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**Abstract**

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**KEYWORDS:** commutative Banach algebras, ring homomorphisms.

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# A REPRESENTATION OF RING HOMOMORPHISMS ON UNITAL REGULAR COMMUTATIVE BANACH ALGEBRAS

Takeshi MIURA

Abstract. We give a complete representation of a ring homomorphism from a unital semisimple regular commutative Banach algebra into a unital semisimple commutative Banach algebra, which need not be regular. As a corollary we give a sufficient condition in order that a ring homomorphism is automatically linear or conjugate linear.

## 1. Introduction and results

Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras. We say that a map $\rho\colon \mathcal{A} \to \mathcal{B}$ is a ring homomorphism if $\rho$ preserves both addition and multiplication. That is,

$$\begin{aligned} \rho(f+g) &= \rho(f)+\rho(g), \\ \rho(fg) &= \rho(f)\rho(g) \end{aligned}$$

for every $f, g \in \mathcal{A}$. Moreover if such $\rho$ preserves scalar multiplication, then we say that $\rho$ is a homomorphism.

In this paper, $C(K)$ denotes the commutative Banach algebra of all complex-valued continuous functions on a compact Hausdorff space $K$. We say that a map $\rho\colon C(X) \to C(Y)$ is a $*$-ring homomorphism if $\rho$ is a ring homomorphism which also preserves complex conjugate: $\rho(\overline{f}) = \overline{\rho(f)}$ for every $f \in C(X)$. Šemrl [6] made a study of $*$-ring homomorphisms on $C(X)$ into $C(Y)$ and remarked that the problem of a general description of all ring homomorphisms on $C(X)$ into $C(Y)$ is much more difficult than the problem of characterizing all $*$-ring homomorphisms. In fact, let $G$ be the set of all surjective ring homomorphisms between the complex number field $\mathbb{C}$. It is well-known that the cardinal number of $G$ is $2^{\mathfrak{c}}$ (cf. [1]). Here $\mathfrak{c}$ denotes the cardinal number of $\mathbb{C}$.

Let $A$ be a unital regular semisimple commutative Banach algebra and $B$ a unital semisimple commutative Banach algebra, which need not be regular. In this paper, we consider a ring homomorphism $\rho\colon A \to B$ and give a representation of $\rho$; hence a description of a ring homomorphism on $C(X)$ into $C(Y)$ is given. This is an answer to the Šemrl's remark above. As a corollary, we can show [5, Theorem 1] and a unital version of [6, Theorem 5.2]. We also prove that an injective or a surjective ring

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homomorphism on $A$ to $B$ is linear or conjugate linear if the maximal ideal spaces of $A$ and $B$ are both infinite and if every constant function is mapped to a constant function.

Throughout this note, $A$ and $B$ denote a unital regular semisimple commutative Banach algebra and a unital semisimple commutative Banach algebra with the maximal ideal spaces $M_A$ and $M_B$, respectively. The units of $A$ and $B$ are denoted by the same symbol $e$. We simply write $f$ for the Gelfand transform of $f$. Before we state our main theorem, we need some terminologies.

**Definition 1.1.** Let $\rho\colon A \to B$ be a ring homomorphism. For each $y \in M_B$ we define the induced ring homomorphism $\rho_y\colon A \to \mathbb{C}$ and $\tilde{\rho}_y\colon \mathbb{C} \to \mathbb{C}$ as

$$\begin{aligned} \rho_y(f) &= \rho(f)(y) \quad (f \in A), \\ \tilde{\rho}_y(z) &= \rho(ze)(y) \quad (z \in \mathbb{C}). \end{aligned}$$

Moreover, $q_y\colon A \to A/\ker\rho_y$ denotes the quotient map for every $y \in M_B$.

A decomposition of a topological space $T$ is a family $\{T_1, T_2, \ldots, T_n\}$ of finitely many subsets $T_1, T_2, \ldots, T_n \subset T$ with the following properties:

$$T = \bigcup_{j=1}^{k} T_j \quad \text{and} \quad T_j \cap T_k = \emptyset \text{ if } j \neq k.$$

Note that each $T_j$ need not be clopen.

Let $\mathcal{A}$ be a commutative algebra with unit. Recall that $\mathcal{P}$ is a prime ideal of $\mathcal{A}$ if $\mathcal{P}$ is a proper ideal which satisfies that $fg \in \mathcal{P}$ implies $f \in \mathcal{P}$ or $g \in \mathcal{P}$. Here and after the term ideal will mean algebra ideal. In particular, every maximal ideal is a prime ideal. By Lemma 2.2, we see that the kernel $\ker\rho_y$ of the map $\rho_y\colon A \to \mathbb{C}$ is a prime ideal if $\ker\rho_y \neq A$. Hence, the quotient algebra $A/\ker\rho_y$ is an integral domain. Therefore, we can define the quotient field $\mathcal{F}_y$ of $A/\ker\rho_y$ if $\ker\rho_y \neq A$.

Now we are in a position to state our results.

**Theorem 1.1.** *Let $\rho\colon A \to B$ be a ring homomorphism. Then there exist a decomposition $\{M_{-1}, M_0, M_1, M_m, M_p\}$ of $M_B$ and a continuous map $\Phi\colon M_B \setminus M_0 \to M_A$ with the following property*:

*For every $y \in M_m \cup M_p$ there exists a non-zero field homomorphism $\tau_y\colon \mathcal{F}_y \to \mathbb{C}$ such that*

$$\rho(f)(y) = \begin{cases} \overline{f(\Phi(y))} & y \in M_{-1} \\ 0 & y \in M_0 \\ f(\Phi(y)) & y \in M_1 \\ \tau_y(f(\Phi(y))) & y \in M_m \\ \tau_y(q_y(f)) & y \in M_p \end{cases}$$

*for every $f \in A$.*

*Moreover, if $\rho$ is surjective then the map $\Phi$ is an injection defined on $M_B$ into $M_A$.*

**Corollary 1.2.** *Let $\rho\colon A \to B$ be an injective or a surjective ring homomorphism satisfying $\rho(\mathbb{C}e) \subset \mathbb{C}e$. If $M_A$ and $M_B$ are both infinite, then $\rho$ is linear or conjugate linear.*

Recall that a subset $S$ of $C(X)$ is separating if for each $x, y \in X$ with $x \neq y$ there corresponds an $f \in S$ so that $f(x) \neq f(y)$. We say that $S$ vanishes nowhere if for every $x \in X$ there exists a function $g$ of $S$ such that $g(x) \neq 0$.

**Corollary 1.3** (cf. Molnar, [5]). *Let $\rho\colon C(X) \to C(Y)$ be a ring homomorphism whose range contains a separating subalgebra of $C(Y)$. If the range $\rho(C(X))$ vanishes nowhere, then $\rho$ is surjective.*

**Corollary 1.4** (Šemrl, [6]). *Let $\rho\colon C(X) \to C(Y)$ be a $*$-ring homomorphism. Then there exist a clopen decomposition $\{Y_{-1}, Y_0, Y_1\}$ of $Y$ and a continuous map $\Phi\colon Y_{-1} \cup Y_1 \to X$ such that*

$$\rho(f)(y) = \begin{cases} \overline{f(\Phi(y))} & y \in Y_{-1} \\ 0 & y \in Y_0 \\ f(\Phi(y)) & y \in Y_1 \end{cases}$$

*for every $f \in C(X)$.*

## 2. Lemmas

Let $\tau\colon \mathbb{C} \to \mathbb{C}$ be a ring homomorphism. We simply say that $\tau$ is a ring homomorphism on $\mathbb{C}$. For example, $\tau(z) = 0$ $(z \in \mathbb{C})$, $\tau(z) = z$ $(z \in \mathbb{C})$ and $\tau(z) = \overline{z}$ $(z \in \mathbb{C})$ are ring homomorphisms on $\mathbb{C}$; we call them trivial ring homomorphisms.

**Proposition 2.1.** *Let $\tau$ be a ring homomorphism on $\mathbb{C}$. Then the following conditions are equivalent.*

(i) *$\tau$ is trivial.*
(ii) *There exist $m_0, L_0 > 0$ such that $|z| < m_0$ implies $|\tau(z)| \leq L_0$.*
(iii) *$\tau$ is continuous at $0$.*
(iv) *$\tau$ is continuous at every point of $\mathbb{C}$.*
(v) *$\tau$ preserves complex conjugate.*

*Proof.* (i) $\Rightarrow$ (ii) It is obvious.

(ii) $\Rightarrow$ (iii) It is enough to consider the case where $\tau$ is non-zero. Then by a simple calculation, we see that $\tau(r) = r$ for every $r \in \mathbb{Q}$, the rational number field of real numbers. For every $\varepsilon > 0$ fix an $r_0 \in \mathbb{Q}$ with $L_0 < r_0\varepsilon$. If $|z| < m_0/r_0$ then we have $|\tau(r_0 z)| \leq L_0$ by hypothesis. Since $\tau$ fixes every rational number, we obtain $|\tau(z)| \leq L_0/r_0 < \varepsilon$ if $|z| < m_0/r_0$. Thus $\tau$ is continuous at 0.

(iii) $\Rightarrow$ (iv) Let $\{z_n\}$ be a sequence converging to $z$. Since $\tau$ is continuous at 0, we see that $\tau(z_n - z) \to 0$ as $n \to \infty$. Hence $\tau(z_n)$ converges to $\tau(z)$.

(iv) $\Rightarrow$ (v) We consider the case where $\tau$ is non-zero. Then $\tau(r) = r$ for every $r \in \mathbb{Q}$. Since $\tau$ is continuous, we have that $\tau(t) = t$ for every $t \in \mathbb{R}$, the real number field. We also have that $\tau(i) = \pm i$ since $\tau(-1) = -1$. This implies that $\tau(\bar{z}) = \overline{\tau(z)}$ for every $z \in \mathbb{C}$.

(v) $\Rightarrow$ (i). By hypothesis, we have $\tau(\mathbb{R}) \subset \mathbb{R}$, and hence $\tau(x+h^2) - \tau(x) = \{\tau(h)\}^2 \geq 0$ for every $x, h \in \mathbb{R}$. It follows that $\tau(x) \geq \tau(y)$ for $x, y \in \mathbb{R}$ with $x \geq y$. If $\tau$ is non-zero, then $\tau$ fixes all $r \in \mathbb{Q}$. Therefore, we obtain $\tau(x) = x$ for $x \in \mathbb{R}$, so that $\tau$ is trivial. □

As remarked in the previous section, there exist non-trivial ring homomorphisms on $\mathbb{C}$. By Proposition 2.1, non-trivial ring homomorphisms are discontinuous at each point of $\mathbb{C}$. Moreover a non-trivial ring homomorphism $\tau$ on $\mathbb{C}$ has the following property:

> For every pair $m, L > 0$ there exists a $z \in \mathbb{C}$ such that $|z| < m$ but $|\tau(z)| > L$.

It is well-known that the kernels of non-zero complex homomorphisms on a unital commutative Banach algebra are maximal ideals. Let $\mathbb{N}$ be the space of all natural numbers and $K_0 = \{0\} \cup \{1/n; n \in \mathbb{N}\}$ with its usual topology. Šemrl showed the existence of a non-zero complex ring homomorphism $\varphi$ on $C(K_0)$ whose kernel $\ker\varphi$ is not a maximal ideal of $C(K_0)$ ([6, Example 5.4]). We show that the kernel $\ker\phi$ of a non-zero complex ring homomorphism $\phi$ on $A$ is a prime ideal that is contained in a unique maximal ideal. De Marco and Orsatti [4] gave a characterization of a commutative ring with unit of which each prime ideal containing the Jacobson radical is contained in a unique maximal ideal.

**Lemma 2.2.** *Let* $\phi\colon A \to \mathbb{C}$ *be a non-zero ring homomorphism. Then the kernel* $\ker\phi$ *is a prime ideal which is contained in a unique maximal ideal of* $A$*.*

*Proof.* As a first step, we show that $\ker\phi$ is an ideal of $A$. Since $\phi$ preserves both addition and multiplication, it is enough to show that $zf$ belongs to $\ker\phi$ for every $z \in \mathbb{C}$ and $f \in \ker\phi$. Note that $\phi(e) = 1$ since $\phi$ is non-zero.

Therefore, we have

$$\phi(zf) = \phi(zf)\phi(e) = \phi(f)\phi(ze) = 0$$

for every $z \in \mathbb{C}$ and $f \in \ker\phi$. Hence $\ker\phi$ is an ideal of $A$. It is now obvious that $\ker\phi$ is a prime ideal.

Since $\ker\phi$ is a proper ideal, there corresponds an $x_0 \in M_A$ such that $\ker\phi \subset \{f \in A; f(x_0) = 0\}$. We show that $\{f \in A; f(x_0) = 0\}$ is the unique maximal ideal containing $\ker\phi$. To this end, assume to the contrary that there exists an $x_1 \in M_A$ such that $x_0 \neq x_1$ and $\ker\phi \subset \{f \in A; f(x_1) = 0\}$. Let $V_j$ be an open neighborhood of $x_j$ for $j = 0, 1$ so that $V_0 \cap V_1 = \emptyset$. Since $A$ is regular, there corresponds an $f_j \in A$ such that

$$f_j(x_j) = 1 \quad \text{and} \quad f_j(M_A \setminus V_j) = 0 \quad (j = 0, 1).$$

Then $f_0 f_1 = 0$ on $M_A$. Since $\ker\phi$ is a prime ideal, $f_0$ or $f_1$ belongs to $\ker\phi$. This is a contradiction since $f_j(x_j) = 1$ for $j = 0, 1$. Hence $\ker\phi$ is contained in the unique maximal ideal $\{f \in A; f(x_0) = 0\}$. □

**Lemma 2.3.** *Let $\phi\colon A \to \mathbb{C}$ be a non-zero ring homomorphism and $q\colon A \to A/\ker\phi$ the quotient map. Then $\phi$ is of the form $\phi = \tau \circ q$ for some non-zero field homomorphism $\tau$ on the quotient field $\mathcal{F}$ of $A/\ker\phi$. If, in addition, $\ker\phi$ is a maximal ideal of $A$, then we may consider $\tau$ a non-zero ring homomorphism on $\mathbb{C}$ and $q \in M_A$.*

*Proof.* Note that the quotient field $\mathcal{F}$ of $A/\ker\phi$ is well-defined since $\ker\phi$ is a prime ideal of $A$, by Lemma 2.2. We define the map $\tau\colon \mathcal{F} \to \mathbb{C}$ by

$$(\sharp) \qquad \tau([f]/[g]) = \frac{\rho(f)}{\rho(g)} \quad ([f]/[g] \in \mathcal{F}).$$

Here $[f] \in A/\ker\phi$ denotes the equivalence class of $f \in A$ with respect to $\ker\phi$. Then $\tau$ is a well-defined non-zero field homomorphism on $\mathcal{F}$. If we identify $[f]$ with $[f]/[e]$, it is obvious that $\phi$ is of the form $\phi = \tau \circ q$.

Moreover if $\ker\phi$ is a maximal ideal of $A$, then the quotient algebra $A/\ker\phi$ is isometrically isomorphic to $\mathbb{C}$. Thus, we may identify $A/\ker\phi$ with the quotient field $\mathcal{F}$ of $A/\ker\phi$. Let $I$ be the isomorphism on $A/\ker\phi$ onto $\mathbb{C}$. Then $\tau \circ I^{-1}$ is a ring homomorphism on $\mathbb{C}$ and $I \circ q$ a non-zero complex homomorphism on $A$ with $\rho = \tau \circ q = (\tau \circ I^{-1}) \circ (I \circ q)$. This completes the proof. □

**Definition 2.1.** Let $\rho\colon A \to B$ be a ring homomorphism. Put $M_0 = \{y \in M_B; \ker\rho_y = A\}$. We define the subsets $M_{B(m)}$ and $M_{B(p)}$ of $M_B \setminus M_0$ as

$$M_{B(m)} = \{y \in M_B \setminus M_0; \ker\rho_y \text{ is a maximal ideal of } A\},$$
$$M_{B(p)} = \{y \in M_B \setminus M_0; \ker\rho_y \text{ is not a maximal ideal of } A\}.$$

Let $M_{-1}$, $M_1$, $M_{m,-1}$ and $M_{m,1}$ be as follows:

$$\begin{aligned} M_{-1} &= \{y \in M_{B(m)}; \tilde{\rho}_y(z) = \overline{z}\ (z \in \mathbb{C})\}, \\ M_1 &= \{y \in M_{B(m)}; \tilde{\rho}_y(z) = z\ (z \in \mathbb{C})\}, \\ M_{m,-1} &= \{y \in M_{B(m)}; \tilde{\rho}_y \text{ is non-trivial and } \tilde{\rho}_y(i) = -i\}, \\ M_{m,1} &= \{y \in M_{B(m)}; \tilde{\rho}_y \text{ is non-trivial and } \tilde{\rho}_y(i) = i\}. \end{aligned}$$

The subsets $M_{p,-1}$ and $M_{p,1}$ of $M_{B(p)}$ are defined by

$$\begin{aligned} M_{p,-1} &= \{y \in M_{B(p)}; \tilde{\rho}_y(i) = -i\}, \\ M_{p,1} &= \{y \in M_{B(p)}; \tilde{\rho}_y(i) = i\}. \end{aligned}$$

Then we write $M_{d,j} = M_{m,j} \cup M_{p,j}$ $(j = -1, 1)$ and $M_d = M_{d,-1} \cup M_{d,1}$.

Note that $\tilde{\rho}_y$ is a non-trivial ring homomorphism on $\mathbb{C}$ for every $y \in M_d$. For if $\tilde{\rho}_y$ is trivial then

$$\rho_y(zf) = \tilde{\rho}_y(z)\rho_y(f) \quad (z \in \mathbb{C}, f \in A)$$

implies that $\ker \rho_y$ is maximal for every $y \in M_B \setminus M_0$. By definition, the subsets $M_{-1}$, $M_0$, $M_1$ and $M_d$ of $M_B$ are mutually disjoint and $M_B = M_{-1} \cup M_0 \cup M_1 \cup M_d$. Hence, $\{M_{-1}, M_0, M_1, M_d\}$ above is a decomposition of $M_B$. We call $\{M_{-1}, M_0, M_1, M_d\}$ *the decomposition of $M_B$ with respect to $\rho$.*

Until the end of this section, $\rho\colon A \to B$ denotes a ring homomorphism and $\{M_{-1}, M_0, M_1, M_d\}$ the decomposition of $M_B$ with respect to $\rho$.

**Lemma 2.4.** *The sets $M_0$, $M_{-1} \cup M_{d,-1}$ and $M_1 \cup M_{d,1}$ are clopen in $M_B$. Also $M_{-1}$ and $M_1$ are both closed in $M_B$.*

*Proof.* By definition, it is easy to see that

$$\begin{aligned} M_0 &= \{y \in M_B; \tilde{\rho}_y(i) = 0\}, \\ M_{-1} \cup M_{d,-1} &= \{y \in M_B; \tilde{\rho}_y(i) = -i\}, \\ M_1 \cup M_{d,1} &= \{y \in M_B; \tilde{\rho}_y(i) = i\}. \end{aligned}$$

Therefore, $M_0$, $M_{-1} \cup M_{d,-1}$ and $M_1 \cup M_{d,1}$ are clopen since the function $\rho(ie)$ is continuous on $M_B$.

Next, we show that $M_1$ is closed in $M_B$. For every $y \in M_{d,1}$ we can find a $z_0 \in \mathbb{C}$ such that $\tilde{\rho}_y(z_0) \neq z_0$ since $\tilde{\rho}_y$ is non-trivial. Put

$$V = \{w \in M_B; |\rho(z_0e)(w) - \rho(z_0e)(y)| < |z_0 - \tilde{\rho}_y(z_0)|/2\}.$$

Then $V$ is an open neighborhood of $y$ with $V \cap M_1 = \emptyset$. Since $M_1 \cup M_{d,1}$ is clopen, this implies that $M_1$ is closed. In a way similar to the above, we see that $M_{-1}$ is closed and the proof is omitted. □

**Definition 2.2.** By Lemma 2.2, for every $y \in M_B \setminus M_0$ there exists a unique $x \in M_A$ such that $\ker \rho_y \subset \{f \in A; f(x) = 0\}$. We denote the correspondence defined on $M_B \setminus M_0$ into $M_A$ as $\Phi$; That is, $\ker \rho_y$ is contained in the unique maximal ideal $\{f \in A; f(\Phi(y)) = 0\}$ for every $y \in M_B \setminus M_0$. We call $\Phi$ the representing map for $\rho$.

**Lemma 2.5.** *Let $r \in \mathbb{Q}$, $G$ open in $M_A$ and $\Phi$ the representing map for $\rho$. Suppose that $h \in A$ satisfies $h(G) = r$ then $\rho_y(h) = r$ for every $y \in \Phi^{-1}(G)$.*

*Proof.* Put $h_r = h - re \in A$ and fix $y \in \Phi^{-1}(G)$. Since $A$ is regular, there exists a function $g \in A$ such that $g(\Phi(y)) = 1$ and $g(M_A \setminus G) = 0$. Then $gh_r = 0$ on $M_A$. Since $\ker \rho_y$ is a prime ideal, $g$ or $h_r$ belongs to $\ker \rho_y$. On the other hand, $g$ does not belong to $\{f \in A; f(\Phi(y)) = 0\}$ since $g(\Phi(y)) = 1$. So we conclude that $h_r \in \ker \rho_y$. Therefore we have $\rho_y(h) = r$ for every $y \in \Phi^{-1}(G)$. □

**Lemma 2.6.** *Let $\Phi$ be the representing map for $\rho$. Then the range $\Phi(M_d)$ is at most finite.*

*Proof.* Assume to the contrary that $\Phi(M_d)$ has a countable subset $\{x_n\}_{n=1}^{\infty}$ such that $x_j \neq x_k$ if $j \neq k$. Without loss of generality, we may assume that each $x_j$ is an isolated point of $\{x_n\}_{n=1}^{\infty}$. By definition, for every $n \in \mathbb{N}$ there exists a $y_n \in M_d$ such that $x_n = \Phi(y_n)$. By induction, we can find an open neighborhood $U_j$ of $x_j$ with

$$(\overline{U}_j \setminus \{x_j\}) \cap \{x_n\}_{n=1}^{\infty} = \emptyset \quad \text{and} \quad \overline{U}_{j+1} \subset M_A \setminus \bigcup_{k=1}^{j} \overline{U}_k$$

for every $j \in \mathbb{N}$. Here $\overline{U}_j$ denotes the closure of $U_j$ in $M_A$. Let $V_j$ be an open neighborhood of $x_j$ so that $\overline{V}_j \subset U_j$. Since $A$ is regular, $A$ is normal (cf. [2, Theorem 6.3 of Chapter I]). That is, there exists a $g_j \in A$ such that $g_j(\overline{V}_j) = 1$ and $g_j(M_A \setminus U_j) = 0$. Since $\tilde{\rho}_{y_j}$ is non-trivial, there corresponds a $z_j \in \mathbb{C}$ so that

$$|z_j| < (2^j \|g_j\|)^{-1} \quad \text{and} \quad |\tilde{\rho}_{y_j}(z_j)| > 2^j,$$

by Proposition 2.1. Here $\|\cdot\|$ denotes the Banach norm on $A$. Put $f_j = z_j g_j \in A$. Then $\rho_y(f_j) = \tilde{\rho}_y(z_j)\rho_y(g_j)$ for every $y \in M_B$. Therefore, by Lemma 2.5 we see that $\rho_{y_j}(f_j) = \tilde{\rho}_{y_j}(z_j)$. Since $\|f_j\| < 2^{-j}$, the series $\sum_{n=1}^{\infty} f_n$ converges in $A$, say $f_0$. Note that $f_j = 0$ on $V_k$ if $k \neq j$. Thus we see that $f_0 = f_j$ on $V_j$ for every $j \in \mathbb{N}$. By Lemma 2.5, we obtain $\rho_{y_j}(f_0 - f_j) = 0$. Therefore,

$$|\rho_{y_j}(f_0)| = |\rho_{y_j}(f_j)| = |\tilde{\rho}_{y_j}(z_j)| > 2^j \quad (j \in \mathbb{N}).$$

This is a contradiction since $\rho(f_0)$ is bounded on $M_B$. Hence we have proved that the range $\Phi(M_d)$ is at most finite. □

## 3. A proof of main result

*Proof of Theorem* 1.1. Let $\{M_{-1}, M_0, M_1, M_d\}$ and $\Phi$ be the decomposition of $M_B$ with respect to $\rho$ and the representing map for $\rho$, respectively. For every $y \in M_B \setminus M_0$, let $q_y\colon A \to A/\ker\rho_y$ denote the quotient map. Recall that $M_{B(m)}$ is the set of all $y \in M_B$ so that $\ker\rho_y$ is a maximal ideal of $A$. By Lemma 2.3, we can find a field homomorphism $\tau_y$ on the quotient field $\mathcal{F}_y$ of the integral domain $A/\ker\rho_y$ into $\mathbb{C}$ such that $\rho_y = \tau_y \circ q_y$. If, in addition, $y \in M_{B(m)}$, then we may consider that $\tau_y$ is a ring homomorphism on $\mathbb{C}$ and $q_y \in M_A$. In this case, we therefore have $\ker q_y = \ker\rho_y = \ker\Phi(y)$. Hence, we see that $q_y = \Phi(y)$ for every $y \in M_{B(m)}$. By the formula ($\sharp$), we also have $\tau_y = \tilde{\rho_y}$ for every $y \in M_{B(m)}$. That is, $\tau_y(z) = \overline{z}$ if $y \in M_{-1}$, $\tau_y(z) = z$ if $y \in M_1$ and $\tau_y$ is non-trivial if $y \in M_{m,-1} \cup M_{m,1}$. Therefore, we have

$$\rho(f)(y) = \begin{cases} 0 & y \in M_0 \\ \tau_y(f(\Phi(y))) & y \in M_{B(m)} \\ \tau_y(q_y(f)) & y \in M_{B(p)} \end{cases}$$
$$= \begin{cases} \overline{f(\Phi(y))} & y \in M_{-1} \\ 0 & y \in M_0 \\ f(\Phi(y)) & y \in M_1 \\ \tau_y(f(\Phi(y))) & y \in M_{m,-1} \cup M_{m,1} \\ \tau_y(q_y(f)) & y \in M_{p,-1} \cup M_{p,1} \end{cases}$$

for every $f \in A$.

By Lemma 2.6, we may put $\Phi(M_d) = \{x_1, x_2, \ldots, x_m\}$. Then we see that the set $M_d(x_j) = \{y \in M_d; \Phi(y) = x_j\}$ is open in $M_B$ for $j = 1, 2, \ldots, m$. Indeed, assume to the contrary that $M_d(x_j)$ is not open. Then there exist a $y_j \in M_d(x_j)$ and a net $\{y_\alpha\}$ in $M_B \setminus M_d(x_j)$ such that $y_\alpha$ converges to $y_j$. Since $M_{-1} \cup M_0 \cup M_1$ is closed in $M_B$ by Lemma 2.4, we see that $M_d$ is an open subset of $M_B$. Therefore, without loss of generality we may assume $\{y_\alpha\} \subset M_d \setminus M_d(x_j)$. Fix open neighborhoods $O_1, O_2$ of $x_j$ with $\overline{O}_1 \subset O_2$ and $\overline{O}_2 \cap \Phi(M_d) = \{x_j\}$. Here, $\overline{\cdot}$ denotes the closure in $M_A$. Since $A$ is regular, we can find a function $h_j \in A$ so that $h_j(\overline{O}_1) = 1$ and $h_j(M_A \setminus O_2) = 0$. By Lemma 2.5, we have that $\rho_{y_j}(h_j) = 1$ and $\rho_{y_\alpha}(h_j) = 0$ for every $\alpha$. This is a contradiction since $\rho(h_j)$ is continuous on $M_B$. Therefore, the set $M_d(x_j) = \{y \in M_d; \Phi(y) = x_j\}$ is open in $M_B$ for $j = 1, 2, \ldots, m$.

Finally we show that the map $\Phi$ on $M_B \setminus M_0$ into $M_A$ is continuous. Indeed, we see that $\Phi$ is continuous at each $y_0 \in M_d$ since $M_d(\Phi(y_0)) = \{y \in M_d; \Phi(y) = \Phi(y_0)\}$ is open as proved above. We show that $\Phi$ is continuous on $M_{-1} \cup M_1$. Let $y_1$ be a point of $M_1$ and $\{y_\beta\}_{\beta\in\Gamma}$ an arbitrary net in $M_B \setminus M_0$ converging to $y_1$. Since $M_0 \cup M_{-1}$ is closed in $M_B$, we see that $M_1 \cup M_d$ is an open subset of $M_B$. Hence, without loss of generality we may assume $\{y_\beta\}_{\beta\in\Gamma} \subset M_1 \cup M_d$. We assert that there exists a $\beta_0 \in \Gamma$ such that $y_\beta \in M_1 \cup \{y \in M_d; \Phi(y) = \Phi(y_1)\}$ for every $\beta \in \Gamma$ with $\beta \geq \beta_0$. In fact, let $W_1$ be an open neighborhood of $\Phi(y_1)$ and $W_2$ an open subset containing $\Phi(M_d) \setminus \{\Phi(y_1)\}$ so that $\overline{W}_1 \cap \overline{W}_2 = \emptyset$. Then we can find a $g_0 \in A$ such that $g_0(\overline{W}_1) = 1$ and $g_0(\overline{W}_2) = 0$. By Lemma 2.5, we see that $\rho_{y_1}(g_0) = 1$ and $\rho_y(g_0) = 0$ for every $y \in \Phi^{-1}(W_2)$. By the continuity of $\rho(g_0)$, there exists a $\beta_0 \in \Gamma$ such that $\beta \geq \beta_0$ implies $|\rho(g_0)(y_\beta) - 1| < 1/2$. That is, $\Phi(y_\beta) \notin \Phi(M_d) \setminus \{\Phi(y_1)\}$ if $\beta \geq \beta_0$. Therefore, we see that $y_\beta \in M_1 \cup \{y \in M_d; \Phi(y) = \Phi(y_1)\}$ for every $\beta \in \Gamma$ with $\beta \geq \beta_0$. Hence, if $\beta \geq \beta_0$ then we have

$$f(\Phi(y_\beta)) = \begin{cases} \rho(f)(y_\beta) & y_\beta \in M_1 \\ f(\Phi(y_1)) & \Phi(y_\beta) = \Phi(y_1) \end{cases}$$

for every $f \in A$. Consequently, $\beta \geq \beta_0$ implies that

$$|f(\Phi(y_\beta)) - f(\Phi(y_1))| \leq |\rho(f)(y_\beta) - \rho(f)(y_1)|$$

for every $f \in A$. Thus $\Phi(y_\beta)$ converges to $\Phi(y_1)$. This implies that $\Phi$ is continuous on $M_1$. In a way similar to the above, we can show that $\Phi$ is continuous on $M_{-1}$ and the proof is omitted. Thus, we have proved that the map $\Phi$ is continuous on $M_B \setminus M_0$.

Suppose that $\rho$ is surjective. Then $M_0$ is an empty set. Hence $\Phi$ is the map defined on $M_B$ into $M_A$. We show that $\ker \rho_y = \{f \in A; f(\Phi(y)) = 0\}$. Recall that $\ker \rho_y \subset \{f \in A; f(\Phi(y)) = 0\}$. So it is enough to show that $\rho_y(f) \neq 0$ implies $f(\Phi(y)) \neq 0$. Let $a \in A$ satisfy $\rho_y(a) \neq 0$. Since $\rho_y(A) = \mathbb{C}$, there corresponds a $b \in A$ such that $\rho_y(a)\rho_y(b) = 1$. Therefore, $ab - e$ belongs to $\ker \rho_y$. We conclude that $a(\Phi(y)) \neq 0$ since $(ab - e)(\Phi(y)) = 0$. Thus, we have proved that $\ker \rho_y = \{f \in A; f(\Phi(y)) = 0\}$. Hence $M_B = M_{-1} \cup M_1 \cup M_{m,-1} \cup M_{m,1}$.

Let $w_1, w_2 \in M_B$ satisfy $w_1 \neq w_2$. Since $\rho$ is surjective, there exists an $a_0 \in A$ such that $\rho(a_0)(w_1) = 1$ and $\rho(a_0)(w_2) = 0$. By the formula for $\rho$, it is easy to see that

$$a_0(\Phi(w_1)) = 1 \quad \text{and} \quad a_0(\Phi(w_2)) = 0.$$

Therefore, we have $\Phi(w_1) \neq \Phi(w_2)$. This implies that $\Phi$ is injective. □

*Proof of Corollary* 1.2. Let $\{M_{-1}, M_0, M_1, M_{d,-1}, M_{d,1}\}$ be the decomposition of $M_B$ with respect to $\rho$ and $\Phi$ the representing map for $\rho$. Since $\rho(\mathbb{C}e) \subset \mathbb{C}e$, we have $M_B = M_{-1} \cup M_{d,-1}$ or $M_B = M_0$ or $M_B = M_1 \cup M_{d,1}$. It is enough to consider the case where $M_B = M_{-1} \cup M_{d,-1}$ or $M_B = M_1 \cup M_{d,1}$.

Suppose that $M_B = M_1 \cup M_{d,1}$. First, we show that $M_1 \neq \emptyset$. Suppose not. Then $M_B = M_{d,1}$. If $\rho$ is surjective, the map $\Phi$ is injective by Theorem 1.1. Since $\Phi(M_{d,1})$ is finite by Lemma 2.6, so is $M_{d,1} = M_B$. This is a contradiction. Therefore, $M_1 \neq \emptyset$ if $\rho$ is surjective. Consider the case where $\rho$ is injective. Since $M_A$ is infinite, there exists an $x_0 \in M_A \setminus \Phi(M_{d,1})$. We can find an open subset $V$ of $M_A$ so that $\Phi(M_{d,1}) \subset V$ and $x_0 \notin \overline{V}$. Since $A$ is regular, there corresponds an $f_0 \in A$ such that $f_0(x_0) = 1$ and $f_0(\overline{V}) = 0$. By Lemma 2.5 we see that $\rho_y(f_0) = 0$ for every $y \in M_{d,1} = M_B$. Since $f_0$ is not identically zero, this contradicts that $\rho$ is injective. Consequently, we have that $M_1 \neq \emptyset$.

Now we show that $M_B = M_1$. Suppose that there exists a $y_1 \in M_{d,1}$. Since $\tilde{\rho}_{y_1}$ is non-trivial, we can find a $z_1 \in \mathbb{C}$ such that $\tilde{\rho}_{y_1}(z_1) \neq z_1$. Note that $\tilde{\rho}_y(z_1) = z_1$ for every $y \in M_1$. This is a contradiction since $\rho(\mathbb{C}e) \subset \mathbb{C}e$. Therefore, we have proved that $M_B = M_1$ if $M_B = M_1 \cup M_{d,1}$. In a way similar to the above, we see that $M_B = M_{-1}$ if $M_B = M_{-1} \cup M_{d,-1}$. Hence, $\rho$ is linear or conjugate linear. □

*Proof of Corollary* 1.3. Let $\{Y_{-1}, Y_0, Y_1, Y_d\}$ be the decomposition of $Y$ with respect to $\rho$ and $\Phi$ the representing map for $\rho$. Since the range $\rho(C(X))$ vanishes nowhere, we see that $Y_0$ is an empty set. Since $\rho(C(X))$ contains a separating subalgebra, in a way similar to the proof of Theorem 1.1, we can prove that $\ker \rho_y$ is a maximal ideal for every $y \in Y$ and that $\Phi \colon Y \to X$ is injective. Hence, $Y$ is homeomorphic to the range $\Phi(Y)$. Let $\varphi \colon \Phi(Y) \to Y$ be the homeomorphism defined by

$$\varphi(x) = \Phi^{-1}(x) \quad (x \in \Phi(Y)).$$

Note that

$$\rho(f)(y) = \begin{cases} \overline{f(\Phi(y))} & y \in Y_{-1} \\ f(\Phi(y)) & y \in Y_1 \\ \tau_y(f(\Phi(y))) & y \in Y_d \end{cases}$$

for every $f \in C(X)$. Here $\tau_y$ denotes a non-trivial ring homomorphism on $\mathbb{C}$. We define the continuous function $h \colon \Phi(Y) \to \mathbb{C}$ by

$$h(x) = \begin{cases} \overline{g(\varphi(x))} & x \in \Phi(Y_{-1}) \\ g(\varphi(x)) & x \in \Phi(Y_1) \\ {\tau_{\varphi(x)}}^{-1}(g(\varphi(x))) & x \in \Phi(Y_d) \end{cases}$$

for each $g \in C(Y)$. Since $\Phi(Y_{-1})$, $\Phi(Y_1)$ and $\Phi(Y_d)$ are disjoint closed subsets of the compact Hausdorff space $X$, there exists an $\tilde{h}$ of $C(X)$ such that $\tilde{h}|_{\Phi(Y)} = h$. Then it is easy to see that $\rho(\tilde{h}) = g$. Hence $\rho$ is surjective. □

*Proof of Corollary* 1.4. Let $\{Y_{-1}, Y_0, Y_1, Y_d\}$ be the decomposition of $Y$ with respect to $\rho$ and $\Phi$ the representing map for $\rho$. Since $\rho$ preserves complex conjugate, by Proposition 2.1 we have that $\tilde{\rho}_y$ is trivial for every $y \in Y$.

Therefore, $Y_d$ is an empty set. By Lemma 2.4, we see that $Y_{-1}$, $Y_0$ and $Y_1$ are all clopen. This completes the proof. □

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