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Degrees of Self-maps

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DEGREES OF SELF-MAPS

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0. Introduction. As well-known, one of the basic problems in algebraic topology is to determine the homology representation

$$H : [X, Y] \to \mathrm{Hom}(H_*(X), H_*(Y))$$

where $[X, Y]$ denotes the set of homotopy classes of maps from X to Y.

In this paper we shall consider it for the case $X = Y$. Previously, Sullivan and Quillen proved the famous theorem for $X = \mathrm{HP}^\infty$ in [7] and C. A. Mcgibbon determined the image of H for $X = \mathrm{RP}^n$, CP^n and HP^n in the stable case in [3]. Furthermore D. M. Davis investigated it for $X = \mathrm{CP}^{n+2}/\mathrm{CP}^{n-1}$, the stunted projective space in [1] and also S. Sasao and M. Nagaishi determined it for $X = \mathrm{HP}^3$ in [6]. In this paper we shall consider the case $X = S^n \cup e^{n+2} \cup e^{n+4}$, which is a generalization of Davis's case and contains the following:

(1) The total spaces of S^2-bundles over S^4.

(2) the Thom complexes of real n-vector bundles over a 2-cell complex $S^2 \cup e^4$, which contain the stunted complex projective spaces $\mathrm{CP}^{n+2}/\mathrm{CP}^{n-1}$.

(3) the iterated suspension of (1) or (2).

Let X be a 3-cell complex of the form $S^n \cup e^{n+2} \cup e^{n+4}$, and e_j be the corresponding generator of $H_j(X) \simeq \boldsymbol{Z}$ for $j = n, n+2, n+4$. Then for each self-map $f \in [X, X]$, the endomorphism

$$H(f) = f_* : H_*(X) \to H_*(X)$$

is uniquely determined by a triple of degrees (d_1, d_2, d_3) which is defined by

$$f_*(e^{(n-2)+2j}) = d_j e^j$$

for $j = 1, 2, 3$. We call f a self-map of X of degrees (d_1, d_2, d_3).

Hence our problem is reduced to characterize a triple of integers (d_1, d_2, d_3) which is a triple of degrees of a self-map of X. This note is organized as follows: In §1, we shall consider the case $n = 2$, and investigate the case $n = 3, 4$ in §2. In §3, we shall treat the case $n > 4$ which is belonging to the stable range, and some examples shall be given in §4.

Remark. *In a subsequent paper, we shall consider the kernel of H.*

Here we state a part of our results.

Theorem A. *For $X = S^2 \cup e^4 \cup e^6$, there exists a self-map of X of degrees (d_1, d_2, d_3) if and only if the followings hold:*

(0) *If $e^2 \cdot e^2 \neq 0$, $Sq^2(e^4) = 0$, then $d_2 = d_1^2$ and $d_3 = d_1^3 \bmod h_6(X)$.*

(1) *If $e^2 \cdot e^2 \neq 0$, $Sq^2(e^4) \neq 0$, $e^2 \cdot e^4 \neq 0$, then $d_2 = d_1^2$ and $d_3 = d_1^3$.*

(2) *If $e^2 \cdot e^2 \neq 0$, $Sq^2(e^4) \neq 0$, $e^2 \cdot e^4 = 0$, then $d_2 = d_1^2$ and $d_3 \equiv d_1 \bmod 2$.*

(3) *If $e^2 \cdot e^2 = 0$, $e^2 \cdot e^4 \neq 0$, $Sq^2(e^4) = 0$, then $d_3 = d_1 d_2$ and $d_3 \equiv d_1 \bmod h_7(X)$.*

(4) *If $e^2 \cdot e^2 = 0$, $e^2 \cdot e^4 \neq 0$, $Sq^2(e^4) \neq 0$, then $d_3 = d_1 d_2$ and $d_3 \equiv d_2 \bmod 2$.*

(5) *If $e^2 \cdot e^2 = e^2 \cdot e^4 = 0$, $Sq^2(e^4) = 0$, then $d_3 \equiv d_1 \bmod h_6(X)$.*

(6) *If $e^2 \cdot e^2 = e^2 \cdot e^4 = 0$, $Sq^2(e^4) \neq 0$, then $d_3 \equiv d_2 \bmod 2$.*

Here $\cdot$ denotes the cup product in the cohomology ring and Sq is the Steenrod squaring operation, and $h_n(X)$ denotes the image of the Hurewicz homomorphism at dimension n.

Corollary E. *For $X = S^n \cup e^{n+2} \cup e^{n+4}$ $(5 \leqq n)$, a triple of integers (d_1, d_2, d_3) is realizable by a self-map of X if and only if the followings hold:*

(1) *$d_3 \equiv d_2 \bmod 2$ and $2d_3 \equiv 2d_1 \bmod h_{n+4}(X)$ if $Sq^2(e^{n+2}) \neq 0$.*

(2) *$d_3 \equiv d_1 \bmod h_{n+4}(X)$ if $Sq^2(e^{n+2}) = 0$ and $Sq^2(e^n) = 0$.*

(3) *$d_2 \equiv d_1 \bmod 2$ and $d_3 \equiv d_1 \bmod h_{n+4}(X)$ if $Sq^2(e^{n+2}) = 0$ and $Sq^2(e^n) \neq 0$.*

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1. The case $n = 2$. Let X be a 3-cell complex of the form $S^2 \cup e^4 \cup e^6$. Let a and b be integers and $\varepsilon \in \{0, 1\}$. Then we call X to be of type (a, b, ε) if and only if

$$e^2 \cdot e^2 = ae^4, \quad e^2 \cdot e^4 = be^6, \quad \text{and} \quad Sq^2(e^4) = \varepsilon e^6$$

where e^j denotes the generator of $H^j(X, \boldsymbol{Z})$ (or $H^j(X, \boldsymbol{Z}/2)$). From now on in this section, we assume that X is of type (a, b, ε).

First we consider the sub-case

1.1 $a = 0$ (i.e. $e^2 \cdot e^2 = 0$) By the assumption we may consider that X has a form $X = (S^2 \vee S^4) \cup e^6$. Then, the attaching class β for the cell e^6 is the following

$$\beta = x\iota_2(\eta_2\eta_3\eta_4) + \iota_4(\eta_4) + b[\iota_2, \iota_4]$$

where η_n denotes the Hopf class of $\pi_{n+1}(S^n)$.

Now define two maps $S^2 \vee S^4 \to S^2 \vee S^4$ as follows:

$$\psi_{k,\ell}|S^2 = k\iota_2 \quad\text{and}\quad \psi_{k,\ell}|S^4 = \ell\iota_4,$$
$$\phi_{k,\ell}|S^2 = k\iota_2 \quad\text{and}\quad \phi_{k,\ell}|S^4 = \ell\iota_4+\iota_2\eta_2\eta_3.$$

Lemma 1. *When $b \neq 0$, there exists a self-map of X of degrees (k, ℓ, m) if and only if $m = k\ell$ and $m \equiv \ell \bmod 2$.*

Proof. By the standard argument we can easily know that there is a self-map of X of degrees (k, ℓ, m) if and only if $\psi_{k,\ell *}(\beta) = m\beta$ or $\phi_{k,\ell *}(\beta) = m\beta$. Since two endomorphisms

$$\psi_{k,\ell *} \text{ and } \phi_{k,\ell *}: \pi_5(S^2\vee S^4)\to \pi_5(S^2\vee S^4)$$

are clearly given by

$$\phi_{k,\ell *}(\beta) = kx\iota_2\eta_2\eta_3\eta_4+\iota_2\eta_2\eta_3\eta_4+\ell\iota_4\eta_4+k\ell b[\iota_2, \iota_4],$$
$$\psi_{k,\ell *}(\beta) = kx\iota_2\eta_2\eta_3\eta_4+\ell\iota_4\eta_4+k\ell b[\iota_2, \iota_4] \qquad (1\text{-}1),$$

the condition is equivalent to $mb = k\ell b$, $\ell \equiv m \bmod 2$, and $kx\eta_2\eta_3\eta_4 \equiv mx\eta_2\eta_3\eta_4 \bmod b\eta_2\eta_3\eta_4$. Then the assumption $b \neq 0$ completes the proof.

Now assume that $b = 0$ and $c \neq 0$. Then, from (1-1) we can obtain the following:

Lemma 2. *When $\varepsilon = 0$ and $b \neq 0$, there is a self-map of X of degrees (k, ℓ, m) if and only if $m = k\ell$ and $m \equiv k \bmod h_7(\Sigma X)$, where Σ denotes the suspension functor.*

Analogously we have

Lemma 3. *When $b = 0$, there exists a self-map of X of degrees (k, ℓ, m) if and only if*

(1) $m \equiv k \bmod h_6(X)$ *if* $\varepsilon = 0$.

(2) $m \equiv \ell \bmod 2$ *if* $\varepsilon \neq 0$.

1.2 $a \neq 0$ (i.e. $e^2\cdot e^2 \neq 0$). Let A_a be the 2 cell-complex $S^2\cup e^4$ which has a $a\eta_2$ as the attaching class for the cell e^4. First we quote the following ((2.13) of [8]):

Lemma 4.

$$\pi_5(A_a) = \begin{cases} \boldsymbol{Z} & a \equiv 1 \pmod 2 \\ \boldsymbol{Z}+\boldsymbol{Z}/4 & a \equiv 2 \pmod 4 \\ \boldsymbol{Z}+\boldsymbol{Z}/2+\boldsymbol{Z}/2 & a \equiv 0 \pmod 4 \end{cases}$$

Let $\psi_k : A_a \to A_a$ be a map of degrees (k, k^2), then other self-maps of A_a of degrees (k, k^2) is only one, which is given by the composite

$$\psi'_k : A_a \underset{c}{\to} A_a \vee S^4 \to A_a = \psi_k \vee \iota_2\eta_2\eta_3 \cdot c$$

Let β be the attaching class for the cell e^6 of X and β' be the image of β by the pinching map $A_a \to S^4 = A_a/S^2$. Clearly β' is 0 or η_4, which is determined by $Sq^2(e^4)$. Using the fact $[\iota_2, \eta_2\eta_3] = 0$, we can easily obtain the following:

$$\psi_{k*}(\beta) = k^3\beta + \iota_2\gamma \quad \text{and} \quad \psi'_{k*}(\beta) = \psi_{k*}(\beta) + \iota_2\eta_2\eta_4\beta' \tag{1-2}$$

where γ is an element of $\pi_5(S^2)$, i.e. 0 or $\eta_2\eta_3\eta_4$.

Lemma 5. *If $a \equiv 1 \bmod 2$, then we have $\psi_{k*}(\beta) = \psi'_{k*}(\beta) = k^3\beta$.*

Proof. Lemma 4 implies that γ and β' in the formula (1-2) is always 0. Hence the proof is complete.

Next we investigate the case $a \equiv 0 \bmod 2$ $(a \neq 0)$. First we prove

Lemma 6. *For $a = 2$, we can choose ψ_k satisfying $\psi_{k*}(\beta) = k^3\beta$.*

Proof. We may regard A_2 as the 4-skelton of the reduced product S^2_∞ of S^2 in [2]. The map $k\iota_2 : S^2 \to S^2$ induces the map $S^2_\infty \to S^2_\infty$ whose restriction on its 4-skelton is the desired map ψ_k, and $\psi_{k*}(\alpha) = k^3\alpha$ holds for α, the attaching class for the 6-cell of S^2_∞. On the other hand, from lemma 4 and the homotopy exact sequence of the pair (A_a, S^2) we can know that there is a class α_1 of $\pi_5(A_2)$ satisfying $j_*(\alpha_1) = [\chi_4, \iota_2]_r$, where $[\ ,\ ]_r$ denotes the relative Whitehead product and χ_4 is the characteristic map for the cell e^4 of S^2_∞. Since we have $e^2 \cdot e^4 = 3e^6$ in $H(S^2_\infty)$ (see [2]), the following relations hold

$$3\alpha_1 = \alpha \quad \text{or} \quad 3\alpha_1 = \alpha + \eta_2\eta_3\eta_4.$$

Then we have

$$3\psi_{k*}(\alpha_1) = \psi_{k*}(\alpha) = k^3\alpha = 3k^3\alpha_1 \quad \text{or}$$
$$3\psi_{k*}(\alpha_1) = \psi_{k*}(\alpha) + k\eta_2\eta_3\eta_4 = k^3\alpha + k\eta_2\eta_3\eta_4 = k^3(\alpha + \eta_2\eta_3\eta_4) = 3k^3\alpha_1 \quad \text{i.e.}$$

$\psi_{k*}(\alpha_1) = k^3\alpha_1.$

Hence, for $\beta(\in \pi_5(A_2))$ with $\beta' = 0$, it holds $\psi_{k*}(\beta) = k^3\beta$. Moreover, if $\beta' = 0$, we obtain from the formula (1-2) that $\psi_{k*}(\beta) = k^3\beta$ or $\psi'_{k*}(\beta) = k^3\beta$. Thus the proof is complete.

Secondly, we prove the general case.

Lemma 7. *If* $a \equiv 0 \bmod 2$, *then there is a map* $\phi_k : A_a \to A_a$ *satisfying* $\phi_{k*}(\beta) = k^3\beta$ *for any* β *of* $\pi_5(A_a)$.

Proof. For β with $\beta' \neq 0$, the proof follows from the formula (1-2). If $\beta' = 0$ we put $a = 2a'$. Since, for any map $\phi(1, a') : A_a \to A_2$ of degrees $(1, a')$, there exists a map $\phi_k : A_a \to A_2$ which makes the following diagram commutative:

$$\begin{array}{ccc} A_a = A_{2a} & \xrightarrow[\phi(1,a')]{} & A_2 \\ \downarrow \phi_k & & \downarrow \psi_k \\ A_a & \xrightarrow[\phi(1,a')]{} & A_2 \end{array}$$

Then, the proof follows from the formula (1-1), lemma 6, and the restriction $\phi(1, a)|S^2 =$ the identity.

Proposition 1. *For* $a \neq 0$, *a triple* (d_1, d_2, d_3) *is realizable by a self-map of* X *if and only if the followings hold*:

(0) $d_2 = k^2$.

(1) *If* $\varepsilon = 0$, *then* $d_3 \equiv d_1^3 \bmod h_6(X)$.

(2) *If* $\varepsilon \neq 0$ *and* $b \neq 0$, *then* $d_3 = d_1^3$.

(3) *If* $\varepsilon \neq 0$ *and* $b = 0$, *then* $d_3 \equiv d_1^3 \bmod 2$.

Proof. (0) is easy by the assumption $a \neq 0$. Using lemma 5 and 7, we may take $\gamma = 0$ in the formula (1-1). Then our desired condition is

$$m\beta = k^3\beta \quad \text{or} \quad m\beta = k^3\beta + \iota_2\eta_2\eta_3\beta'.$$

Thus the proof of (1), (2), and (3) follows from using that β' is determined by $Sq^2(e^4)$.

Now we prove Theorem A. Namely, (3) follows from lemma 2. (4) follows from lemma 1. (5) and (6) follow from lemma 3. The others follow from lemma 5 and proposition 1.

2. The case $n = 3, 4$. The case can be divided into two ones by the formula

Mathematical Journal of Okayama University, Vol. 34 [1992], Iss. 1, Art. 21

$Sq^2Sq^2 = Sq^3Sq^1$:
(1) $Sq^2(e^{n+2}) \neq 0$, then $X = (S^n \vee S^{n+2}) \cup e^{n+4}$ because of $Sq^2(e^n) = 0$.
(2) $Sq^2(e^{n+2}) = 0$, then $X = e^{n+4} \cup S^n \cup e^{n+2}$.

Let β be the attaching class for the $(n+4)$-cell of X. First we consider the case (1). Define two maps $\psi_{k,\ell}$ and $\psi'_{k,\ell}$ as follows:

$$\psi_{k,\ell},\ \psi'_{k,\ell}:\ S^n \vee S^{n+2} \to S^n \vee S^{n+2},$$
$$\psi_{k,\ell}|S^n = k\iota_n \quad \text{and} \quad \psi_{k,\ell}|S^{n+2} = \ell\iota_{n+2},$$
$$\psi'_{k,\ell}|S^n = k\iota_n \quad \text{and} \quad \psi'_{k,\ell}|S^{n+2} = \ell\iota_{n+2} + \iota_n\eta_n\eta_{n+1}.$$

Then we have

$$\psi_{k,\ell *}(\beta) = (k\iota_n)_*(\beta_1) + \iota_{n+2}(\ell\eta_{n+2}),$$
and
$$\psi'_{k,\ell *}(\beta) = (k\iota_n)_*(\beta_1) + \iota_n\eta_n\eta_{n+1}\eta_{n+2} + \iota_{n+2}(\ell\eta_{n+2}),$$

where $\beta = \beta_1 + \iota_{n+2}\eta_{n+2}$ $(\in \pi_{n+3}(S^n \vee S^{n+2}) = \pi_{n+3}(S^n) + \pi_{n+3}(S^{n+2}))$. Hence, a triple (k, ℓ, m) is realizable by a self-map of X if and only if the following equality holds:

$$m\beta_1 + m\iota_{n+2}\eta_{n+2} = m\beta = (k\iota_n)_*\beta_1 + \ell\iota_{n+2}\eta_{n+2} \mod \iota_n\eta_n\eta_{n+1}\eta_{n+2}. \qquad (2\text{-}1)$$

Lemma 8. *For $n = 3$ $(Sq^2(e^5) \neq 0)$, there exists a self-map of X of degrees (k, ℓ, m) if and only if $\ell \equiv m \mod 2$ and $2m \equiv 2k \mod h_7(X)$.*

Proof. Since we can replace $(k\iota_n)_*(\beta_1)$ with $k\beta_1$ in the formula (2-1) we have that $\ell \equiv m \mod 2$ and $(m-k)\beta_1 \equiv 0 \mod \eta_3\eta_4\eta_5$. On the other hand, $\eta_3\eta_4\eta_5$ is the only one element of order 2 in $\pi_6(S^3)$. Therefore the latter condition is equivalent to $2(m-k)\beta_1 = 0$. Thus the proof is completed by $2\beta_1 = 2\beta$.

Lemma 9. *For $n = 4$ $(Sq^2(e^6) \neq 0)$, our conditions are as follows:*
(1) *If $e^4 \cdot e^4 = 0$, then $\ell \equiv m \mod 2$ and $2(m-k) \equiv 0 \mod h_8(X)$.*
(2) *If $e^4 \cdot e^4 \neq 0$, then $m = k^2$, $\ell \equiv m \mod 2$, and $m \equiv k \mod h_9(\Sigma X)$.*

Proof. Since β_1 has a representation

$$\beta_1 = x\nu + y\Sigma\omega \quad \text{for some integers } x \text{ and } y,$$

where ν denotes the Hopf map $S^7 \to S^4$ and ω is the Blaker-Massey map, we get

$$(k\iota_4)_*(\beta_1) = k\beta_1 + k(k-1)/2[\iota_4, \iota_4]H(\beta_1)$$
$$= k^2x\nu + \{ky + k(k-1)x/2\}\Sigma\omega.$$

Hence, the formula (2-1) is equivalent to

$$mx\nu + my\Sigma\omega = k^2x\nu + [\{2ky + k(k-1)x\}/2]\Sigma\omega \quad \text{mod } \eta_4\eta_5\eta_6.$$

Moreover this gives that

if $x \neq 0$ (i.e. $e^4 \cdot e^4 \neq 0$), then $m = k^2$ and $\{2my - 2ky - k(k-1)x\}/2\Sigma\omega \equiv 0$ mod $\eta_4\eta_5\eta_6$, and that

if $x = 0$ (i.e. $e^4 \cdot e^4 = 0$), then $(m-k)y\Sigma\omega \equiv 0$ mod $\eta_4\eta_5\eta_6$.

Thus (1) is obtained from the same argument as lemma 8. Next, we consider the case (2). From $m = k^2$ we have that

$$2my - 2ky - k(k-1)x = (m-k)2y - k(k-1)x = k(k-1)(2y-x).$$

On the other hand, we know that $\Sigma\beta_1 = (x-2y)\nu$. Thus the proof of (2) follows from $k(k-1)\Sigma\beta_1 = k(k-1)\Sigma\beta$.

Secondly, we prove the case $Sq^2(e^{n+2}) = 0$. Let A be the subcomplex, $S^n \cup e^{n+2}$, of X and let $\phi_k(\gamma)$ be the map defined by

$$\phi_k(\gamma) = (k1_A \vee \gamma)C_A : A \to A \vee S^{n+2} \to A$$

for $\gamma \in \pi_{n+2}(A)$ where $k1_A$ denotes the k time of the identity of A in the sense of the suspension-addition and C_A is the co-action map of A. Here we note that

(0) $\beta = i_*(\beta')$, where i is the inclusion $S^n \to A$.

(1) $\phi_k(\gamma)$ is of degrees $(k, k + h_{n+2}(X))$.

(2) $\phi_k(\gamma)_*(\beta) = i_*((k\iota_n)_*(\beta'))$.

Now consider the following diagram:

$$\begin{array}{ccccc} & & \pi_{n+3}(S^n) & \xrightarrow[i_*]{} & \pi_{n+3}(A) \\ & & \downarrow (k\iota_n)_* & & \downarrow (k1_A)_* \\ \pi_{n+4}(A, S^n) & \xrightarrow[\partial]{} & \pi_{n+3}(S^n) & \xrightarrow[i_*]{} & \pi_{n+3}(A) \end{array}$$

Then, it is easy from $\phi_k(\gamma)_*(\beta) = (k1_A)_*(\beta)$ to obtain the following:

Lemma 10. *There exists a self-map of X of degrees (k, ℓ, m) if and only if $\ell \equiv k$ mod $h_{n+2}(X)$ and $m(\beta') \equiv (k\iota_n)_*(\beta')$ mod ∂-image.*

Remark. *$h_{n+2}(X) = \mathbf{Z}$ if $Sq^2(e^n) = 0$ and $h_{n+2}(X) = 2\mathbf{Z}$ if $Sq^2(e^n) \neq 0$.*

Since $(k\iota_3)_*(\beta') = k\beta'$ (for $n = 3$), we have

Lemma 11. *For $n = 3$, a triple (k, ℓ, m) is realizable by a self-map of X*

if and only if $\ell \equiv k \bmod h_5(X)$ *and* $m \equiv k \bmod h_7(X)$.

If $n = 4$ we can take $x\nu + y\Sigma\omega$ as β' and use the formula,

$$(k\iota_4)_*(\beta') = k(\beta') + \{k(k-1)/2\}x[\iota_4, \iota_4]. \tag{2-2}$$

Lemma 12. *If* $e^4 \cdot e^4 = 0$, *then there is a self-map of* X *of degrees* (k, ℓ, m) *if and only if* $\ell \equiv k \bmod h_6(X)$ *and* $m \equiv k \bmod h_8(X)$.

Proof. Since $e^4 \cdot e^4 = 0$ is equivalent to $x = 0$, we have

$$(k\iota_4)_*(\beta') = k(\beta'), \quad \text{i.e.} \quad (k1_A)_*(\beta) = k\beta$$

from (2-2). Hence $(m-k)(\beta) = 0$, which completes the proof.

Lemma 13. *If* $x \neq 0$, *then there is a self-map of* X *of degrees* (k, ℓ, m) *if and only if* $\ell \equiv k \bmod h_6(X)$, $m = k^2$, *and* $m \equiv k \bmod h_9(\Sigma X)$.

Proof. First, we have $m = k^2$ from (2-2) and $[\iota_4, \iota_4] = 2\nu + \Sigma\omega$. Then, it holds $i_*\{(m-k)(x-2y)/2\}\Sigma\omega = 0$, which gives

$$i_*\{(m-k)(x-2y)\nu\} = i_*\{(m-k)\Sigma\beta'\} = (m-k)\Sigma\beta = 0$$

from applying the suspension functor. Thus the proof is complete.

Now from lemmas 8, 9, 11, 12, and 13 we have

Theorem B. *Let* X *be a complex of the form* $S^3 \cup e^5 \cup e^7$. *Then a triple* (d_1, d_2, d_3) *is realizable by a self-map of* X *if and only if*

(1) *If* $Sq^2(e^5) \neq 0$, *then* $d_3 \equiv d_2 \bmod 2$ *and* $2d_3 \equiv 2d_2 \bmod h_7(X)$.

(2) *If* $Sq^2(e^5) = 0$ *and* $Sq^2(e^3) = 0$, *then* $d_3 \equiv d_1 \bmod h_7(X)$.

(3) *If* $Sq^2(e^5) = 0$ *and* $Sq^2(e^3) \neq 0$, *then* $d_3 \equiv d_1 \bmod h_7(X)$ *and* $d_2 \equiv d_1 \bmod 2$.

Theorem C. *Let* X *be a complex of the form* $S^4 \cup e^6 \cup e^8$. *Then a triple* (d_1, d_2, d_3) *is realizable by a self-map of* X *if and only if*

(1) *If* $Sq^2(e^6) \neq 0$ *and* $e^4 \cdot e^4 = 0$, *then* $d_3 \equiv d_2 \bmod 2$ *and* $2d_3 \equiv 2d_1 \bmod h_8(X)$.

(2) *If* $Sq^2(e^6) \neq 0$ *and* $e^4 \cdot e^4 \neq 0$, *then* $d_3 \equiv d_2 \bmod 2$, $d_3 \equiv d_1^2$, *and* $d_3 \equiv d_2 \bmod h_9(\Sigma X)$.

(3) *If* $Sq^2(e^6) = 0$ *and* $e^4 \cdot e^4 = 0$, *then* $d_2 \equiv d_1 \bmod h_6(X)$ *and* $d_3 \equiv d_1 \bmod h_8(X)$.

(4) *If* $Sq^2(e^6) = 0$ *and* $e^4 \cdot e^4 \neq 0$, *then* $d_2 \equiv d_1 \bmod h_6(X)$, $d_3 = d_1^2$, *and* $d_3 \equiv d_1 \bmod h_9(\Sigma X)$.

3. The case $X = \Sigma X'$. First we note that this case contains the case $5 \leqq n$. Next, let Y be the complex X/S^n and let us consider maps from Y to X.

Lemma 14. *If* $Sq^2(e^{n+2}) = 0$ *in* $\mathrm{H}^*(X; \boldsymbol{Z}/2)$, *then there exists a map from* Y *to* X *with degrees* (ℓ, m) *if and only if* $\ell \equiv 0 \bmod h_{n+2}(X)$ *and* $m \equiv 0 \bmod h_{n+4}(X)$.

Proof. Since the assumption implies $Y = S^{n+2} \vee S^{n+4}$, the proof is clear.

Lemma 15. *If* $Sq^2(e^{n+2}) \neq 0$, *then there is a map from* Y *to* X *with degrees* (ℓ, m) *if and only if* $m \equiv \ell \bmod 2$ *and* $2m \equiv 0 \bmod h_{n+4}(X)$.

Proof. Since $Sq^2(e^{n+2}) \neq 0$ implies $Sq^2(e^n) = 0$, X has a decomposition

$$X = (S^n \vee S^{n+2}) \cup e^{n+4}.$$

Let $f: S^{n+2} \to X$ be the map defined by $f = n\gamma + \ell \iota_{n+2}$ for $\gamma \in \pi_{n+2}(S^n)$ and $\beta = \iota_n \beta' + \iota_{n+2}\eta_{n+2}$ be the attaching class for the $(n+4)$-cell of X. Then, we have

$$f_*(\eta_{n+2}) = \iota_n(\gamma\eta_{n+2}) + \ell\iota_{n+2}\eta_{n+2}.$$

Hence the proof follows from $m\beta \equiv m\iota_n\beta' + m\iota_{n+2}\eta_{n+2} = f_*(\eta_{n+2})$,

i.e. $m \equiv \ell \bmod 2$ and $m\beta' \equiv 0 \bmod \iota_n\eta_n\eta_{n+1}\eta_{n+2}$.

Secondly, using the group structure of $[X, X] = [\Sigma X', \Sigma X']$ we have the map $k1_X$, which is of degrees (k, k, k). Let $g: X \to X$ be a self-map of degrees (k, ℓ, m). Then the map $g - k1_X$ is of degrees $(0, \ell - k, m - k)$, and lemmas 14 and 15 give the following

Theorem D. *Assume that* X *is a suspended space. Then a triple* (d_1, d_2, d_3) *is realizable by a self-map of* X *if and only if*

(1) *If* $Sq^2(e^{n+2}) = 0$, *then* $d_2 \equiv d_1 \bmod h_{n+2}(X)$ *and* $d_3 \equiv d_1 \bmod h_{n+4}(X)$.

(2) *If* $Sq^2(e^{n+2}) \neq 0$, *then* $d_2 \equiv d_1 \bmod 2$ *and* $2d_3 \equiv 2d_1 \bmod h_{n+4}(X)$.

Remark. *If* $Sq^2(e^n) = 0$, *then* $h_{n+2}(X) = \boldsymbol{Z}$, *and if* $Sq^2(e^n) \neq 0$, *then* $h_{n+2}(X) = 2\boldsymbol{Z}$.

4. Examples.

Example 1. Let X_n be the space $\mathrm{CP}^{n+2}/\mathrm{CP}^{n-1}$ $(1 \leqq n)$. Then, using the results in [5] and $Sq^2(e^i) = ie^{i+1}$ in $H^*(\mathrm{CP}^N)$, we have that degrees of self-maps of X_n, (d_1, d_2, d_3), is characterlized as follows:

(1) For $3 \leqq n$,

if $n \equiv 1 \bmod 2$, then $d_2 \equiv d_1 \bmod 2$ and $d_3 \equiv d_1 \bmod 24/(n+3, 24)$,

if $n \equiv 0 \bmod 8$, then $d_2 \equiv d_3 \bmod 2$ and $2d_3 \equiv 2d_1 \bmod 96/(n, 48)$,

if $n = 2, 4, 6 \bmod 8$, then $d_2 \equiv d_3 \bmod 2$ and $2d_3 \equiv 2d_1 \bmod 48/(n, 48)$,

where (,) denotes the greatest common diviser of integers.

Remark. These results also hold for the iterated-suspension $\Sigma^s X_n$.

(2) For $n = 2$, $d_3 \equiv d_2 \bmod 2$ and $d_3 \equiv d_1^2$, i.e. $(k, k^2+2\boldsymbol{Z}, k^2)$, and for the space $\Sigma^s X_2$ $(1 \leqq s)$ we have $(d_1, d_2, d_3) = (k, k+2\boldsymbol{Z}, k+12\boldsymbol{Z})$.

(3) For $n = 1$, $(d_1, d_2, d_3) = (k, k^2, k^3)$, and for $\Sigma^s X_1$ $(1 \leqq s)$ we have $(d_1, d_2, d_3) = (k, k+2\boldsymbol{Z}, k+6\boldsymbol{Z})$.

Example 2. Let X be the 6-skelton of the reduced product of S^2. Since it is clear that $\Sigma^s X$ is decomposed into $S^{2+s} \vee S^{4+s} \vee S^{6+s}$ $(1 \leqq s)$ we can know that

(1) For $s = 0$, $(d_1, d_2, d_3) = (k, k^2, k^3)$,

(2) For $1 \leqq s$, (d_1, d_2, d_3) for any d_1, d_2 and d_3.

Example 3. Let X_r be the 2-sphere bundle over S^4 whose characteristic class is r times of a generator $(\in \pi_3(SO(3)) = \boldsymbol{Z})$. Then it is easy to see

$$e^2 \cdot e^2 = re^4 \quad \text{and} \quad e^2 \cdot e^4 = e^6.$$

Furthermore the suspension $\Sigma^N X_r$ has a decomposition

$$\Sigma^N X_r = S^{N+2} \cup e^{N+4} \cup e^{N+6}$$

in which the attaching class β_N for the $(N+6)$-cell is given by

$$\beta_N = i_*(\Sigma^{N-1} J(\xi_r)) = i_*(r\Sigma^{N-1}\omega) = ri_*(\Sigma^{N-1}\omega)$$

where ξ_r is the real vector bundle associated with X_r.

On the other hand, it is easy to show that

$$Sq^2(e^{N+4}) = 0 \ (0 \leqq N) \text{ and } Sq^2(e^{N+2}) = 0 \text{ for even } r, \neq 0 \text{ for odd } r \ (1 \leqq N).$$

Since $Sq^2(e^{N+4}) = 0$ implies that i^* is injective, we get

$$h_{N+6}(\Sigma^N X_r) = (r, 12)\boldsymbol{Z}.$$

These facts give the following:

A triple (d_1, d_2, d_3) is realizable by a self-map of $\Sigma^N X_r$ if and only if

(1) The case $1 \leqq N$.

$d_3 \equiv d_1 \bmod (r, 12)$ if $r \equiv 0 \bmod 2$,

$d_2 \equiv d_1 \bmod 2$ and $d_3 \equiv d_1 \bmod (r, 12)$ if $r \equiv 1 \bmod 2$.

(2) The case $N = 0$.

$d_3 \equiv d_1 d_2$ if $r = 0$,

$d_2 = d_1^2$ and $d_3 = d_1^3$ if $r \neq 0$.

Example 4. Let Y_a be the CW-complex $S^2 \cup e^4$ which has $a\eta_2$ as the attaching class for the cell e^4. Assume $5 \leqq N$ and consider an N-dim real vector bundle ξ over Y_a. For simplicity we suppose that its Stiefel-Whitney class $w_2(\xi)$ is trivial. Since $w_2(\xi) = 0$ implies that the restriction $\xi | S^2$ is trivial, there is a commutative diagram

$$\begin{array}{ccc} \xi & \rightarrow & \xi' \\ \downarrow & & \downarrow \\ Y_a & \rightarrow & S^4 = Y_a/S^2 \end{array}$$

for some ξ'.

Then, from $Sq^2(e^{N+2}) = ae^{N+4}$ ([4]), we can know that the Thom complex $T(\xi)$ has a decomposition

$$T(\xi) = (S^N \vee S^{N+2}) \cup e^{N+4}$$

in which $\beta = \iota_n J(\xi') + \iota_{n+2}(a\eta_{n+2})$ $(\in \pi_{N+3}(S^N) + \pi_{N+3}(S^{N+2}))$. Thus, from Theorem D, we have that a triple (d_1, d_2, d_3) is realizable by a self-map of $T(\xi)$ if and only if

(1) *if* $a \equiv 0 \bmod 2$, *then* $d_3 \equiv d_1 \bmod (b, 24)$,

(2) *if* $a \equiv 1 \bmod 2$, *then* $d_3 \equiv d_2 \bmod 2$ *and* $2d_3 \equiv 2d_1 \bmod (b, 12)$,

where $2be^4 = p_1(\xi)$, *the first Pontrijagin class of* ξ.

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