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On Azumaya's exact rings

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ON AZUMAYA'S EXACT RINGS

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With best wishes to Professor Hisao Tominaga for his 60th birthday

Azumaya [2] and Morita [8] proved that there is a duality between the finitely generated left modules over a ring R and the finitely generated right modules over some ring S if and only if R is left artinian and the indecomposable injective left R-modules are all finitely generated. To this day there are no known examples of artinian rings with indecomposable injective modules having the correct composition lengths that do not have *self duality*, i. e., a duality between their categories of finitely generated left and right modules; and there are very few (besides artin algebras, quasi-Frobenius rings and serial rings [13], [14]) that are actually known to possess such a duality. In this paper we consider Azumaya's exact rings and his conjecture that they have self duality [3].

Azumaya [3] called a ring R *exact* in case R is left artinian and has a composition series of two-sided ideals

$$_R R_R = I_0 > I_1 > \cdots > I_n = 0$$

such that for each $i = 1, \ldots, n$ every left endomorphism of I_{i-1}/I_i is given by right multiplication by an element of R. As he observed, commutative artinian rings (well known to have self duality) are exact. He proved that this notion is left-right symmetric, that indecomposable injective modules do have the correct composition lengths (i. e., $c(Re/Je) = c(E(eR/eJ))$, $e = e^2 \in R$) over exact rings, and that Nakayama's serial (or generalized uniserial) rings and split algebras are exact rings.

After adapting Azumaya's exactness to bimodules, we prove in Section 1 that exactness is a Morita invariant for rings. Thus in subsequent work on the subject we need only consider basic rings. In Section 2 we present several characterizations of exact rings; for example, an artin ring is exact whenever it is exact modulo the square of its radical. One consequence of these is that any artinian ring R with $R/\mathrm{rad}(R)$ a finite dimensional algebra over an algebraically closed field is an exact ring. In the remaining section, we verify Azumaya's conjecture for certain trivial extensions of semisimple rings.

We freely use the terminology of [1]. Also we will always denote the Jacobson radical of a ring R by J.

We are indebted to Steve Landsburg for Examples 2.9 and 3.3.

1. Exact bimodules and Morita equivalence. Azumaya's argument [3] showing that his notion of exactness is left-right symmetric in fact serves to prove the following result on bimodules over rings that are semilocal (i. e., artinian modulo their radical).

1.1. Proposition. *Let R and S be semilocal rings and let $_RM_S$ be a bimodule with composition series*

$$_RM_S = M_0 > M_1 > \cdots > M_n = 0$$

of bi-submodules. Then the following are equivalent:
    (a) $_RM$ *has a composition series and for each $i = 1,\ldots, n$* $\text{End}(_RM_{i-1}/M_i)$ *consists of multiplications by elements of S;*
    (b) M_{i-1}/M_i *is a balanced R-S bimodule for each $i = 1,\ldots, n$;*
    (c) M_S *has a composition series and for each $i = 1,\ldots, n$* $\text{End}(M_{i-1}/M_{iS})$ *consists of multiplications by elements of R.*

If the bimodule $_RM_S$ has a composition series

$$_RM_S = M_0 > M_1 > \cdots > M_n = 0$$

we shall indicate R and S modulo the left and right annihilators of M by $\overline{R} = R/\text{ann}_R(M)$ and $\overline{S} = S/\text{ann}_S(M)$, and we shall write $\overline{R}_i$ and $\overline{S}_i$ for R and S modulo the left and right annihilators of M_{i-1}/M_i. As Azumaya noted, if R and S are semilocal then $\overline{R}_i$ and $\overline{S}_i$ are simple artinian; and Proposition 1.1 follows from the observation that if $_RM_{i-1}/M_i$ is finitely generated and $\overline{S}_i \cong \text{End}(_RM_{i-1}/M_i)$ canonically, then M_{i-1}/M_1 is a progenerator over both $\overline{R}_i$ and $\overline{S}_i$.

We shall say that a bimodule that has a composition series whose composition factors are balanced is an *exact bimodule* and that R is an *exact ring* in case the regular bimodule $_RR_R$ is exact. Thus the rings that Azumaya studied in [3] (and the ones that we shall be principally concerned with) are exact artinian rings. According to the Jordan-Holder Theorem any (two-sided) composition factor of an exact bimodule is balanced, and if $K \leq {}_RM_S$ then M is an exact R-S bimodule iff K and M/K are.

As we shall see, exact bimodules are useful tools in the study of exact rings. First, however, we note that they are preserved by Morita equivalence.

1.2. Lemma. *If $F: R\text{-Mod} \to S\text{-Mod}$ is an equivalence of categories and $_RM_T$ is exact then so is the canonically induced bimodule $_SF(M)_T$.*

Proof. The equivalence F induces an isomorphism between the lattices of submodules of $_RM$ and $_SF(M)$ under which R-T submodules of M correspond to S-T submodules of $F(M)$. Thus we may assume that $_RM_T$ is (simple and) balanced, so that $_RM$ is a balanced R-module. But then by [9] or [4], $_SF(M)$ is a balanced S-module; and since $_RM_T$ is balanced, so is $_RF(M)_T$.

Now it is a simple matter to show that exactness is preserved under Morita equivalence of rings.

1.3. Theorem. *If R is an exact ring then so is every ring that is Morita equivalent to R.*

Proof. Suppose that S is Morita equivalent to R. Then (see [1, Section 22]) there is a balanced bimodule $_SP_R$ (with $_SP$ and P_R progenerators) such that $(_SP\otimes_R-) : R$-Mod $\to$ S-Mod and $\mathrm{Hom}_R(_SP_R, -) :$ Mod-$R \to$ Mod-S are category equivalences. If R is an exact ring then by Lemma 1.2, $_SP_R \cong (_SP \otimes_R R_R)$ is exact and so is $_SS_S \cong \mathrm{Hom}_R(_SP_R, {}_SP_R)$.

This last proof shows that over exact rings progenerators are exact. It follows that so are finitely generated projective modules, as we shall see after the next lemma.

1.4. Lemma. *Let $_RM_S$ be an exact bimodule. If e is an idempotent in S such that $Me \neq 0$ then $_RMe_{eSe}$ is an exact bimodule.*

Proof. Let

$$_RM_S = M_0 > M_1 > \cdots > M_n = 0$$

be a composition series for $_RM_S$ and suppose that e is an idempotent in S with $Me \neq 0$. Then choosing $i_0 < \cdots < i_k$ so that $M_{i_j}e$ are the distinct members of $M_1e,\ldots, M_ne$ we obtain a composition series

$$_RMe_{eSe} = M_{i_0}e > M_{i_1}e > \cdots > M_{i_k}e = 0$$

for $_RMe_{eSe}$ with each

$$M_{i_{j-1}}e/M_{i_j}e \cong (M_{i_{j-1}}/M_{i_j})e$$

as R-eSe bimodules. Thus we need only show that if $_RM_S$ is simple and balanced then the simple bimodule $_RMe_{eSe}$ is also balanced. If $_RM_S$ is balanced then multiplication by the elements of S comprises the endomorphism ring of $_RM$ so $_RM$ is a balanced module; and if $_RM_S$ is simple then $MeS = M$ and

$\text{ann}_M(Se) = 0$, i.e., the direct summand $_RMe$ of $_RM$ generates and cogenerates $_RM$, so by $[1, 14.1]$ $_RMe$ is balanced. Therefore, if $_RM_S$ is simple and balanced then so is $_RMe_{eSe}$.

1.5. Corollary. *If R is an exact ring then any finitely generated projective module $_RP$ is an exact R-$\text{End}(_RP)$ bimodule.*

Proof. Let $_RF$ be a finitely generated free module with $_RF \cong P \oplus P'$, let $S = \text{End}(_RF)$ and let e be an idempotent in S with $P = Fe$. Then by Theorem 1.3 S is an exact ring, by Lemma 1.2 $_RF_S \cong F \otimes_S S$ is exact, and by Lemma 1.4 $_RP_{\text{End}(_RP)}$ is exact since $\text{End}(_RP) = eSe$ canonically.

2. Exact artinian rings. In this section we turn to Azumaya's exact artinian rings, obtaining several characterizations. Azumaya $[3]$ proved that split finite dimensional algebras and artinian serial rings are exact. We apply our characterizations to extend these results and to show that any artinian ring R such that R/J is a finite dimensional algebra over an algebraically closed field is an exact ring.

Henceforth, we shall assume all rings under consideration to be artinian; so if $_RM_S$ is a simple bimodule then $\bar{R}$ and $\bar{S}$ are simple artinian, and a ring R is basic in case R/J is a direct sum of division rings.

2.1. Lemma. *Let R and S be basic rings. Then a simple bimodule $_RM_S$ is balanced if and only if $_RM$ and M_S are simple modules, i.e., $\dim(_{\bar{R}}M) = 1 = \dim(M_{\bar{S}})$.*

Proof. $(\Rightarrow)$ If $\text{End}(_RM) \cong \bar{S}$ and $\text{End}(M_S) \cong \bar{R}$ are division rings then $_{\bar{R}}M$ and $M_{\bar{S}}$ must be (indecomposable and hence) one dimensional.

$(\Leftarrow)$ Suppose that $_{\bar{R}}M$ and $M_{\bar{S}}$ are one dimensional vector spaces. Let $T = \text{End}(_{\bar{R}}M)$. Then T is a division ring and $\text{End}(M_T) \cong \bar{R}$ is too, so $M_T \cong T_T$. Now the right multiplication map $\rho : \bar{S} \to T$ is an injective ring homomorphism and yields $M_{\bar{S}} \cong T_{\bar{S}}$. But then $\dim(T_{\bar{S}}) = \dim(M_{\bar{S}}) = 1$, so $\cdot\rho$ is surjective and $_RM_S$ is balanced.

Theorem 1.3 allows us to restrict our attention to basic rings when considering exact artinian rings. We shall now use this fact to show that exact rings possess a property that is well known to be shared by serial rings and split algebras — if they are indecomposable their simple modules have isomorphic endomorphism rings.

2.2. Proposition. *If R is an indecomposable exact artinian ring then R/J is isomorphic to a direct sum of matrix rings over the same division ring.*

Proof. Assume that R is basic and exact. If e and f are primitive idempotents in R such that $eRf \neq 0$ then there must be a simple subquotient $_R M_R = I_{i-1}/I_i$ of R such that $M = eMf \neq 0$. But then $_{\overline{eRe}} M_{\overline{fRf}}$ is simple and balanced, and so one dimensional on each side by Lemma 2.1. Clearly now $\overline{eRe} \cong \overline{fRf}$. Thus by the Block Decomposition Theorem [1, 7.9] R/J is a direct sum of pairwise isomorphic division rings.

Azumaya [3, Theorem 2] proved that exact rings satisfy condition (b) of the following theorem.

2.3. Theorem. *The following statements about an artinian ring R are equivalent:*

(a) *R is exact;*

(b) *Every left and every right indecomposable projective R-module has a composition series whose terms are stable under endomorphisms;*

(c) *$_R Re_{eRe}$ and $_{eRe} eR_R$ are exact bimodules for every primitive idempotent e in R.*

Proof. Assume that R is basic.

(a) $\Rightarrow$ (c)   By Lemma 1.4.

(c) $\Rightarrow$ (b)   If $_R Re_{eRe}$ is exact, then by Lemma 2.1 a composition series for this bimodule must have factors that are left simple, and so must be a composition series for $_R Re$ with terms stable under endomorphisms.

(b) $\Rightarrow$ (a)   Assuming (b) we see that if e is a primitive idempotent in R then every composition factor of the bimodule $_R Re_{eRe}$ is simple on the left. But since R is basic, if $K < I$ are ideals of R such that I/K is a composition factor of $_R R_R$, there is a primitive idempotent e in R with

$$I/K = (I/K)e \cong Ie/Ke,$$

and the latter is a composition factor of $_R Re_{eRe}$. Thus I/K is left, and similarly right, simple over R, so R is exact by Lemma 2.1.

Over a serial ring every submodule of an indecomposable projective module is clearly stable under endomorphisms, and by Theorem 2.3 any artinian ring satisfying this condition (e.g., one whose indecomposable projective modules have distributive submodule lattices [12]) is an exact ring.

One of Nakayama's first observations about serial rings [11] was that R

V. P. CAMILLO, K. R. FULLER and J. K. HAACK

is serial if R/J^2 is. Employing part of the following lemma we shall show that an analogous result holds for exact rings.

2.4. Lemma. *If $_RL_S$, $_SM_R$ and $_RN_T$ are exact (resp., simple balanced) bimodules, then so are*

$$_S(M \otimes_R N)_T \text{ and } _S\mathrm{Hom}_R(L, N)_T$$

provided they are not zero.

Proof. Let $_SM_R = M_0 > M_1 > \cdots > M_m = 0$ and $_RN_S = N_0 > N_1 > \cdots > N_n = 0$ be composition series. If $M_{m-1} \otimes_R N_{n-1} \neq 0$ then since R is semilocal $\mathrm{ann}_R(N_{n-1}) = \mathrm{ann}_R(M_{m-1})$, and since

$$(_{\overline{S}}M_{m-1} \otimes_{\overline{R}_-}) : \overline{R}\text{-Mod} \to \overline{S}\text{-Mod}$$

is an equivalence, $M_{m-1} \otimes_R N_{n-1} = M_{m-1} \otimes_{\overline{R}} N_{n-1}$ is an exact (and simple) S-T-bimodule by Lemma 1.2. Now assuming, inductively, that $_S(M_{m-1} \otimes_R N/N_{n-1})_T$ is exact or zero and considering the exact sequence of S-T-homomorphisms

$$M_{m-1} \otimes_R N_{n-1} \mapsto M_{m-1} \otimes_R N \to M_{m-1} \otimes_R N/N_{n-1} \to 0$$

we see that $_S(M_{m-1} \otimes_R N)_T$ is exact or zero. Induction on m now shows that $_S(M \otimes_R N)_T$ is exact or zero. Similarly, one argues that $_S\mathrm{Hom}_R(L, N)_T$ is exact or zero.

2.5. Theorem. *If R is an artinian ring such that R/J^2 is exact then so is R.*

Proof. There is an R-R-epimorphism

$$J/J^2 \otimes J^k/J^{k+1} \to J^{k+1}/j^{k+2} \to 0$$

with $(a+J^2) \otimes (b+J^{k+1}) \to ab + J^{k+2}$. Thus by induction and Lemma 2.4, if the first upper Loewy factor J/J^2 is exact then so are all the rest, and so is R.

If R is a split algebra over a field K then for each primitive idempotent e in R, $eRe = eKe + eJe$. Thus Azumaya's observation that split algebras are exact is immediate from the following corollary of Theorems 1.3 and 2.5.

2.6. Corollary. *An artinian ring R is exact if and only if for each primitive idempotent e in R, the one-sided R-modules Je/J^2e and eJ/eJ^2 have com-*

position series whose terms are also $\overline{eRe}$-subspaces.

Now we turn, for the moment, to rings with radical squared zero. At this juncture it should be noted that if C and D are division rings then by Lemma 2.1 a bi-vector space $_CV_D$ is exact if and only if it has a (left or, equivalently, right) basis $v_1,\ldots, v_n$ (which we call an *exact basis*) such that

$$\sum_{i=1}^{k} Cv_i = \sum_{i=1}^{k} v_iD \qquad (K = 1,\ldots, n),$$

2.7. Proposition. *An artinian ring R with $J^2 = 0$ is exact if and only if for each pair of primitive idempotents e and f in R, the bi-vector space $_{\overline{eRe}}eJf_{\overline{fRf}}$ is exact or zero.*

Proof. Necessity is by Lemma 1.4. For sufficiency, assume that R is basic. Then $eJf = ReJf$ is just the Re/Je-homogeneous component of Jf, so by hypothesis $_RJf_{\overline{fRf}}$, being the two-sided direct sum of its homogeneous components, is exact. Now with a similar observation about eJ we see by Theorem 2.3 that R is exact.

The most intriguing aspect of exact rings is Azumaya's conjecture that they have self duality. From this point of view split algebras (or any artin algebras) are, of course, not very interesting. However, our next result provides a large class of exact rings that are not artin algebras.

2.8. Theorem. *If R is an artinian ring such that R/J is a finite dimensional algebra over an algebraically closed field then R is an exact ring.*

Proof. By Theorem 2.5 we may assume that $J^2 = 0$, and by hypothesis we may also assume that $eRe/eJe = K$, an algebraically closed field, for each primitive idempotent e in R. Then by Proposition 2.7 it will suffice to prove that any bi-vector space $_KV_K$ that is finite dimensional on each side is exact. We do this by induction on $n = \dim(_KV)$: Suppose $V = Kv$ and $\dim(V_K)$ is finite. Then $vk = f(k)v\ (k \in K)$ defines a field homomorphism $f: K \to K$. If $a_j \in K$ and $a_1v,\ldots, a_mv$ form a dependent set of vectors in V_K then for some $k_j \in K$ (not all zero) we have

$$0 = \sum_{j=1}^{m} a_jvk_j = \left(\sum_{j=1}^{m} a_jf(k_j)\right)v$$

so $a_1,\cdots, a_m$ are dependent in K over $f(K)$. Thus $\dim(_{f(K)}K)$ is finite, so since K is algebraically closed $f(K) = K$. Therefore $vK = Kv$, as desired. Now suppose that $\dim(_KV) = n$. Then since the right multiplication functions $\rho(k)$

48   V. P. CAMILLO, K. R. FULLER and J. K. HAACK

$(k \in K)$ form a commutative set of linear transformations of $_K V$ there is, according to [6, page 134, Theorem 7], a basis $v_1, \dots, v_n$ relative to which the matrix of every $\rho(k)$ is lower triangular. Thus for each k in K there are a_{ij} in K such that

$$
\begin{aligned}
v_1 k &= a_{11} v_1 \\
v_2 k &= a_{21} v_1 + a_{22} v_2 \\
&\;\;\vdots \\
v_n k &= a_{n1} v_1 + a_{n2} v_2 + \cdots + a_{nn} v_n.
\end{aligned}
$$

Then $K v_1 K \subseteq K v_1$, so by what we have just shown $K v_1 = v_1 K$; and by induction $_K W_K = V/K v_1$ is exact. But then v_1 together with a set of representatives of an exact basis for $_K W_K$ form an exact basis for $_K V_K$.

Like split algebras, the rings of Theorem 2.8 have basic rings that are commutative modulo their radical. Also Azumaya [3] proved that exact rings have the property that their left and right Cartan matrices are transposes of each other. These two conditions are not enough to insure exactness.

2.9. Example. Let $\varphi \colon \mathbb{C} \to \mathbb{C}$ be an automorphism of the complex numbers such that $\varphi(\mathbb{R}) \not\subseteq \mathbb{R}$ [7, page 157, Exercise 5]. Let $_\mathbb{R} V_\mathbb{R}$ be the bi-vector space with $_\mathbb{R} V = {}_\mathbb{R}\mathbb{C}$ with ordinary multiplication, and right multiplication defined by $vr = v\varphi(r)$ in $\mathbb{C}$. Then $\dim({}_\mathbb{R} V) = \dim(V_\mathbb{R}) = 2$, but $_\mathbb{R} V_\mathbb{R}$ is not exact. For if $\varphi(r) = a + bi$ with $b \neq 0$, then the matrix of right multiplication by r with respect to the basis $1, i$ of $_\mathbb{R} V$ is

$$
\begin{bmatrix}
a & b \\
-b & a
\end{bmatrix}
$$

which has no real eigenvalue. Thus, right multiplication (by r) holds no 1-dimensional subspace of $_\mathbb{R} V$ invariant. Now the trivial extension $R = \mathbb{R} \ltimes V$ is commutative modulo its radical and has identical left and right Cartan matrices [3], but R is not exact.

3. Split exact trivial extensions of semisimple rings. According to Azumaya [2] and Morita [8] an artinian ring R has self duality if and only if it has a finitely generated injective cogenerator $_R E$ whose endomorphism ring is isomorphic to R. If R is basic then $_R E$ must be the minimal cogenerator in R-Mod.

A ring R with ideal X and subring S such that $X^2 = 0$ and $R = S \oplus X$ is called a *trivial extension of S by X*. The usual notation for this is $R = S \ltimes$

X. If $_sX_s$ is a bimodule then one constructs such a ring from $S \times X$ with multiplication $(s, x)(t, y) = (st, sy + xt)$.

Let S be a semisimple ring. Then $_sS$ is an injective cogenerator with $\text{End}(_sS) \cong S$, so according to Mueller's [10, Theorem 10], if $_sX_s$ is both left and right finitely generated then $R = S \ltimes X$ has self duality if and only if $_s\text{Hom}_S(_sX, _sS)$ is finitely generated and there is an isomorphism

$$R \cong S \ltimes \text{Hom}_S(_s\text{Hom}_S(X, _sS), _sS).$$

It is not difficult to see that this is equivalent to the existence of an automorphism θ of S and a θ-semilinear S-S-bimodule isomorphism

$$_s\text{Hom}_S(_sX, _sS)_s \to _s\text{Hom}_S(X_s, S_s)_s.$$

In the proof that follows, we shall actually find an ordinary isomorphism between these bimodules.

A bimodule $_RM_S$ such that

$$_s\text{Hom}_R(_RM, _RR)_R \cong _s\text{Hom}_S(M_S, S_S)_R$$

is said to satisfy the *duality condition*.

A trivial extension of a semisimple ring is Morita equivalent to a ring $R = S \ltimes J$ (its basic ring) with $S = D_1 \oplus \cdots \oplus D_n$ a direct sum of division rings whose identity elements $e_i \in D_i$ form a basic set of idempotents for R; of course J is the radical of R. Here, each e_iJe_j is an ideal of R, $J = \bigoplus_{ij} e_iJe_j$,

$$_s\text{Hom}_S(_sJ, _sS)_s \cong \bigoplus_{ij} {_s}\text{Hom}_{D_i}(_{D_i}e_iJe_j, _{D_i}D_i)_s$$

and

$$_s\text{Hom}_S(J_s, S_s)_s \cong \bigoplus_{ij} {_s}\text{Hom}_{D_j}(e_iJe_j {_{D_j}}, D_{j D_j})_s.$$

Thus, to show that R has self duality it would suffice to prove that the $_{D_i}e_iJe_{j D_j}$ satisfy the duality condition.

We call a bimodule *split exact* in case it is a finite direct sum of simple balanced bimodules. Thus by Lemma 2.1 a bi-vector space $_cV_D$ is split exact in case it has a basis $v_1, \ldots, v_n$ such that

$$Cv_i = v_iD \, (i = 1, \ldots, n).$$

3.1. Lemma. *Every split exact bi-vector space satisfies the duality condition.*

Proof. If $_cV_D = V_1 \oplus \cdots \oplus V_n$ then $\text{Hom}_C(_cV, _cC) \cong \text{Hom}_C(_cV_1, _cC) \oplus \cdots \oplus \text{Hom}_C(_cV_n, _cC)$ as D-C- bi-vector spaces, etc. Thus we may assume

that $_cV_D$ is one dimensional on each side. Let $x \in V$ such that $Cx = V = xD$, and for each γ in $\mathrm{Hom}_C(_cV, _cC)$ let

$$h(\gamma) \in \mathrm{Hom}_D(V_D, D_D)$$

be the unique D-map with

$$x(h(\gamma)(\mathrm{x})) = \gamma(x)x.$$

Then, checking that $h(\gamma)$ is the unique map in $\mathrm{Hom}_D(V_D, D_D)$ such that

$$v(h(\gamma)(w)) = \gamma(v)w$$

for all v, w in V, we see that

$$h : \mathrm{Hom}_C(_cV, _cC) \to \mathrm{Hom}_D(V_D, D_D)$$

is a D-C-isomorphism.

An artinian ring R is called *split exact* in case each of its upper Loewy factors J^{k-1}/J^k is a split exact R-R-bimodule. All of the results of the preceding sections except Theorem 2.8 have analogues for split exact rings. In particular R is split exact if and only if $_{\overline{eRe}}e(J/J^2)f_{\overline{fRf}}$ is a split exact bi-vector space for each pair of primitive idempotents e, f, in R. (It follows that split algebras and serial rings, for example, are split exact.) Thus by Lemma 3.1 and the discussion preceding it we have

3.2. Proposition. *Every split exact trivial extension of a semisimple ring has self duality.*

Even over a field, exact bi-vector spaces need not be split exact.

3.3. Example. Let F be a field and let $K = F(X)$ be the field of rational functions over F with formal derivative $(\ )'$. Let $V = K \times K$ with scalar multiplications

$$k(a, b) = (ka, kb) \text{ and } (a, b)k = (ak + bk', bk)$$

and standard basis $v_1 = (1, 0)$, $v_2 = (0, 1)$. Then $_KV_K$ is a bi-vector space with

$$Kv_1 = v_1K \text{ and } Kv_1 + Kv_2 = v_1K + v_2K$$

so $_KV_K$ is exact. However, it is not hard to show that Kv_1 is its only proper bi-subspace, so $_VV_K$ is not split exact. In particular $K \ltimes V$ is an exact ring that is not split exact.

A. H. Schofield has pointed out to us that if $_KV_L$ is a bivector space finite dimensional over fields K and L, then $_KV_L$ satisfies the duality condition. Indeed, he ovserved that this follows because the subring R of $\text{End}(V_Z)$ generated by the K and L scalar multiplications is a commutative finite dimensional algebra over both K and L ($R \subseteq \text{End}(_KV)^{\text{op}} \cap \text{End}(V_L)$), and so the K and L duals are isomorphic on R-mod. Thus all trivial extensions of commutative semisimple rings (in particular rings of Examples 2.9 and 3.3) have self duality.

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