Mathematical Journal of Okayama University

Volume 45, *Issue* 1 2003 *Article* 3

JANUARY 2003

When is $\mathrm{RP}^n \times$Spin(n) Diffemorphic to $\mathrm{S}^n \times$SO(n) and how

Thomas Puttmann* Alcibiades Rigas†

*Ruhr-Universitat Bochum

†Instituto de Matemática, Estatística e Computação Científica

When is $\mathrm{RP}^n \times$Spin(n) Diffemorphic to $\mathrm{S}^n \times$SO(n) and how

Thomas Puttmann and Alcibiades Rigas

Abstract

We show that the spaces in the title, whose corresponding homotopy groups are isomorphic, are homotopy equivalent only when n = 3 or n = 7. We produce an explicit diffeomorphism in the only non trivial case, n = 7.

Math. J. Okayama Univ. **45** (2003), 111–115

WHEN IS $RP^n \times Spin(n)$ DIFFEOMORPHIC TO $S^n \times SO(n)$ AND HOW

Thomas PÜTTMANN and A. RIGAS

Abstract. We show that the spaces in the title, whose corresponding homotopy groups are isomorphic, are homotopy equivalent only when $n = 3$ or $n = 7$. We produce an explicit diffeomorphism in the only non trivial case, $n = 7$.

Introduction

For $n \geq 3$, $Spin(n)$ is the universal covering group of the rotation group $SO(n)$, whose fundamental group is Z_2 (see [5]). This implies that $S^n \times Spin(n)$ is the fundamental cover of both $S^n \times SO(n)$ and of $RP^n \times Spin(n)$, for all $n \geq 3$ and that the corresponding homotopy groups of these two spaces are isomorphic. For $n = 3$ the algebra of quaternions implies that $Spin(3)$ is isomorphic with S^3 and $SO(3)$ is isomorphic with RP^3 (see [5]). A simple switching of the factors provides the diffeomorphism $RP^3 \times Spin(3) \cong S^3 \times SO(3)$. Two obvious question arise: The one in the title and "To what extend does the existence of an algebra structure on R^{n+1} describe adequately the solution to the first question?".

In section 1 we show that if $RP^n \times Spin(n)$ is homotopy equivalent to $S^n \times SO(n)$ then S^n is an H-space and therefore $n = 3$ or 7 (see [1]) (remember, here $n \geq 3$).

In section 2 we use the Cayley algebra and the principle of triality (see [2]) to produce an explicit formula for a diffeomorphism in the case $n = 7$.

The second author is indebted to Zig Fiedorowicz for his help in the first part. We also want to thank Wolfgang Ziller for his hospitality during our visit to the University of Pennsylvania. Our joint work was supported by the CNPq-GMD agreement.

1. Topological obstructions

If n is even RP^n is not orientable and there is no homotopy equivalence between $RP^n \times Spin(n)$ and $S^n \times SO(n)$. So, let n be odd and let $h : S^n \times SO(n) \to RP^n \times Spin(n)$ be a homotopy equivalence. Composing with the obvious inclusions and projections we have:

$$RP^n \to RP^n \times Spin(n) \to S^n \times SO(n) \to SO(n) \to$$
$$\to S^n \times SO(n) \to RP^n \times Spin(n) \to RP^n$$

where we have employed first h^{-1} and then h. The induced maps in rational cohomology compose to

$$F : H^*(RP^n; Q) \to H^*(RP^n; Q).$$

From [5], p. 177, Th. 2.19 (2) and Cor. 3.15 (2), p. 122 we see that the projection induces an isomorphism $H^*(SO(n); Q) \cong H^*(Spin(n); Q)$ which is, for odd n, isomorphic to the exterior algebra in the generators e_3, e_7, ..., e_{2n-3}. The projection $S^n \to RP^n$ also induces an isomorphism in cohomology with rational coefficients and $H^*(RP^n; Q)$ is the exterior algebra in one generator, s, of degree n. If we follow the composition F around we easily conclude that $F(s) = \lambda s$.

Claim: λ is an odd integer.

Proof: It is easy to see that the maps $f : RP^n \to SO(n)$ and $g : SO(n) \to RP^n$, composed as is obvious from some of the maps that make up F, induce isomorphisms on the fundamental groups that are isomorphic to Z_2. Consequently, $(g \circ f)^* : H^*(RP^n; Z_2) \to H^*(RP^n; Z_2)$ is an isomorphism since $H^*(RP^n; Z_2)$ is generated by an element of degree 1, the dual of the generator of the fundamental group. In particular, $(g \circ f)^*$ is an isomorphism.

Corollary: $(g \circ f)^* : H^*(RP^n; Z) \to H^*(RP^n; Z)$ is multiplication by an odd integer.

Corollary: $(g \circ f)^* : H^*(RP^n; Q) \to H^*(RP^n; Q)$ is multiplication by an odd integer.

As a consequence we have that the map $g \circ f$ is a homotopy equivalence on the 2-primary localizations of RP^n and $SO(n)$, which implies that $RP^n_{(2)}$ is an H-space (see [4]). Localization is a functor that preserves coverings, so $S^n_{(2)}$ is an H-space. Now apply the 2-primary localization to the Hopf construction (see [6]) to obtain a map $S^{2n+1}_{(2)} \to S^{n+1}_{(2)}$, whose Hopf invariant is unit in $Z_{(2)}$, the integers localized at 2. Corollary 5.13, p. 89 of [4] implies now that some odd integer multiple of this must arise from localizing an actual map $S^{2n+1} \to S^{n+1}$, Corollary 15.14, p. 409 of [5] implies now, using [1], that $n = 3$ or 7 (recall that $n \geq 3$).

2. The diffeomorphism

Recall (see e.g. [3]) that $Spin(8)$ is identified with the subgroup of all triples $(A, B, C) \in SO(8) \times SO(8) \times SO(8)$ with the property

$$\text{(T)} \qquad A(xy) = B(x)C(y), \text{ for all } x, y \in Ca, \text{ the Cayley field.}$$

One really needs just two copies of $SO(8)$ as C is determined from A and the sign of B, but it seems to be more convenient to use all three to express the triality automorphisms.

The subgroup $Spin(7) \subset Spin(8)$ can be identified with all $(N, M, \widetilde{M})$, where $\widetilde{M}(x) = \overline{M(\overline{x})}$, for all $x \in Ca$, the bar denoting the usual conjugation of a Cayley number. This is equivalent to $N(1) = 1$.

If γ is the usual triality automorphism of order 3, then

$$\gamma(Spin(7)) = \{(M, N, M) \text{ in (T), with } N(1) = 1\}$$

Lemma 1. *The map $\gamma(Spin(7)) \to SO(8)$ with $(M, N, M) \mapsto M$ is an injective group morphism.*

Proof. It is a group morphism by its definition and the kernel is (I, I, I), because if $(I, N, I) \mapsto I$, then $I(y) = I(y1) = N(y)I(1) = N(y)$ for all $y \in Ca$, which implies $N = I$. □

From now on $Spin(7)$ is the subgroup of $SO(8)$ with $(M, N, M) \in \gamma(Spin(7))$, equivalently, $N(1) = 1$.

Lemma 2. *The map $\pi : SO(8) \to RP^7$ with $\pi(X) = \pm Y(1)$ is well defined.*

Proof. Note that $(X, \pm(Y, Z))$ is a well defined pair of points in $Spin(8)$, namely the fiber of the projection onto the first $SO(8)$ factor. □

Claim 3. *The fiber $\pi^{-1}(1)$ consists of all $X \in SO(8)$ with $(X, \pm(Y, Z)) \in Spin(8)$ and $\pm Y(1) = 1$, i.e., $Y \in O(7) = SO(7) \cup -SO(7)$.*

Proof. $Y(1) = 1$. The element $(X, Y, X) \in \gamma(Spin(7))$is represented by $X \in SO(8)$. The element $-Y(1) = 1$ is $(X, -Y, -X) \in Spin(8)$, for it is (X, Y, Z) for some Z, so $X(y) = Y(1)Z(y) = -1Z(y)$ and $Z = -X$. □

Note that the image in $SO(8)$ is the same: X. Also that $\pi^{-1}(\pm 1)$ is a subgroup of $SO(8)$ as the first factor projection of

$$Pin(7) = \{(X, Y, X)\} \cup \{(X, -Y, -X)\}$$

into $SO(8)$. This projection coincides with the inclusion of $Spin(7) \subset SO(8)$ of Lemma 1.

Proposition 4. *The map π of Lemma 2 is the projection of the fibration*

$$Spin(7) \cdots SO(8) \to SO(8)/Spin(7).$$

Proof. Consider the right action by a subgroup multiplication $SO(8) \times Spin(7) \to SO(8)$ with $X(M, N, M) \mapsto XM$. Then $(X, \pm(Y, Z))(M, N, M) = (XM, \pm(YN, ZM))$ and the whole orbit XM is mapped through π to $\pm YN(1) = \pm Y(1)$: the point $\pi(X) \in RP^7$. □

Consider now the following map $\chi : RP^7 \to SO(8)$ defined by $\chi(\pm\alpha) = L_{\pm\alpha} \circ R_{\pm\alpha} = L_\alpha \circ R_\alpha$, where $L_\alpha(x) = \alpha x$ and $R_\alpha(x) = x\alpha$, Cayley products.

Proposition 5. *χ is a well defined section of the principal bundle π.*

Proof. It is clearly well defined. From the Moufang identity $\alpha(xy)\alpha = (\alpha x)(y\alpha)$ (see e.g. [3]) we see that $(L_\alpha \circ R_\alpha, \pm(L_\alpha, R_\alpha)) \in Spin(8)$ and $\pi(\chi(\pm\alpha)) = \pm L_\alpha(1) = \pm\alpha$. □

Corollary 6. *$SO(8)$ is diffeomorphic to $RP^7 \times Spin(7)$ as follows:*
$RP^7 \times Spin(7) \ni (\pm\alpha, M) \mapsto (L_\alpha \circ R_\alpha)M \in SO(8)$ whose inverse is
$SO(8) \ni X \mapsto (\pm Y(1), (L_{\overline{Y(1)}} \circ R_{\overline{Y(1)}})X) \in RP^7 \times Spin(7)$.

Proof. To X corresponds $(X, \pm(Y,Z))$, we have also

$$(L_{Y(1)} \circ R_{Y(1)}, \pm(L_{Y(1)}, R_{Y(1)}))$$

and their product in $Spin(8)$ is

$$(L_{\overline{Y(1)}} \circ R_{\overline{Y(1)}}, \pm(L_{\overline{Y(1)}}, R_{\overline{Y(1)}}))(X, \pm(Y,Z)) \\ = ((L_{\overline{Y(1)}} \circ R_{\overline{Y(1)}})X, \pm(L_{\overline{Y(1)}}Y, R_{\overline{Y(1)}}Z)).$$

But $\pm(L_{\overline{Y(1)}}Y)(1) = \pm 1$, so $(L_{\overline{Y(1)}} \circ R_{\overline{Y(1)}})X$ is in $Spin(7) \subset SO(8)$. □

On the other hand, $SO(8)$ is diffeomorphic to $S^7 \times SO(7)$ as follows:

$$SO(8) \ni W \mapsto (W(1), L_{\overline{W(1)}} \circ W) \in S^7 \times SO(7)$$

whose inverse is $S^7 \times SO(7) \ni (\beta, A) \mapsto L_\beta \circ A \in SO(8)$. Now we can compose these two diffeomorphisms, i.e., given (β, A) in $S^7 \times SO(7)$ we look for its image in $RP^7 \times Spin(7)$. Note that $A(1) = 1$, $(A, \pm(B, \widetilde{B})) \in Spin(7) \subset Spin(8)$ and $L_\beta \circ A = X$ will go to $(\pm Y(1), L_{\overline{Y(1)}} \circ R_{\overline{Y(1)}} \circ X)$. From the Moufang identity $\beta(xy) = (\beta x\beta)(\overline{\beta}y)$ (see e.g. [3]) we obtain $(L_\beta, \pm(L_\beta \circ R_\beta, L_{\overline{\beta}})) \in Spin(8)$. So the triality triple $(X, \pm(Y,Z))$ will be the product

$$(L_\beta, \pm(L_\beta \circ R_\beta, L_{\overline{\beta}}))(A, \pm(B, \widetilde{B})) = (L_\beta \circ A, \pm(L_\beta \circ R_\beta \circ B, L_{\overline{\beta}} \circ \widetilde{B})).$$

Through the identification of $SO(8)$ with $RP^7 \times Spin(7)$ this will go to

$$(\pm L_\beta \circ R_\beta \circ B(1), L_{\overline{\beta B(1)\beta}} \circ R_{\overline{\beta B(1)\beta}} \circ L_\beta \circ A) \\ = (\pm\beta B(1)\beta, L_{\overline{\beta B(1)\beta}} \circ R_{\overline{\beta B(1)\beta}} \circ L_\beta \circ A),$$

which we denote by λ.

The following little calculation now

$$\xi \mapsto (\overline{\beta B(1)\beta})(\beta A(\xi))(\overline{\beta B(1)\beta}) \\ = (\overline{\beta B(1)})[A(\xi)(\overline{\beta B(1)\beta})] = (L_{\overline{\beta B(1)}})(R_{\overline{\beta B(1)\beta}}(A(\xi)))$$

and the associativity of the subalgebra generated by the two elements β and $B(1)$ imply that the operators $L_{\overline{\beta B(1)}}$ and $R_{\overline{\beta B(1)\beta}}$ commute and therefore

$$\lambda = (\pm\beta B(1)\beta, R_{\overline{\beta B(1)\beta}} \circ L_{\overline{\beta B(1)}} \circ A) \in RP^7 \times Spin(7)$$

is the image of $(\beta, A) \in S^7 \times SO(7)$.

The inverse of this map is $RP^7 \times Spin(7) \ni (\pm\alpha, M) \mapsto W \in S^7 \times SO(7)$, where

$$W = (L_\alpha \circ R_\alpha) \circ M \mapsto ((L_\alpha \circ R_\alpha)(M(1)), (L_{\overline{\alpha M(1)\alpha}} \circ L_\alpha \circ R_\alpha) \circ M)$$

To verify that the matrix coordinate is really in $SO(7)$:

$$((L_{\overline{\alpha M(1)\alpha}} \circ L_\alpha \circ R_\alpha) \circ M)(1) = (\overline{\alpha M(1)\alpha})(\alpha M(1)\alpha) = 1.$$

References

[1] J. F. Adams, *On the non existence of elements of Hopf invariant one*, Ann. of Math. **72**(1960), 20–104.

[2] E. Cartan, *Le principe de dualité et la theorie des groups simples e semisimples*, Bull. Sci. Math. **49**(1925), 361–374.

[3] F. R. Harvey, *Spinors and Calibrations*, Perspectives in Mathematics, v. 9, Academic Press (1990).

[4] P. Hilton, G. Mislin and J. Roitberg, *Localization of Nilpotent Groups and Spaces*, Notas de Matemática, v. 55, North-Holland Math. Studies 1975.

[5] M. Mimura and H. Toda, *Topology of Lie Groups I and II*, Transl. Math. Monogr. v. 91, AMS 1991.

[6] G. Whitehead, *Elements of Homotopy Theory*, GTM, v. 61, Springer 1979.

Thomas Püttmann
Ruhr-Universität Bochum
Germany
e-mail address: puttmann@math.ruhr-uni-bochum.de

A. Rigas
IMECC - Unicamp
Brazil
e-mail address: rigas@ime.unicamp.br

(*Received December 24, 2002*)