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LIE IDEALS AND SEMI-DERIVATIONS OF PRIME RINGS

Dedicated to Professor Hisao Tominaga on his 60th birthday

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Throughout, R will represent a prime ring of characteristic not 2 with center C. Let $d : r \mapsto r'$ be a *non-zero* semi-derivation of R associated with a surjective function $\sigma : R \to R$, namely d be a non-zero additive endomorphism of R such that $d\sigma = \sigma d$ and $(xy)' = x'\sigma(y)+xy' = x'y+\sigma(x)y$ for all $x, y \in R$. Then, as was shown in [3, Theorem 1], σ becomes a ring homomorphism. Let U be a Lie ideal of R, and put $W = [U,U]$ and $S = [W,W]$. Obviously, $\sigma(U)$ and both W and S are Lie ideals of R. We consider the following conditions:

(1-U) $U \subseteq C$.

(2-U) $U' \subseteq C$.

(3-U) $U'' \subseteq C$.

(4-U) $[U',U'] \subseteq C$.

(5-U) There exists some $a \in R\backslash C$ such that $[a,U'] \subseteq C$.

(6-U) There exists a non-zero $a \in R$ such that $aU' \subseteq C$.

(7-U) $[u,u'] \in C$ for all $u \in U$.

(8-U) There exists a non-zero semi-derivation $\delta : r \mapsto r^*$ of R associated with a bijective function $\tau : R \to R$ such that $\delta\sigma = \sigma\delta$ and $(U')^* \subseteq C$.

All the results in [1] and [5] proved for Lie ideals of prime rings of characteristic not 2 with derivations have been summarized in Theorem A of [4]. The objective of this paper is to prove the following theorem, which will lead us to a generalization of [4, Theorem A] (Corollary 1).

Theorem 1. (1-U), (1-W), (1-S) *and* (2-U) *are equivalent. and each of* (3-U)$-$(8-U) *implies* (1-$\sigma(U)$).

In preparation for proving our theorem, we state several preliminary lemmas.

Lemma 1. (1) $W' \subseteq [U,U']+[\sigma(U),U'] \subseteq U+\sigma(U)$; *in particular, if* $U' \subseteq C$ *then* $W' = 0$.

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(2) *If* $U' = 0$ *then* $U \subseteq C$.
(3) $\sigma(a) = a$ *for every* $a \in R$ *with* $a' = 0$.

Proof. (1) This is clear by the equality $[u, v]' = [u', v] + [\sigma(u), v']$ $(u, v \in U)$.

(2) Since $[U, R'] \subseteq [U, R]' + [U', \sigma(R)] = 0$, [3, Theorem 4] gives $U \subseteq C$.

(3) Let r be an arbitrary element of R. Then $\sigma(a)r' = (ar)' = ar'$, and so $(a - \sigma(a))R' = 0$. Hence $a - \sigma(a) = 0$ by [3, Lemma 1].

Lemma 2. $(1\text{-}U)$, $(1\text{-}W)$, $(1\text{-}S)$ *and* $(2\text{-}U)$ *are equivalent.*

Proof. By [2, Lemma 3], $(1\text{-}U)$, $(1\text{-}W)$ and $(1\text{-}S)$ are equivalent. It is easy to see that $(1\text{-}U)$ implies $(2\text{-}U)$. If $(2\text{-}U)$ is satisfied then $W' = 0$ by Lemma 1 (1). Hence, by Lemma 1 (2), we have $(1\text{-}W)$.

Lemma 3. *If there exists a non-zero* $a \in R$ *such that* $aU' = 0$ (*or* $U'a = 0$) *then* $U \subseteq C$.

Proof. Suppose, to the contrary, that $U \not\subseteq C$. Let $u, v \in U$, and $r \in R$. Then $0 = a[u, v'ru]' = a([u, v'r]u)' = a[u, v'r]'\sigma(u) + a[u, v'r]u' = auv'ru'$, i.e., $auU'Ru' = 0$. Since $U' \neq 0$ (Lemma 1 (2)), by making use of Brauer's trick, we can easily see that $aUU' = 0$. But this is impossible by [2, Lemma 4].

Lemma 4. *If* $U'' = 0$ *then* $U \subseteq C$.

Proof. If not, there exists an ideal M of R such that $[M, R] \subseteq U$ but $[M, R] \not\subseteq C$, by [2, Lemma 1]. Let $m \in [M, R]$, $w \in W$ and $r \in R$. Note that $m, [m, r] \in U$, $\sigma(m') = m'$ (Lemma 1 (3)), $w' \in U'$ and $[w', r] \in U + \sigma(U)$ (Lemma 1 (1)). Then $0 = [mw', r]'' = (m[w', r] + [m, r]w')'' = m''[w', r] + 2\sigma(m')[w', r]' + \sigma^2(m)[w', r]'' + 2\sigma([m, r]')w'' + \sigma^2([m, r])w''' = 2m'[w', r]'$, i.e., $0 = [M, R]'[W', R]' = [M, R]'[W', R']$. Since $[M, R] \not\subseteq C$, Lemma 3 gives $[W', R'] = 0$, and so $W' \subseteq C$ by [3, Theorem 4]. But this contradicts Lemma 2.

Lemma 5. (1) *If* $C' \neq 0$ *and* $[a, U'] \subseteq C$ *for some* $a \in R \backslash C$, *then* $U \subseteq C$.

(2) *If* $[a, U'] = 0$ *for some* $a \in R \backslash C$, *then* $\sigma(U) \subseteq C$.

Proof. (1) There exists a $c \in C$ such that $c' \neq 0$. Then, for any

$u \in U$ and $r \in R$, $c'[a,[u,r]] = [a,[u,cr]'] - \sigma(c)[a,[u,r]'] \in C$. Hence $[a,[a,[U,R]]] = 0$, and so [5, Theorem 4] gives $[U,R] \subseteq C$; in particular, $[a,[a,U]] = 0$. Again by [5, Theorem 4], we get $U \subseteq C$.

(2) In view of (1), we may assume that $C' = 0$. Suppose, to the contrary, that $\sigma(U) \nsubseteq C$. Since $0 = [a,[a,u]'] = [a,[a',\sigma(u)]]$ for any $u \in U$, we have $[a,[a',\sigma(U)]] = 0$, and so [5, Theorem 4] proves that $a' \in C$ and $\sigma(a') = a'$ (Lemma 1 (3)). Then, for any $u \in U$, $0 = [a,[a^2,u]'] = [a,[(a^2)',\sigma(u)]] = [a,[a'\sigma(a)+aa',\sigma(u)]] = a'[a,[a+\sigma(a),\sigma(u)]]$, i.e., $a'[a,[a+\sigma(a),\sigma(U)]] = 0$. Hence either $a' = 0$ or $[a,[a+\sigma(a),\sigma(U)]] = 0$. In the latter case, we have $a+\sigma(a) \in C$ by [5, Theorem 4], and $0 = (a+\sigma(a))' = 2a'$. We have thus seen that $a' = 0$, in either case. Now, by [2, Lemma 1], there exists an ideal M of R such that $[M,R] \subseteq U$ but $[M,R] \nsubseteq C$. Let $m \in [M,R]\,(\subseteq M \cap U)$, and $u \in U$. Then $m'[a,[a,u]] = [a,m'[a,u]] = [a, m'[a,u] + \sigma(m)([a,u'] + [a',\sigma(u)])] = [a, m'[a,u] + \sigma(m)[a,u]'] = [a,(m[a,u])'] = [a,[ma,u]'] - [a,([m,u]a)'] = -[a,[m,u]']a = 0$, i.e., $[M,R]'[a,[a,U]] = 0$. Hence, Lemma 3 shows that $[a,[a,U]] = 0$, and therefore $U \subseteq C$ by [5, Theorem 4]. But this is a contradiction.

Lemma 6. *If $\sigma(U) \nsubseteq C$ and $U''' = 0$ then $R''' = 0$.*

Proof. First, we claim that $R''' \subseteq C$. It is easy to see that $S' \subseteq V + \sigma(V)$ and $S'' \subseteq U + \sigma(U) + \sigma^2(U)$ (see Lemma 1 (1)). Then, for any $s \in S$ and $r \in R$, $[s'',r'''] = [s'',r]''' \in [U,r]''' + [\sigma(U),r]''' + [\sigma^2(U),r]''' = 0$ and $[s',r^{(4)}] = 3\sigma([s'',r''']) + [s',r^{(4)}] = 3[s'',\sigma(r''')] + [s',r^{(4)}] = [s',r']''' \in [V,r]''' + [\sigma(V),r]''' = 0$ (see Lemma 1 (3)). Hence $[S',R^{(4)}] = 0$. Since $\sigma(S) \nsubseteq C$ (Lemma 2), Lemma 5 (2) gives $R^{(4)} \subseteq C$. Now, let $u \in U$ and $r \in R$. Then $0 = [u,r]^{(4)} = 6[u'',\sigma^2(r'')] + 4[u',\sigma(r''')]$ and $0 = [u,r']''' = 3[u'',\sigma^2(r'')] + 3[u',\sigma(r''')]$, so that $0 = 2[u,r']''' - [u,r]^{(4)} = 2[u',\sigma(r''')]$, i.e., $[U',R'''] = 0$. Thus $R''' \subseteq C$ by Lemma 5 (2). In particular, $R'''U'' = (RU'')''' \subseteq C$, and so either $R''' = 0$ or $U'' \subseteq C$. If $U'' \subseteq C$ then $r^{(4)}u' = (ru')^{(4)} - 4\sigma(r''')u'' \in C$, i.e., $R^{(4)}U' \subseteq C$. Since $U' \nsubseteq C$ (Lemma 2), we get $R^{(4)} = 0$. Hence $4\sigma(r''')u'' = (ru')^{(4)} - r^{(4)}u' = 0$, i.e., $R'''U'' = 0$. Combining this with $U'' \neq 0$ (Lemma 4), we get $R''' = 0$. This completes the proof.

We are now ready to complete the proof of our theorem.

Proof of Theorem 1. By Lemma 2, it remains to prove that each of (3-U)—(8-U) implies (1 - $\sigma(U)$).

(7-U) $\Rightarrow$ (1-$\sigma(U)$). Linearizing the relations $[u,u'] \in C$ and

$[u', \sigma(u)] = [u, u]' - [u, u'] \in C$, we get $[u, v'] - [u', v] \in C$ and $[u', \sigma(v)] - [\sigma(u), v'] \in C$ $(u, v \in U)$. Then, by making use of Jacobi identity, we see that $[u, [u', \sigma(v)]] = [u, [\sigma(u), v']] = [u, [u, v]'] - [u, [u', v]] = [u, [u, v]'] - [u', [u, v]] \in C$, i.e., $[u, [u', \sigma(U)]] \subseteq C$. If $\sigma(U) \not\subseteq C$ then $u' \in C$ by [5, Theorem 4]; $U' \subseteq C$. But this contradicts Lemma 2.

(3-U) $\Rightarrow$ (1-$\sigma(U)$). Let $u, v \in U$. Then $2[\sigma(u'), v'] = [u, v]'' \in C$, i.e., $[\sigma(U'), U'] \subseteq C$. In case $C' \neq 0$, Lemma 5 (1) proves that R satisfies (1-$\sigma(U)$). Thus, we assume henceforth that $C' = 0$. Suppose, to the contrary, that $\sigma(U) \not\subseteq C$. Since $U''' \subseteq C' = 0$, Lemma 6 proves that $R''' = 0$. Now, let u be an arbitrary element of U with $u'' = 0$. Then, for any $v \in U$, $[u, v']' = [u', v'] = [\sigma(u'), v'] \in C$ and $2u'[u', v'] = 2u'[u, v']' = 2u'\sigma([u, v']') = (u[u, v'])'' = [u, uv']'' \in C$ (Lemma 1 (3)). Hence $[u', U'] = 0$, and $u' \in C$ (Lemma 5 (2)). In what follows, let u be an arbitrary element of U with $u'' \neq 0$. Since $[u, R'']'' = 0$, by what we have just shown above we see that $[u', R''] = [u, R'']' \subseteq C$; in particular, $[u', [u', R'']] = 0$. Thus $u''[u', [u', R']] = [u', [u', u''\sigma^2(R')]] \subseteq [u', [u', (uR')'']] + u'[u', [u', \sigma(R'')]] = 0$, whence $[u', [u', R']] = 0$ follows, and therefore $u''[u', [u', R]] = [u', [u', u''R]] \subseteq [u', [u', (u'R)']] + u'[u', [u', \sigma(R')]] = 0$. Hence $[u', [u', R]] = 0$, and so $u' \in C$ by [5, Theorem 4]. We have thus seen that $U' \subseteq C$, which contradicts Lemma 2.

(5-U) $\Rightarrow$ (1-$\sigma(U)$). Suppose, to the contrary, that $\sigma(U) \not\subseteq C$. Let u, v be arbitrary elements of U. Then $[a, [u'', \sigma(v)]] = [a, [u, v]'] - [a, [u', v']] = [a, [u', v]'] + [u', [v', a]] + [v', [a, u']] \in C$, i.e., $[a, [u'', \sigma(U)]] \subseteq C$. Hence $u'' \in C$ by [5, Theorem 4]; R satisfies (3-U). This is a contradiction.

(4-U) $\Rightarrow$ (1-$\sigma(U)$). (4-U) implies either (2-U) or (5-U), and hence (1-$\sigma(U)$).

(6-U) $\Rightarrow$ (1-$\sigma(U)$). If $a \in C$ then $U \subseteq C$ (Lemma 2). We assume henceforth that $a \notin C$. If there exists a $c \in C$ such that $c' \neq 0$, then $c'a[u, r] = a[u, cr]' - \sigma(c)a[u, r]' \in C$ $(u \in U, r \in R)$, and so $a[a, [U, R]] = [a, a[U, R]] = 0$. This implies $[U, R] \subseteq C$ by [2, Lemma 7], and hence $U \subseteq C$ by [2, Lemma 6]. Thus, in what follows, we assume that $C' = 0$. Suppose, to the contrary, that $\sigma(U) \not\subseteq C$. Since $a[a, U'] = [a, aU'] = 0$ and $a[a', \sigma(U)] = a[a, U]' \subseteq C$, we get $a' \in C$ by [5, Theorem 6]. Now, let $M = RWR$. Then $[M, R] \subseteq U$ (see the proof of [2, Lemma 1]). Since $\sigma(W) \not\subseteq C$ (Lemma 2), $\sigma(M) = R\sigma(W)R \neq 0$, and therefore, as is well known, $\sigma(M)$ is not commutative. Furthermore, in view of [2, Lemma 6], we see that $[\sigma(M), R] \not\subseteq C$. Let $m \in [M, R]$ $(\subseteq M \cap U)$. Then $a[a, m]'$

$= 0$ and $a[a,m]a' = a[a,m]'\sigma(a)+a[a,m]a' = a([a,m]a)' = a[a,ma]' \in C$. Hence $(a')^2[a,\sigma(m)] = a'[a,(am)'] = (a'[a,am])' = (a[a,m]a')' = 0$; $(a')^2[a,[\sigma(M),R]] = 0$. Since $[a,[\sigma(M),R]] \not\subseteq C$ by [2, Lemma 6], we get $a' = 0$. Thus $aU'' = (aU')' = 0$ and $aU''' = 0$. Since $\sigma(a) = a$ (Lemma 1 (3)), it is easy to see that $a\sigma(U'') = 0$, $a[U',\sigma(U)]' \subseteq C$ and $a[U',\sigma^2(U)]' \subseteq C$. Noting here that $S'' \subseteq U+\sigma(U)+\sigma^2(U)$ (see Lemma 1 (1)), we can easily see that $aU'S''' = a[U',S'']' \subseteq C$. Since $aU' \neq 0$ (Lemma 3), we get $S''' \subseteq C$, and so $aU'S'' = a[U,S'']' \subseteq C$. Hence R satisfies (3-S). This is impossible by Lemma 2.

(8-U) $\Rightarrow$ (1-$\sigma(U)$). Noting that $W' \subseteq U+\sigma(U)$, we can easily see that $[(\sigma(U))^*, W''] = ([U,W']')^* \subseteq C$. If $W'' \not\subseteq C$ then R satisfies (5-$\sigma(U)$) with respect to δ, and therefore $\tau\sigma(U) \subseteq C$. Hence $\sigma(U) \subseteq C$. On the other hand, if $W'' \subseteq C$ then $\sigma(W) \subseteq C$ by the above, and $\sigma(U) \subseteq C$ (Lemma 2).

Corollary 1. *If R is not commutative and σ is bijective, then each of* (2-S)—(8-S) *is equivalent to* (1-U).

Remark 1. Obviously, Theorem 1 gives generalizations of Theorems 3—7 and 9 in [3], as well. Furthermore, it is easy to see that if U is either d-stable or σ-stable and there exists a non-zero semi-derivation δ: $r \mapsto r^*$ of R associated with a surjective function $\tau: R \to R$ such that $(U')^* \subseteq C$ then $\tau\sigma(U) \subseteq C$. This is a generalization of Theorem 8 in [3].

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