{a}{b} & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})^6\bar{a}^8|a|^{-12}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

$$w_8 = \begin{cases} -(b\bar{a})\bar{b}^8|b|^{-2}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ -(b\bar{a})\bar{b}^8|b|^{-2}(L_1L_2L_3L_4L_5L(9))^{-1}(F\circ\alpha)\binom{a}{b} & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})^7\bar{a}^8|a|^{-14}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

$$w_9 = \begin{cases} -\bar{b}^8(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ -\bar{b}^8(L_1L_2L_3L_4L_5L(9))^{-1}(F\circ\alpha)\binom{a}{b} & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})^8\bar{a}^8|a|^{-16}(L_1L_2L_3L_4L_5L(9))^{-1} & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}. \end{cases}$$

The reasoning employed above shows us that if the transition function g^n_{VU} is null-homotopic then the bundle $\tilde{P}_n$ is trivial.

Let us consider the following commutative diagram:

$$\begin{array}{ccc} S^3 \times S^3 & \xrightarrow{[\bar{p}_2,\bar{p}_1]} & S^3 \\ \wedge\big\downarrow & & \big\uparrow\omega \\ S^6 = S^3 \wedge S^3 & \xrightarrow{\text{id}} & S^3 \wedge S^3, \end{array}$$

diagram 3

that is, ω is defined here by $\omega(A\wedge B) = [\bar{B}, \bar{A}] = BA\bar{B}\bar{A}$.

Remark 3. It is well known that ω defined above is a generator of $\pi_6(S^3)$ (cf. [J], [Mc] or remark 2). With the aid of this fact and (16) above we note that the transition functions $g^n_{VU} : U \cap V \longrightarrow S^3$, $g^n_{VU}\binom{a}{b} = \frac{b^{n-1}(a\bar{b})^{n-1}\bar{a}^{n-1}}{(|a||b|)^{2(n-1)}}$ of $\tilde{P}_n$ are such that $g^n_{VU}\circ\beta : S^3 \times S^3 \longrightarrow S^3$ all factor through $S^3 \wedge S^3$ where β is the diffeomorphism between $S^3 \times S^3 \times (0, \frac{\pi}{2})$ and $U \cap V$ given above, that is, there exists $\omega_n : S^3 \wedge S^3 \longrightarrow S^3$ such that $g^n_{UV}\circ\beta = \omega_n \circ \wedge$. Moreover if the equivalence class of the transition function g^n_{VU} classifies the bundle ξ then the homotopy class of ω_n in $\pi_6(S^3)$ also classifies the same bundle ξ. We have thus the following homotopy-commutative diagram:

$$\begin{array}{ccc} S^3 \times S^3 & \xrightarrow{g^n_{VU}\circ\delta} & S^3 \\ \big\downarrow\wedge & & \big\| \\ S^3 \wedge S^3 & \xrightarrow{\omega_n} & S^3, \end{array}$$

diagram 4

where $\omega_n(A\wedge B) \simeq [\bar{B}, \bar{A}]^{\varphi(n-1)}$.

Remark 4. Recently Carlos E. Duran [D] has exhibited the following a priori smooth formula for the Blakers-Massey element $\omega : S^6 \subseteq \mathrm{Im}(\mathbb{H}) \oplus \mathbb{H} \longrightarrow S^3$,

$$\omega\begin{pmatrix} p \\ u \end{pmatrix} = \begin{cases} \frac{\bar{u}}{|u|} \exp(\pi p) \frac{u}{|u|} & \text{if } u \neq 0 \\ -1 & \text{if } u = 0, \end{cases}$$

where $\exp(\theta p) = \cos(\theta|p|) + \sin(\theta|p|)\frac{p}{|p|}$.

4. The Diffeomorphism $Sp(2) \times S^3 = E_{7\omega} \times S^3$

Now we try writing explicitly (in terms of transition functions) a diffeomorphism $Sp(2) \times S^3 = E_{7\omega} \times S^3$ following the same steps as above.

Lemma 3. *There exists a diffeomorphism* $\delta_n : S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2}) \longrightarrow \tilde{\mathbf{U}}_\mathbf{n} \cap \tilde{\mathbf{V}}_\mathbf{n}$ *for all* $n \in \mathbb{N}$, $n \geq 2$.

Proof. We define δ_n as follows:

$$\delta_n(A, B, C, \theta) = M_n(\cos\theta A, \sin\theta B, (\cos^{n-1}\theta)(l_1 l_2 \ldots l_{n-4}) l(n)^{-1} C),$$

where $l_k = L_k$ changing a by $\cos\theta A$, b by $\sin\theta B$ $(1 \leq k \leq n-4)$ and $l(n) = L(n)$ making the same changes. As examples we have

$$\delta_2(A,B,C,\theta) = \begin{pmatrix} \cos\theta A & -\sin\theta A\bar{B}C \\ \sin\theta B & \cos\theta C \end{pmatrix},$$

$$\delta_3(A,B,C,\theta) = \begin{pmatrix} \cos\theta A & -\sin^3\theta B & -\cos\theta\sin\theta\sqrt{1+\sin^2\theta}A\bar{B}C \\ \sin\theta B & \cos\theta\sin^2\theta B\bar{A}B & \cos^2\theta\sqrt{1+\sin^2\theta}C \\ 0 & \cos\theta\sqrt{1+\sin^2\theta}A & -\cos^{-1}\theta\sin\theta A\bar{B}C \end{pmatrix},$$

$$\delta_4(A,B,C,\theta) = \begin{pmatrix} \cos\theta A & -\sin^3\theta B l^{-1} & 0 & -\cos^2\theta\sin\theta A\bar{B}Cl^{-1} \\ \sin\theta B & \sin^2\theta\cos\theta B\bar{A}Bl^{-1} & 0 & \cos^3\theta C l^{-1} \\ 0 & \cos^3\theta A l^{-1} & -\sin\theta B & -\cos\theta\sin^2\theta(A\bar{B})^2 C l^{-1} \\ 0 & \cos^2\theta\sin\theta A\bar{B}Al^{-1} & \cos\theta A & -\sin^3\theta(A\bar{B})^3 C l^{-1} \end{pmatrix},$$

where $l = l(4) = \sqrt{\sin^4\theta + \cos^4\theta}$.

We can easily verify then

$$\delta_n^{-1}(M_n(a,b,y_2)) = \left(\frac{a}{|a|}, \frac{b}{|b|}, \frac{y_2}{|y_2|}, \cos^{-1}|a|\right)$$

and with this the Lemma is proved. □

Lemma 4. *Let $f : S^3 \times S^3 \times S^3 \longrightarrow S^3$ and $p : S^3 \times S^3 \times S^3 \longrightarrow S^3 \times S^3$ be continuous functions.*

i) *If $f = c_2 \circ p$, where $c_2(x, y) = [x, y] = x^{-1}y^{-1}xy$, then the order of f in $[S^3 \times S^3 \times S^3, S^3]$ is a divisor of 12,*

ii) *If $f = c_3 \circ p$, where $c_3(x, y) = [x, [x, y]]$ or $[y, [x, y]]$, then $f \simeq 1$.*

Proof. It is suffices to observe that, by Theorem 3 applied to the group $[S^3 \times S^3, S^3]$ we have that c_2 represents a homotopy class of order 12, and $c_3 \simeq 1$. □

With this Lemma, we note that for $x, y \in G_3 = [S^3 \times S^3 \times S^3, S^3]$ we have

$$[x, y] = x^{-1}y^{-1}xy = x^{-1}xy^{-1}[y^{-1}, x]y = y^{-1}y[y^{-1}, x][[y^{-1}, x], y] = [y^{-1}, x].$$

In a similar manner as in N1 we obtain

Lemma 5. *Given $x, y \in G_3 = [S^3 \times S^3 \times S^3, S^3]$, then*

$$[x, y] = [y^{-1}, x] = [y, x^{-1}] = [x^{-1}, y^{-1}].$$

With the aid of the last two lemmas we observe that N4, N5 and N6 applied to the group $G_3 = [S^3 \times S^3 \times S^3, S^3]$ transform to Lemma 6 and Theorem 4 bellow.

Lemma 6. *Given $x, y \in G_3 = [S^3 \times S^3 \times S^3, S^3]$, then*

$$[x, y]^n = [x^n, y] = [x, y^n], \ n \in \mathbb{Z}.$$

Proof. For $n \geq 0$ the result follows directly from N1 and N5. If $n < 0$ we have $[x, y]^n = [y, x]^{-n} = [y^{-n}, x] = [y, x^{-n}]$, but $[y^{-n}, x] = [(y^n)^{-1}, x] = [x, y^n]$ and $[y, x^{-n}] = [y, (x^n)^{-1}] = [x^n, y]$, hence the result follows. □

Theorem 4. *Given $x, y \in G_3 = [S^3 \times S^3 \times S^3, S^3]$, then*

$$(xy)^n = y^n[x, y]^{\varphi(n)}x^n, \ n \in \mathbb{Z}.$$

We observe that $[S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2}), S^3] = [S^3 \times S^3 \times S^3, S^3]$. It follows from Theorem 3 that $\Gamma = [S^3 \times S^3 \times S^3, S^3]$ is a nilpotent group of class ≤ 3 with central chain $\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 = \{1\}$ such that

$$\frac{\Gamma_0}{\Gamma_1} \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad \frac{\Gamma_1}{\Gamma_2} \cong \mathbb{Z}_{12} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{12}, \quad \frac{\Gamma_2}{\Gamma_3} = \Gamma_2 \cong \mathbb{Z}_3.$$

We denote by A, B, $C : S^3 \times S^3 \times S^3 \times (0, \frac{\pi}{2}) \longrightarrow S^3$ the projections $A(x_1, x_2, x_3, \theta) = x_1$, $B(x_1, x_2, x_3, \theta) = x_2$, $C(x_1, x_2, x_3, \theta) = x_3$, and by $\bar{A}$, $\bar{B}$, $\bar{C}$ the quaternionic conjugates of A, B, C respectively. To simplify the notation we set $p = \frac{m-tkn+tk-1}{t}$ and using the results above we obtain

Theorem 5. *The transition function $g_{n,m,k,t}\circ\delta_n$ can be written as a product of simple and double commutators, namely,*

$$g_{n,m,k,t}\circ\delta_n \simeq [C,B]^{e_1}[B,A]^{e_2}[A,C]^{e_1}[[B,C],A]^f \text{ for certain } e_1,e_2,f\in\mathbb{Z}.$$

Proof.

$$\begin{aligned}
&g_{n,m,k,t}\circ\delta_n\\
&= (B^p(\bar C(B\bar A)^{n-1})^k)^t(A\bar B)^{m-1}(C^k\bar A^p)^t\\
&\underset{(\text{T. }4)}{\simeq} (\bar C(B\bar A)^{n-1})^{tk}[B^p,(\bar C(B\bar A)^{n-1})^k]^{\varphi(t)}B^{pt}(A\bar B)^{m-1}\bar A^{pt}\\
&\quad [C^k,\bar A^p]^{\varphi(t)}C^{tk}\\
&\underset{(\text{L. }4)}{\simeq} [B^p,(\bar C(B\bar A)^{n-1})^k]^{\varphi(t)}(\bar C(B\bar A)^{n-1})^{tk}B^{pt}(A\bar B)^{m-1}\bar A^{pt}\\
&\quad [C^k,\bar A^p]^{\varphi(t)}C^{kt}\\
&\underset{(\text{T. 4 and L. }6)}{\simeq} [B,\bar C(B\bar A)^{n-1}]^{k\varphi(t)p}(B\bar A)^{tk(n-1)}[\bar C,(B\bar A)^{n-1}]^{\varphi(tk)}\bar C^{tk}B^{pt}\\
&\quad (A\bar B)^{m-1}\bar A^{pt}[C,\bar A]^{k\varphi(t)p}C^{tk}\\
&\underset{(\text{T. }4)}{\simeq} [B,\bar C(B\bar A)^{n-1}]^{k\varphi(t)p}[\bar C,(B\bar A)^{n-1}]^{\varphi(tk)}(B\bar A)^{tk(n-1)}\bar C^{tk}B^{pt}\\
&\quad (A\bar B)^{m-1}\bar A^{pt}[C,\bar A]^{k\varphi(t)p}C^{tk}\\
&\underset{(\text{p}_4)}{\simeq} [B,\bar C(B\bar A)^{n-1}]^{k\varphi(t)p}[\bar C,(B\bar A)^{n-1}]^{\varphi(tk)}\bar C^{tk}(B\bar A)^{tk(n-1)}\\
&\quad [(B\bar A)^{tk(n-1)},\bar C^{tk}]B^{pt}(A\bar B)^{m-1}\bar A^{pt}[C,\bar A]^{k\varphi(t)p}C^{tk}\\
&\underset{(\text{T. }4)}{\simeq} [B,\bar C(B\bar A)^{n-1}]^{k\varphi(t)p}[\bar C,(B\bar A)^{n-1}]^{\varphi(tk)}\bar C^{tk}[(B\bar A)^{tk(n-1)},\bar C^{tk}]\\
&\quad (B\bar A)^{tk(n-1)}B^{pt}(A\bar B)^{m-1}\bar A^{pt}[C,\bar A]^{k\varphi(t)p}C^{tk}\\
&\underset{(\text{p}_3,\,\text{p}_4\text{ and L. }6)}{\simeq} ([B,(B\bar A)^{n-1}][B,\bar C][[B,\bar C],(B\bar A)^{n-1}])^{k\varphi(t)p}[\bar C,B\bar A]^{\varphi(tk)(n-1)}\\
&\quad \bar C^{tk}[B\bar A,\bar C]^{(tk)^2(n-1)}B^{pt}(B\bar A)^{tk(n-1)}[(B\bar A)^{tk(n-1)},B^{pt}]\\
&\quad (A\bar B)^{m-1}\bar A^{pt}[C,\bar A]^{k\varphi(t)p}C^{tk}\\
&\underset{(15\text{ and T. }4)}{\simeq} [B,B\bar A]^{(n-1)k\varphi(t)p}[B,\bar C]^{k\varphi(t)p}[[B,\bar C],B\bar A]^{(n-1)k\varphi(t)p}\\
&\quad [\bar C,B\bar A]^{\varphi(tk)(n-1)}\bar C^{tk}[B\bar A,\bar C]^{(tk)^2(n-1)}[(B\bar A)^{tk(n-1)},B^{pt}]\\
&\quad B^{pt}(A\bar B)^{pt}\bar A^{pt}[C,\bar A]^{k\varphi(t)p}C^{tk}\\
&\underset{(\text{p}_2,\,\text{p}_3\text{ and T. }4)}{\simeq} [B,\bar A]^{(n-1)k\varphi(t)p}[B,\bar C]^{k\varphi(t)p}[[B,\bar C],\bar A]^{(n-1)k\varphi(t)p}\\
&\quad [B\bar A,\bar C]^{\varphi(tk-1)(n-1)}\bar C^{tk}[\bar A,B]^{tk(n-1)pt}[A,\bar B]^{\varphi(pt)}[C,\bar A]^{k\varphi(t)p}C^{tk}\\
&\underset{(\text{p}_2\text{ and p}_4)}{\simeq} [B,\bar A]^{(n-1)k\varphi(t)p}[B,\bar C]^{k\varphi(t)p}[[B,\bar C],\bar A]^{(n-1)k\varphi(t)p}\\
&\quad ([B,\bar C][[B,\bar C],\bar A][\bar A,\bar C])^{\varphi(tk-1)(n-1)}[\bar A,B]^{\varphi(pt)+tk(n-1)pt}\bar C^{tk}\\
&\quad [\bar C,[\bar A,B]]^{tk\varphi(pt)+(tk)^2(n-1)pt}[C,\bar A]^{k\varphi(t)p}C^{tk}\\
&\underset{(15\text{ and L. }6)}{\simeq} [B,\bar C]^{k\varphi(t)p+\varphi(tk-1)(n-1)}[A,\bar B]^{\varphi(pt)+tk(n-1)pt-(n-1)k\varphi(t)p}\\
&\quad [C,\bar A]^{k\varphi(t)p+\varphi(tk-1)(n-1)}[[B,\bar C],\bar A]^{(n-1)k\varphi(t)p+\varphi(tk-1)(n-1)}\\
&\quad [\bar C,[\bar A,B]]^{tk\varphi(pt)+(tk)^2(n-1)pt}\\
&\underset{(\text{p}_1\text{ and L. }6)}{\simeq} [C,B]^{k\varphi(t)p+\varphi(tk-1)(n-1)}[B,A]^{\varphi(pt)+tk(n-1)pt-(n-1)k\varphi(t)p}\\
&\quad [A,C]^{k\varphi(t)p+\varphi(tk-1)(n-1)}[[B,C],A]^{(n-1)k\varphi(t)p+\varphi(tk-1)(n-1)}\\
&\quad [[A,B],C]^{-\{tk\varphi(pt)+(tk)^2(n-1)pt\}}.
\end{aligned}$$

We showed thus that for $k,t\in\mathbb{Z}$ such that $\frac{m-tkn+tk-1}{t}\in\mathbb{Z}$, we have

$$g_{n,m,k,t}\circ\delta_n \simeq [C,B]^{e_1}[B,A]^{e_2}[A,C]^{e_1}[[B,C],A]^{e_3}[[A,B],C]^{-e_4},$$

where $e_1 = \frac{(m-n+p)tk}{2}$, $e_2 = \frac{(m-kn+k)tp}{2}$, $e_3 = \frac{(p+t(p+k)-1)tk(n-1)}{2}$, $e_4 = \frac{(2m-tp-1)t^2pk}{2}$ and $p = p(m,n,k,t) = \frac{m-tkn+tk-1}{t}$.

Now if $\wedge : S^3 \times S^3 \times S^3 \longrightarrow S^3 \wedge S^3 \wedge S^3 = S^9$ is the natural projection, then $[[B,C],A] = \wedge^*(\omega \circ \Sigma^3\omega)$, where ω is the generator of $\pi_6(S^3)$ given in Remark 3 above (cf. [B1]), and as $\omega \circ \Sigma^3\omega$ is a generator of $\pi_9(S^3) \cong \mathbb{Z}_3$ it follows that $[[B,C],A]$ in $[S^3 \times S^3 \times S^3, S^3]$ has order 3. The same occurs with $[[C,A],B]$ and $[[A,B],C]$ and as none of them is nullhomotopic we conclude with the aid of N7 that $[[B,C],A] = [[C,A],B] = [[A,B],C]$ in $[S^3 \times S^3 \times S^3, S^3]$. Thus we have finally

$$g_{n,m,k,t} \circ \delta_n \simeq [C,B]^{e_1}[B,A]^{e_2}[A,C]^{e_1}[[B,C],A]^f,$$

where $f = e_3 - e_4$. □

Remark 5. The choice of the partial sections S_V^n and S_U^n for the bundles $S^3 \cdots \tilde{P}_n \longrightarrow S^7$ given in page 77 enabled us to write the homotopy class of the corresponding transition functions as a power of a unique commutator of weight 2. We would like to have partial sections for the bundles $S^3 \cdots \tilde{P}_{n,m} \longrightarrow \tilde{P}_n$ for which the homotopy class of the corresponding transition functions could be expressed as power of a unique commutator of weight 3 as is suggested by the obstruction in the Hilton-Roitberg formula (Theorem 1). This however, cannot be realized with our choice of transition functions:

If we suppose that $[C,B]^{e_1}[B,A]^{e_2}[A,C]^{e_1}[[B,C],A]^f \simeq [[B,C],A]^r$ for some r then $g = [C,B]^{e_1}[B,A]^{e_2}[A,C]^{e_1} \simeq [[B,C],A]^s$ $(s = r - f)$, and so $g \in [\Gamma_1, \Gamma] \subseteq \Gamma_2$ which implies that $g|_{P_2} \simeq 1$ where $P_2 = X_1 \cup X_2 \cup X_3$, $X_1 = \{1\} \times S^3 \times S^3$, $X_2 = S^3 \times \{1\} \times S^3$, $X_3 = S^3 \times S^3 \times \{1\}$ and this implies $g|_{X_i} \simeq 1$ $(i = 1,2,3)$ in other words, $[C,B]^{e_1}$, $[B,A]^{e_2}$, $[A,C]^{e_1}$: $S^3 \times S^3 \longrightarrow S^3$ are all nullhomotopic, which gives $e_1 \equiv e_2 \equiv 0 \bmod 12$ for every m,n,k,t such that $\frac{m-nkt+kt-1}{t} \in \mathbb{Z}$, but this is not true for example if $m = 5$, $n = 2$, $k = 1$ and $t = 2$ or $m = 14$, $n = 23$, $k = 2$ and $t = 13$. We do not know if there exist such transition functions.

It follows from Theorem 2 that $E_{1,7} = \tilde{P}_{23,14}$ and $E_{7,1} = \tilde{P}_{14,23}$.

From Theorem 3 and Theorem 5 choosing $k = 19$ and $t = 1$ we have that $g_{14,23,19,1} \circ \delta_{14}$ is homotopic to

$$([C,B]^{12})^{-171}([B,A]^{12})^{2100}([A,C]^{12})^{-171}([[B,C],A]^3)^{174591} \simeq 1. \tag{17}$$

Also, choosing $k = 5$ and $t = -13$ we obtain that $g_{23,14,5,-13} \circ \delta_{23}$ is homotopic to

$$([C,B]^{12})^{325}([B,A]^{12})^{-5772}([A,C]^{12})^{325}([[B,C],A]^3)^{-22437350} \simeq 1. \tag{18}$$

Thus, following the same steps of the trivialization of $\tilde{P}_9$, we can exhibit, up to a homotopy of the above commutator powers to the constant 1, the diffeomorphisms

$$\tilde{P}_{23}\times S^3=\tilde{P}_{23,14}\quad\text{and}\quad\tilde{P}_{14}\times S^3=\tilde{P}_{14,23}.$$

Let $H : S^3\times S^3\times S^3\times[0,\frac{\pi}{2}]\longrightarrow S^3$ be a smooth homotopy such that

$$H(A,B,C,\theta)=\begin{cases} g_{23,14,5,-13}\circ\delta_{23}(A,B,C,\theta) & \text{if } \theta\in[0,\frac{\pi}{6}]\\ 1 & \text{if } \theta\in[\frac{\pi}{3},\frac{\pi}{2}].\end{cases}$$

By remembering that $s^{5,-13}_{\tilde{U}_{23}} : \tilde{U}_{23}\longrightarrow\tilde{P}_{23,14}$ and $s^{5,-13}_{\tilde{V}_{23}} : \tilde{V}_{23}\longrightarrow\tilde{P}_{23,14}$ are partial sections of $\tilde{P}_{23,14}$ over $\tilde{U}_{23}$ and $\tilde{V}_{23}$ respectively given by

$$s^{5,-13}_{\tilde{U}_{23}}(M_{23}(a,b,y_2))=(M_{23}(a,b,y_2),M_{14}(a,b,Y_2(5,-13))),$$
$$s^{5,-13}_{\tilde{V}_{23}}(M_{23}(a,b,z_{23}))=(M_{23}(a,b,z_{23}),M_{14}(a,b,Z_{14}(5,-13))),$$

where

$$Y_2(5,-13)=(y_2^5\bar{A}^{-111})^{-13}\frac{L_1L_2\ldots L_{10}L(23)^{-65}}{(L_1L_2\ldots L_{19})^{-65}L(14)},$$
$$Z_{14}(5,-13)=(-1)^{-64}(z_{23}^5\bar{B}^{-111})^{-13}\frac{(L_1L_2\ldots L_{19}L(23))^{-65}}{L_1L_2\ldots L_{10}L(14)},$$

we can then construct a global section $s_{23,14} : \tilde{P}_{23}\longrightarrow\tilde{P}_{23,14}$ given by

$$s_{23,14}(M_{23})=\begin{cases} s^{5,-13}_{\tilde{V}_{23}}(M_{23}) & \text{if } \frac{5\pi}{12}\le\cos^{-1}|a|\le\frac{\pi}{2}\\ s^{5,-13}_{\tilde{V}_{23}}(M_{23})(H\circ\delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12}\le\cos^{-1}|a|\le\frac{5\pi}{12}\\ s^{5,-13}_{\tilde{U}_{23}}(M_{23}) & \text{if } 0\le\cos^{-1}|a|\le\frac{\pi}{12}.\end{cases}$$

Therefore, a diffeomorphism $\Phi_{23,14} : \tilde{P}_{23}\times S^3\longrightarrow\tilde{P}_{23,14}$ is given by

$$\begin{aligned}&\Phi_{23,14}(M_{23}(a,b,x_1,x_2,\ldots,x_{23}),q)\\ &\quad=(M_{23}(a,b,x_1,x_2,\ldots,x_{23}),M_{14}(a,b,y_1q,y_2q,\ldots,y_{14}q)),\end{aligned}$$

where if $\theta=\theta(a)=\cos^{-1}|a|$ then

$$y_1=\begin{cases}(b\bar{a})^{12}|b|^{-24}(L_1L_2\ldots L_{10})^2Z_{14} & \text{if } \frac{5\pi}{12}\le\theta\le\frac{\pi}{2}\\ (b\bar{a})^{12}|b|^{-24}(L_1L_2\ldots L_{10})^2Z_{14}.(H\circ\delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12}\le\theta\le\frac{5\pi}{12}\\ -(a\bar{b})|a|^{-2}Y_2 & \text{if } 0\le\theta\le\frac{\pi}{12},\end{cases}$$

$$y_2=\begin{cases}-(b\bar{a})^{13}|b|^{-26}(L_1L_2\ldots L_{10})^2Z_{14} & \text{if } \frac{5\pi}{12}\le\theta\le\frac{\pi}{2}\\ -(b\bar{a})^{13}|b|^{-26}(L_1L_2\ldots L_{10})^2Z_{14}.(H\circ\delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12}\le\theta\le\frac{5\pi}{12}\\ Y_2 & \text{if } 0\le\theta\le\frac{\pi}{12},\end{cases}$$

$$y_k=\begin{cases}(b\bar{a})^{14-k}|b|^{2(14-k)}(L_1L_2\ldots L_{14-(k+2)})^2Z_{14} & \text{if } \frac{5\pi}{12}\le\theta\le\frac{\pi}{2}\\ (b\bar{a})^{14-k}|b|^{2(14-k)}(L_1L_2\ldots L_{14-(k+2)})^2Z_{14}.(H\circ\delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12}\le\theta\le\frac{5\pi}{12}\\ -(a\bar{b})^{k-1}|a|^{-2(k-1)}(L_{14-(k+1)}L_{14-k}\ldots L_{10})^{-2}Y_2 & \text{if } 0\le\theta\le\frac{\pi}{12}\end{cases}$$

for $3\le k\le 11$,

$$y_{12}=\begin{cases}(b\bar{a})^2|b|^{-4}Z_{14} & \text{if } \frac{5\pi}{12}\le\theta\le\frac{\pi}{2}\\ (b\bar{a})^2|b|^{-4}Z_{14}.(H\circ\delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12}\le\theta\le\frac{5\pi}{12}\\ -(a\bar{b})^{11}|a|^{-22}(L_1L_2\ldots L_{10})^{-2}Y_2 & \text{if } 0\le\theta\le\frac{\pi}{12},\end{cases}$$

$$y_{13}=\begin{cases}(b\bar{a})|b|^{-2}Z_{14} & \text{if } \frac{5\pi}{12}\le\theta\le\frac{\pi}{2}\\ (b\bar{a})|b|^{-2}Z_{14}.(H\circ\delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12}\le\theta\le\frac{5\pi}{12}\\ -(a\bar{b})^{12}|a|^{-24}(L_1L_2\ldots L_{10})^{-2}Y_2 & \text{if } 0\le\theta\le\frac{\pi}{12},\end{cases}$$

$$y_{14}=\begin{cases}Z_{14} & \text{if } \frac{5\pi}{12}\le\theta\le\frac{\pi}{2}\\ Z_{14}.(H\circ\delta_{23}^{-1})(M_{23}) & \text{if } \frac{\pi}{12}\le\theta\le\frac{5\pi}{12}\\ -(a\bar{b})^{13}|a|^{-26}(L_1L_2\ldots L_{10})^{-2}Y_2 & \text{if } 0\le\theta\le\frac{\pi}{12},\end{cases}$$

$$Z_{14}=(-1)^{-64}(x_{23}^5\bar{b}^{-111})^{-13}\tfrac{(L_1L_2\ldots L_{19}L(23))^{-65}}{(L_1L_2\ldots L_{10}L(14))},$$
$$Y_2=(x_2^5\bar{a}^{-111})^{-13}\tfrac{L_1L_2\ldots L_{10}L(23)^{-65}}{(L_1L_2\ldots L_{19})^{-65}L(14)}.$$

Let now $G:S^3\times S^3\times S^3\times[0,\frac{\pi}{2}]\longrightarrow S^3$ be a smooth homotopy such that

$$G(A,B,C,\theta)=\begin{cases}g_{14,23,19,1}\circ\delta_{14}(A,B,C,\theta) & \text{if } \theta\in[0,\frac{\pi}{6}]\\ 1 & \text{if } \theta\in[\frac{\pi}{3},\frac{\pi}{2}].\end{cases}$$

Following the same steps of the construction of $\Phi_{23,14}$ we have

$$s^{19,1}_{\tilde{U}_{14}}(M_{14}(a,b,y_2))=(M_{14}(a,b,y_2),M_{23}(a,b,\mathcal{Y}_2(19,1))),$$
$$s^{19,1}_{\tilde{V}_{14}}(M_{14}(a,b,z_{14}))=(M_{14}(a,b,z_{14}),M_{23}(a,b,\mathcal{Z}_{23}(19,1)))$$

are partial sections of $\tilde{P}_{14,23}$ over $\tilde{U}_{14}$ and $\tilde{V}_{14}$ respectively, where

$$\mathcal{Y}_2(19,1)=\mathcal{Y}_2(19,1)(a,b,y_2)=y_2^{19}\bar{a}^{-225}\frac{L_1L_2L_3\ldots L_{19}L(14)}{(L_1L_2\ldots L_{10})^{19}L(23)},$$
$$\mathcal{Z}_{23}(19,1)=\mathcal{Z}_{23}(19,1)(a,b,z_{14})=-z_{14}^{19}\bar{b}^{-225}\frac{(L_1L_2\ldots L_{10}L(14))^{19}}{L_1L_2L_3\ldots L_{19}L(23)}.$$

If $M_{14} = M_{14}(a,b,x_1,x_2,\ldots,x_{14}) \in \tilde{P}_{14}$ then a global section, $s_{14,23} : \tilde{P}_{14} \longrightarrow \tilde{P}_{14,23}$ is given by

$$s_{14,23}(M_{14}) = \begin{cases} s^{19,1}_{\tilde{V}_{14}}(M_{14}) & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ s^{19,1}_{\tilde{V}_{14}}(M_{14}).(G \circ \delta_{14}^{-1})(M_{14}) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ s^{19,1}_{\tilde{U}_{14}}(M_{14}) & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

and a diffeomorphism $\Phi_{14,23} : \tilde{P}_{14} \times S^3 \longrightarrow \tilde{P}_{14,23}$ is given by

$$\begin{aligned} &\Phi(M_{14}(a,b,x_1,x_2,\ldots,x_{14}),q) \\ &\quad = (M_{14}(a,b,x_1,\ldots,x_{14}), M_{23}(a,b,r_1.q,r_2.q,\ldots,r_{23}.q)) \end{aligned}$$

with

$$r_1 = \begin{cases} (b\bar{a})^{21}|b|^{-42}(L_1L_2\ldots L_{19})^2\mathcal{Z}_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ (b\bar{a})^{21}|b|^{-42}(L_1L_2\ldots L_{19})^2\mathcal{Z}_{23}.(G \circ \delta_{14}^{-1})(M_{14}) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})|a|^{-2}\mathcal{Y}_2 & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

$$r_2 = \begin{cases} -(b\bar{a})^{22}|b|^{-44}(L_1L_2\ldots L_{19})^2\mathcal{Z}_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ -(b\bar{a})^{22}|b|^{-44}(L_1L_2\ldots L_{19})^2\mathcal{Z}_{23}.(G \circ \delta_{14}^{-1})(M_{14}) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ x_2^{19}\bar{a}^{-225}\frac{L_1L_2\ldots L_{19}L(14)^{19}}{(L_1L_2\ldots L_{10})^{19}L(23)} & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

$$r_k = \begin{cases} (b\bar{a})^{23-k}|b|^{-2(23-k)}(L_1L_2\ldots L_{21-k})^2\mathcal{Z}_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ \frac{(b\bar{a})^{23-k}}{|b|^{2(23-k)}}(L_1L_2\ldots L_{21-k})^2\mathcal{Z}_{23}.(G \circ \delta_{14}^{-1})(M_{14}) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})^{k-1}|a|^{-2(k-1)}(L_{23-(k+1)}L_{23-k}\ldots L_{19})^{-2}\mathcal{Y}_2 & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12} \end{cases}$$

for $3 \leq k \leq 20$,

$$r_{21} = \begin{cases} (b\bar{a})^2|b|^{-4}\mathcal{Z}_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ (b\bar{a})^2|b|^{-4}\mathcal{Z}_{23}.(G \circ \delta_{14}^{-1})(M_{14}) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})^{20}|a|^{-40}(L_1L_2\ldots L_{19})^{-2}\mathcal{Y}_2 & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

$$r_{22} = \begin{cases} (b\bar{a})|b|^{-2}\mathcal{Z}_{23} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ (b\bar{a})|b|^{-2}\mathcal{Z}_{23}.(G \circ \delta_{14}^{-1})(M_{14}) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})^{21}|a|^{-42}(L_1L_2\ldots L_{19})^{-2}\mathcal{Y}_2 & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

$$r_{23} = \begin{cases} -x_{14}^{19}\bar{b}^{-225}\frac{(L_1L_2\ldots L_{10}L(14))^{19}}{L_1L_2\ldots L_{19}L(23)} & \text{if } \frac{5\pi}{12} \leq \cos^{-1}|a| \leq \frac{\pi}{2} \\ -x_{14}^{19}\bar{b}^{-225}\frac{(L_1L_2\ldots L_{10}L(14))^{19}}{L_1L_2\ldots L_{19}L(23)}.(G \circ \delta_{14}^{-1})(M_{14}) & \text{if } \frac{\pi}{12} \leq \cos^{-1}|a| \leq \frac{5\pi}{12} \\ -(a\bar{b})^{22}|a|^{-44}(L_1L_2\ldots L_{19})^{-2}\mathcal{Y}_2 & \text{if } 0 \leq \cos^{-1}|a| \leq \frac{\pi}{12}, \end{cases}$$

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$$\mathcal{Y}_2 = x_2^{19}\bar{a}^{-225}\frac{L_1L_2...L_{19}L(14)^{19}}{(L_1L_2...L_{10})^{19}L(23)},$$
$$\mathcal{Z}_{23} = -x_{14}^{19}\bar{b}^{-225}\frac{(L_1L_2...L_{10}L(14))^{19}}{L_1L_2...L_{19}L(23)}.$$

We have with this

$$\tilde{P}_{23}\times S^3 \xrightarrow{\Phi_{23,14}} \tilde{P}_{23,14} \xrightarrow{c} \tilde{P}_{14,23} \xrightarrow{\Phi^{-1}_{14,23}} \tilde{P}_{14}\times S^3,$$

where $c(M_2, M_{11}) = (M_{11}, M_2)$.

Remark 6. We also observe that the same procedure provides the trivialization of the bundle $\tilde{P}_{2,11}$, choosing $k = 11$ and $t = 5$.

Exotic Actions

Non-cancellation phenomena related to products $M\times G = N\times G$ of non equivalent spaces M and N by a group G can be seen as exotic actions of the group on, say, $M\times G$ with quotient N. We have treated here two such cases were $G = S^3$. We showed precisely how the specific diffeomorphisms and, equivalently, the corresponding exotic actions depend on the homotopy commutativity of certain powers of commutators in S^3.

In the case of the exotic actions treated above we have S^3 acting freely on $P\times S^3$, where P can be considered as a parameter space on which S^3 acts in a standard way and that parametrizes a complicated action of S^3 on itself, the second factor, so that the action on the product is free and the quotient is not equivalent to P. Investigating properties of prospective P's, like for example, how small such a compact P can be, etc., seems like an interesting way of looking at some classical questions.

On the other hand, specifying explicit homotopies between powers of commutators and constants seems to be a problem of geometric nature [CR], [RC].

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