Mathematical Journal of Okayama University

Volume 24, *Issue* 2 1982 *Article* 3

DECEMBER 1982

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Math. J. Okayama Univ. 24 (1982), 117—132

ON STRONGLY PRIME MODULES AND RELATED TOPICS

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Introduction. The useful notions of prime rings, semiprime rings, SP (strongly prime) rings and STP rings have been extended to modules by Beachy [3], Dauns [6], Desale-Nicholson [7], Handelman-Lawrence [9], Zelmanowitz [23] and others. In the preliminary section of this paper, we state those definitions together with already known relationships among them and with some examples. §1 deals with the further properties of SP modules and related modules. We shall give characterizations of ZP modules, HSP modules and QSP modules, and prove [9, Proposition II.3], [14, Corollary 5.7] and [22, Proposition 2.6] altogether. §2 begins with characterizations of semiprime rings, prime rings, STP rings and others in terms of the existence of certain kinds of faithful modules. Next, we shall give a slight generalization of [7, Proposition 3.1]. The latter part of §2 is concerned with some ring extensions of SP rings and STP rings. In the final section §3, we treat with rings not necessarily containing identity, and extend the results obtained in [18] and [19] for normal classes and special classes of prime rings to those of semiprime rings.

0. Preliminaries. Except in §3, where we deal with weakly special classes, all the rings we consider will be associative rings with identity 1 ($\neq 0$) and all the modules considered will be unital. Let R be a ring. As usual, $_RM$ (resp. M_R) will denote that M is an object in the category R-mod (resp. mod-R) of all left (resp. right) R-modules and we write morphisms on the side opposite to that of scalars. Unless otherwise mentioned, by a module we mean a left R-module and the concepts will be left sided ones, and by an ideal of R a two-sided ideal. By $Z(M)$, $\mathrm{Soc}(M)$ and $J(M)$, we denote the singular submodule, the socle and the Jacobson radical of M, respectively. If X and Y are subsets of $_RM$ (resp. M_R), we set $YX^{-1}=\{a\in R \mid aX\subseteq Y\}$ (resp. $X^{-1}Y=\{a\in R \mid Xa\subseteq Y\}$); in particular we write $l_R(X)=0X^{-1}$ (resp. $r_R(X)=X^{-1}0$).

We now recall some definitions. A subfunctor T of the identity functor on R-mod is called a *preradical*; a preradical T is *left exact* (abbreviated as LE) if $T(N)=N\cap T(M)$ whenever $_RN\subseteq {_RM}$, and T is called a *radical* if $T(M/T(M))=0$ for all $M\in R$-mod (see, e.g. [20]). Let

M be a non-zero left R-module. If M has no non-trivial fully invariant submodules then M is called *endosimple* [7]. If M is endosimple then $J(M)=0$ or M. Following [6], M is called *weakly semiprime* (resp. *semiprime*) (abbreviated as WsP (resp. sP)) if, for any $m \in M$ and $a \in R$, $aRaRm=0$ (resp. $aRam=0$) implies $aRm=0$ (resp. $am=0$), and M is called *compressible* if M can be embedded in each of its non-zero submodules. Following [3], M is called *prime* if one of the following equivalent conditions is satisfied: (1) $aRm=0$ ($m \in M$, $a \in R$) implies $m=0$ or $aM=0$; (2) $l_R(N)=l_R(M)$ for all non-zero submodules N of M, and M is called *monoform* if every non-zero partial endomorphism of M is a monomorphism. Furthermore, M is called *strongly prime* (abbreviated as SP) if one of the following equivalent conditions is satisfied: (1) $T(M)=0$ or M for all LE preradical T on R-mod; (2) given m' and non-zero m in M, there exists a finite set $\{a_1,\cdots,a_n\} \subseteq R$ such that $\bigcap_{i=1}^{n} l_R(a_i m) \subseteq l_R(m')$, and M is called *cofaithful* if there exists a finite subset F of M such that $l_R(F)=0$. Needless to say, every cofaithful module is faithful. If M is SP then $Z(M)=0$ or M, and $\mathrm{Soc}(M)=0$ or M. On the other hand, M is called *strongly prime in the sense of Handelman-Lawrence* (abbreviated as HSP) if for any non-zero $m \in M$ there exists a finite set $\{a_1,\cdots,a_n\} \subseteq R$ such that $\bigcap_{i=1}^{n} l_R(a_i m) =0$ (cf. [9]), and M is called *strictly prime* (abbreviated as STP) if for any non-zero $m \in M$ there exists $a \in R$ such that $l_R(am)=l_R(M)$ ([6]). A faithful module M is STP if and only if R is embedded in each non-zero cyclic submodule of M. When this is the case, M is HSP. We say that M is *quasi-strongly prime* (abbreviated as QSP) if for any non-zero submodule N of M there exists a finite set $\{x_1,\cdots,x_n\} \subseteq N$ and $m \in M \setminus N$ such that $\bigcap_{i=1}^{n} l_R(x_i) \subseteq Nm^{-1}$. Now, let $M^*=\mathrm{Hom}_R(M,R)$, and $S=\mathrm{Hom}_R(M,M)$. Then there exists a derived Morita context (R,M,M^*,S), where $(m,f)=mf$ and $[f,m]$ is defined by $m'[f,m]=(m',f)m$ for m, $m' \in M$ and $f \in M^*$ (see, e.g. [1]). Following [23], M is called *semiprime* (resp. *prime*) *in the sense of Zelmanowitz* (abbreviated as ZsP (resp. ZP)) if $(m,M^*)m=0$ ($m \in M$) (resp. $(m,M^*)m'=0$ (m, $m' \in M$)) implies $m=0$ (resp. $m=0$ or $m'=0$). Finally, M is called *torsionless* if $(m,M^*)=0$ ($m \in M$) implies $m=0$.

We summarize here already known implications among the notions cited above (see also the table given at the end): Every STP module is SP, every SP module is prime and QSP, every prime module is sP, and every sP module is WsP [6, p. 311 and 7, p. 549]. Every simple module is compressible, monoform and endosimple, every compressible module is SP, and every endosimple module is SP [7, pp. 550 and 551]. Every HSP module is cofaithful, SP and non-singular [9, p. 220], every ZP module is prime and ZsP, and every ZsP module is torsionless and sP.

A ring R is called *SP*, *STP* or *self-compressible* according as ${}_RR$ is HSP, STP or compressible. A ring R is SP (resp. simple) if and only if ${}_RR$ is SP (resp. endosimple) [7, p. 550]. Following [21], a ring R is called *fully left idempotent* if every left ideal of R is idempotent, and R is called a *left V-ring* if every left ideal of R is an intersection of maximal left ideals of R. Following [7], a ring R is called *endoprimitive* (resp. *weakly primitive*) if there exists a faithful SP left R-module (resp. a faithful, compressible and monoform left R-module). A ring R is called *CTF* if $T(R)=0$ or R for all LE radicals T on R-mod [4 or 14]. A ring R is called *bisimple* if for any non-zero a, $b\in R$ there exists $c\in R$ such that $aR=cR$ and $Rb=Rc$ [13], and R is called *E2* if R has no non-trivial left strongly idempotent ideals, where an ideal I of R is *left strongly idempotent* if $L=IL$ for all left ideals $L\subseteq I$ [14].

Remark 0.1. Obviously, ${}_{\boldsymbol{Z}}\boldsymbol{Z}$ is not endosimple, but compressible and monoform. Moreover, for any prime p, ${}_{\boldsymbol{Z}}\boldsymbol{Z}_p$ is neither non-singular nor faithful nor torsionless, but STP. If R is the trivial extension $\boldsymbol{Z}\ltimes\boldsymbol{Q}$ of $\boldsymbol{Z}$ by $\boldsymbol{Q}$, then ${}_RR$ is not sP but QSP. Needless to say, for any semiprime ring R which is not prime, ${}_RR$ is not prime but ZsP and cofaithful. ${}_{\boldsymbol{Z}}\boldsymbol{Z}/8\boldsymbol{Z}$ is not monoform but uniform. In what follows, F will represent an arbitrary field.

(1) Let R be the (non-commutative) free algebra $F\{x,y\}$, and $M=R/RxRx$. Since $xRx\cdot\bar{1}=0$ and $x\cdot\bar{1}\neq 0$, ${}_RM$ is not semiprime. Next, we prove that M is WsP. Suppose that $aRaRb\subseteq RxRx$ (a, $b\in R$). Then, $b=rx$ for some $r\in R$, and hence $aRaRr\subseteq RxR$. If neither r nor a is in RxR, then $aar\notin RxR$, a contradiction. Hence either r or a belongs to RxR. Therefore, $aRr\subseteq RxR$, and hence $aRb\subseteq RxRx$.

(2) Let $R=\begin{pmatrix} F & F\\ 0 & F\end{pmatrix}$. Then ${}_RR$ is not sP but torsionless.

(3) Let $R=\begin{pmatrix} F & F\\ F & F\end{pmatrix}$, $M=\begin{pmatrix} F & 0\\ F & 0\end{pmatrix}$, and $M'=\begin{pmatrix} F & F\\ 0 & 0\end{pmatrix}$. Then ${}_RR$ is neither compressible nor STP, but SP; ${}_RM$ is not STP but faithful simple; ${}_FM'$ is neither simple nor compressible nor uniform, but endosimple.

(4) Let V be a countably infinite dimensional vector space over F, $R=\mathrm{Hom}_F(V,V)$, and $I=\{a\in R\mid \mathrm{rank}\ a<\infty\}$. Then ${}_RR$ is not QSP but prime (see Theorem 1.4 below) and non-singular, and ${}_RI$ is not cofaithful but faithful.

Mathematical Journal of Okayama University, Vol. 24 [1982], Iss. 2, Art. 3

1. SP modules and related modules. In this section, we mainly study further properties of modules mentioned in §0. First, we characterize ZP modules.

Theorem 1.1. *A non-zero module M is ZP if and only if it is prime and $[M^*,M]$-unital ($m[M^*,M]=0$ ($m \in M$) implies $m=0$). In particular, a faithful non-zero module M is ZP if and only if it is prime and torsionless.*

Proof. It suffices to prove the if part. Suppose $(m,M^*)m'=0$ for some m, $m' \in M$. If $m \neq 0$ then there exists some $f \in M^*$ such that $(m,f)M \neq 0$, since M is $[M^*,M]$-unital. Now, for any $a \in R$ we have $(m,f)am'=(m,fa)m' \in (m,M^*)m'=0$, i.e. $(m,f)Rm'=0$. Since M is prime and $(m,f)M \neq 0$, we obtain $m'=0$, which proves that M is ZP.

Next, we consider the case that R is a left duo ring ($aR \subseteq Ra$ for all $a \in R$).

Proposition 1.1. *Let R be a left duo ring, and M an R-module. Then the following are equivalent:*

1) *M is prime.*
2) *M is STP.*
3) *$l_R(m)=l_R(M)$ for any non-zero $m \in M$, that is M is a torsion free $R/l_R(M)$-module.*

Proof. It suffices to show that 1) implies 3). Let m be an arbitrary non-zero element in M. If a is in $l_R(m)$, then $aRm \subseteq Ram=0$. Thus, $aM=0$, i.e. $a \in l_R(M)$.

Corollary 1.1 (cf. [8, Lemma 1.9 and Corollary 1.10]). *Let R be a left duo ring. If M is faithful and prime, then it is HSP and torsion free, and R has to be a completely prime ring; in particular, every non-zero submodule of M is faithful.*

Proposition 1.2. *If M is a uniform module, then the following are equivalent:*

1) *M is STP.*
2) *Each non-zero submodule of M is STP.*
3) *Each non-zero cyclic submodule of M is STP.*

Proof. Obviously, 1) ⇒ 2) ⇒ 3).

3) ⇒ 1). Let m be an arbitrary non-zero element in M. Since Rm

is STP, there exists $a \in R$ such that $l_R(am)=l_R(Rm)$. If $l_R(Rm) \supsetneqq l_R(M)$, then there exists $b \in l_R(Rm) \setminus l_R(M)$. Then there is a non-zero element x in M such that $bx \neq 0$. Let y be an arbitrary non-zero element in Rm, and choose a non-zero z in $Rx \cap Ry$. Since $Rx \supseteq Rz \neq 0$ and Rx is STP, we have $l_R(Rz)=l_R(Rx)$, and so $bx \in bRx=0$, a contradiction. Thus, $l_R(am)=l_R(M)$, that is M is STP.

The next characterizes HSP modules.

Theorem 1.2. *If M is a non-zero module, then the following are equivalent*:

1) *M is HSP.*
2) *M is cofaithful and SP.*
3) *Each non-zero cyclic submodule of M is cofaithful.*
4) *For each non-zero $m \in M$, there exists a positive integer n such that R is embedded in the direct sum $(Rm)^{(n)}$ of n copies of Rm.*

Proof. Obviously, 1) is equivalent with 4) and implies 2) and 3).

2) $\Rightarrow$ 1). Let m be an arbitrary non-zero element in M. Since M is cofaithful and SP, there exists a finite set $\{y_1,\cdots,y_n\} \subseteq M$ such that $\bigcap_{j=1}^{n} l_R(y_j)=0$ and we can find a finite set $\{a_1,\cdots,a_k\} \subseteq R$ such that $\bigcap_{i=1}^{k} l_R(a_i m) \subseteq l_R(y_j)$ for all j. Thus, we have $\bigcap_{i=1}^{k} l_R(a_i m)=0$.

3) $\Rightarrow$ 1). Let m be an arbitrary non-zero element in M. Since Rm is cofaithful, there exists a finite set $\{a_1,\cdots,a_n\} \subseteq R$ such that $\bigcap_{i=1}^{n} l_R(a_i m)=0$.

The equivalence of 2) and 4) in the next theorem has been shown in [4, Corollary 2.3]. However, the theorem will provide an alternative proof of the equivalence.

Theorem 1.3. *If M is a non-zero module, then the following are equivalent*:

1) *M is QSP.*
2) *$T(M)=0$ or M for all LE radicals T on R*-mod.
3) *For each non-zero proper snbmodule N of M,* $\mathrm{Hom}_R(Rx, M/N) \neq 0$ *with some $x \in N$.*
4) *For each non-zero proper submodule N of M, $l_R(x) \subseteq Nm^{-1}$ with some $x \in N$ and $m \in M \setminus N$.*

Proof. Obviously, 4) is equivalent with 3) and implies 1).

1) $\Rightarrow$ 2). Suppose that $0 \neq T(M) \neq M$ for some LE radical T, and choose $y \in M \setminus T(M)$. Since M is QSP, there exists a finite set $\{x_1,\cdots,x_n\}$

$\subseteq T(M)$ such that $\bigcap_{i=1}^{n} l_R(x_i) \subseteq T(M)y^{-1}$. Let $x=(x_1,\cdots,x_n) \in (T(M))^{(n)}$. Then we can define a map $f : Rx \to (Ry+T(M))/T(M)$ by $(ax)f=ay+T(M)$ $(a \in R)$. Since $Rx \supseteq T(Rx)=Rx \cap T(M^{(n)})=Rx \cap (T(M))^{(n)}=Rx$, we have $Rx=T(Rx)$, and so

$$\begin{aligned}(Rx+T(M))/T(M)&=(Rx)f=(T(Rx))f\\ &\subseteq T((Ry+T(M))/T(M)) \subseteq T(M/T(M))=0.\end{aligned}$$

Thus we have $y \in Ry \subseteq T(M)$, a contradiction.

2) $\Rightarrow$ 3). Let N be a non-zero proper submodule of M, and choose $y \in M \backslash N$. We set $P(X)=(y)\mathrm{Hom}_R(Ry,X)$ $(X \in R\text{-mod})$. Then, by [20, Proposition VI.1.5], P is an LE preradical and there exists the smallest radical $\bar{P}$ larger than P. Furthermore, $\bar{P}$ is an LE radical by [20, Corollary VI.3.4]. Since $\bar{P}(M)\neq 0$, we have $\bar{P}(M)=M$, and hence $\bar{P}(M/N)=M/N$. Thus $P(M/N)\neq 0$, which proves $\mathrm{Hom}_R(Ry,M/N)\neq 0$.

Corollary 1.2 (cf. [4, Theorem 2.4]). *The following are equivalent*:

1) *R is a CTF ring.*

2) *For each non-zero proper left ideal L of R, there exists $a \in R\backslash L$ and a finite set $\{x_1,\cdots,x_n\} \subseteq L$ such that $\bigcap_{i=1}^{n} l_R(x_i) \subseteq La^{-1}$.*

3) *For each non-zero proper left ideal L of R, there exists $x \in L$ and $a \in R\backslash L$ such that $l_R(x) \subseteq La^{-1}$.*

4) *For each non-zero proper left ideal L of R, there exists $x \in L$ such that* $\mathrm{Hom}_R(Rx,R/L)\neq 0$.

By Hirano [12], the notion of a fully left idempotent ring has been extended to modules: A non-zero module M is called a *fully left idempotent module* if for each $x \in M$ there exist finite sets $\{f_1,\cdots,f_n\} \subseteq M^*$ and $\{r_1,\cdots,r_n\} \subseteq R$ such that $x=\sum_{i=1}^{n} r_i(x)f_i x$. Now, we extend the notion of a left strongly idempotent ideal to modules. A fully invariant submodule of a non-zero module M is called a *left strongly idempotent submodule* if for each $x \in N$ there exist finite sets $\{f_1,\cdots f_n\} \subseteq M^*$, $\{r_1,\cdots,r_n\} \subseteq R$ and $y \in N$ such that $x=\sum_{i=1}^{n} r_i(y)f_i x$. The following is immediate by definition.

Lemma 1.1. *If M is a fully left idempotent module, then every fully invariant submodule of M is left strongly idempotent.*

Lemma 1.2. *Every CTF-ring R is $E2$.*

Proof. Let I be a non-zero proper left strongly idempotent ideal of R. Since R is CTF, by Corollary 1.2 we have $\mathrm{Hom}_R(Rx,R/I)\neq 0$ with some

$x \in I$. But, since $x=ax$ with some $a \in I$, we get $(x)f=a(x)f=0$ for all $f \in \mathrm{Hom}_R(Rx, R/I)$, a contradiction.

Now, combining Lemmas 1.1 and 1.2, we readily obtain [9, Proposition II.3], [14, Corollary 5.7] and [22, Proposition 2.6] altogether.

Theorem 1.4. *The following are equivalent*:
1) *R is a simple ring.*
2) *R is a fully left idempotent SP ring.*
3) *R is a fully left idempotent CTF ring.*
4) *R is a fully left idempotent E2 ring.*

2. SP rings and related rings. First, we characterize semiprime rings, prime rings, STP rings and others by the existence of such faithful modules as treated in the preceding section.

Theorem 2.1. (1) *The following are equivalent*:
1) *R is a semiprime ring.*
2) *There exists a faithful WsP R-module.*
3) *There exists a faithful sP R-module.*
4) *There exists a faithful ZsP R-module.*

(2) *The following are equivalent*:
1) *R is a prime ring.*
2) *There exists a faithful prime R-module.*
3) *There exists a faithful ZP R-module.*

(3) *R is a non-singular ring if and only if there exists a faithful non-singular R-module.*

(4) *R is an SP ring if and only if there exists an HSP R-module.*

(5) *The following are equivalent*:
1) *R is an STP ring.*
2) *There exists a faithful STP R-module.*
3) *There exists a faithful R-module M such that R can be embedded in each non-zero cyclic submodule of M.*
4) *R is a self-compressible ring.*

Proof. (1) Obviously, 4) ⇒ 3) ⇒ 2).

1) ⇒ 4). If R is a semiprime ring then ${}_RR$ is a faithful ZsP module.

2) ⇒ 1). Let M be a faithful WsP module. If $aRa=0$ for some $a \in R$, then $aRaRm=0$ for each $m \in M$. Hence, $aRm=0$, i.e. $aRM=0$. Since M is faithful, we have $a=0$, and so R is a semiprime ring.

(2) Obviously, 3) ⇒ 2).

1) ⇒ 3). If R is a prime ring then ${}_RR$ is a faithful ZP module.

2) ⇒ 1). Let M be a faithful prime module. If $aRb=0$ for some $a, b \in R$, then $aRbm=0$ for all $m \in M$. In case $a \neq 0$, i.e. $aM \neq 0$, we have $bm=0$ for all m, whence it follows that $b=0$.

(3) Let M be a faithful non-singular module, $a \in Z(R)$, and $m \in M$. Since $l_R(a) \subseteq l_R(am)$, $l_R(am)$ is essential in ${}_RR$, and so $am \in Z(M)=0$. Thus, $aM=0$, and so $a=0$.

(4) By definition, R is an SP ring if and only if ${}_RR$ is HSP. Thus, this is a claim stated in [9, p. 220].

(5) Obviously, 1) ⇒ 2) and 4), and 4) ⇒ 1).

2) ⇒ 3). For each non-zero $m \in M$, there exists $a \in R$ such that $l_R(am)=0$. Then, the map $f: R \to Rm$ defined by $(x)f=xam$ $(x \in R)$ is a monomorphism.

3) ⇒ 2). Let m be a non-zero element in M, and $f: R \to Rm$ a monomorphism. Then $l_R(am)=0$, where $(1)f=am$.

2) ⇒ 1). Let M be a faithful STP module, and b a non-zero element of R. Since $bm \neq 0$ for some $m \in M$, we have $l_R(abm)=0$ for some $a \in R$, and then $l_R(ab)=0$.

Proposition 2.1. *Let M be a non-zero R-module, and e an idempotent of R such that $eM \neq 0$. Then the properties WsP, sP, prime, SP, HSP and STP are inherited by the left eRe-module eM.*

Proof. It suffices to show that if M is SP then so is eM. Let $em \neq 0$ and em' be in eM, and choose a finite set $\{a_1,\cdots,a_n\} \subseteq R$ such that $\bigcap_{i=1}^n l_R(a_iem) \subseteq l_R(em')$. Then we have $\bigcap_{i=1}^n l_{eRe}(ea_ieem) \subseteq l_{eRe}(em')$, which implies that ${}_{eRe}eM$ is SP.

As a combination of Theorem 2.1 and Proposition 2.1, we readily obtain

Corollary 2.1. *Let e be an arbitrary non-zero idempotent of R. Then the ring properties semiprime, prime, endoprimitive, SP and STP are inherited by the ring eRe.*

Next, we improve [7, Proposition 3.1] as follows:

Theorem 2.2. *If R is a fully left idempotent, endoprimitive ring then R is primitive.*

Proof. By [7, Theorem 2.1], there exists a left ideal L of R such that ${}_RR/L$ is a faithful SP module. Choose a maximal left ideal K containing L. We prove that ${}_RR/K$ is faithful, which will imply that R is primitive. Suppose to the contrary that $A=l_R(R/K)\neq 0$. Then A is a non-zero ideal of R with $A \nsubseteq L$, since R/L is faithful. For any $x\in A\backslash L$, we can find a finite set $\{a_1,\cdots,a_n\}\subseteq R$ such that $\bigcap_{i=1}^n l_R(a_ix+L)\subseteq l_R(1+L)=L$. Since R is fully left idempotent, A is left s-unital ($a\in Aa$ for all $a\in A$), and so by [21, Theorem 1] there exists $e\in A$ such that $ea_ix=a_ix$ for all i. Thus $1-e\in\bigcap_{i=1}^n l_R(a_ix+L)\subseteq L$, which yields a contradiction $1\in L+A\subseteq K$.

Every left V-ring is fully left idempotent by [21, Proposition 6], and so the next is immediate by Theorem 2.2.

Corollary 2.2 (cf. [7, Proposition 3.1]). *Let R be a left V-ring (or regular ring). If R is an endoprimitive ring, then R is primitive. In particular, a left non-singular, self-injective ring is primitive if and only if it is endoprimitive.*

Now, we treat with some types of ring extensions. The next can be easily seen.

Proposition 2.2. *Let $R\subseteq S$ be rings and let S be generated by elements which centralize R. If S is either a semiprime ring; a prime ring; an endoprimitive ring; an SP ring, then so is R.*

Proposition 2.2 together with Theorem 2.2 gives the following

Corollary 2.3 (cf. [7, Remark]). *Let R be a semiprime ring, $M(R)$ the Martindale (left) quotient ring of R, and C the center of $M(R)$. Let S be the central closure RC of R.*

(1) *If R is a fully left idempotent ring and S is an endoprimitive ring, then R is primitive.*

(2) *If R is a strongly regular ring and S is a prime ring, then R is a division ring.*

Let $R\subseteq S$ be rings. If $Ss\cap R\neq 0$ for each non-zero $s\in S$, S is called a (left) *intrinsic extension* of R. The next will be almost evident.

Proposition 2.3. *Let S be an intrinsic extension of R. If R is either a semiprime ring; a prime ring; an SP ring; an STP ring, then so is S.*

In advance of proving the final result of this section, we state the following lemma.

Lemma 2.1. *Every bisimple ring R is an STP ring.*

Proof. Given non-zero $a \in R$, there exists $c \in R$ such that $Rc = Ra$ and $cR = R$. Write $c = xa$ and $1 = cy$ with some $x, y \in R$. Then, $l_R(xa) = l_R(c) \subseteq l_R(1) = 0$. Thus R is an STP ring.

We can now prove the following that includes [17, Theorem 3.3(1)], [9, Corollary 1 to Proposition IV.1] and [22, Proposition 2.8 and Corollary 2.9].

Theorem 2.3. (1) *If R is a semiprime ring, then so is $M(R)$.*

(2) *A ring R is simple if and only if R is a fully left idempotent ring and the central closure S of R is an SP ring.*

(3) *Let R be an SP ring, and $Q(R)$ the maximal left quotient ring of R. Then, every ring S with $R \subseteq S \subseteq Q(R)$ is an SP ring. Furthermore, if S is a fully left idempotent ring, then S is simple; in particular, $Q(R)$ is a simple ring.*

(4) *Let R be an STP ring. Then, every ring S with $R \subseteq S \subseteq Q(R)$ is an STP ring and $Q(R)$ is bisimple. In particular, if R is a bisimple ring, then so is $Q(R)$.*

(5) *A left self-injective ring R is an SP ring if and only if it is simple.*

(6) *A left self-injective ring R is an STP ring if and only if it is bisimple.*

Proof. (1) This is immediate by Proposition 2.3, since R is essential in ${}_RM(R)$.

(2) If R is a simple ring, then R is SP and fully left idempotent. Since R is essential in ${}_RS$, S is an SP ring by Proposition 2.3. Conversely, if R is a fully left idempotent ring and S is an SP ring, then R is SP by Proposition 2.2, and so simple by Theorem 1.4.

(3) By Proposition 2.3 and Theorem 1.4.

(4) By Proposition 2.3, S and $Q(R)$ are STP rings. Since $Q(R)$ is a left self-injective ring, $Q(R)$ is bisimple by [13, Corollary 3]. The rest of the proof is immediate by Lemma 2.1.

(5) By (3) and Theorem 1.4.

(6) By (4) and Lemma 2.1.

3. Normal classes and weakly special classes. Throughout this last section, rings need not have identity. We denote by R^1 the Dorroh extension of R obtained by adjoining identity in the usual way. Every left R-module is a left R^1-module in an obvious way. We define *HSP modules, torsionless modules, monoform modules* and *ZsP modules* in the same way as in §0, and *sP modules, SP modules* and *STP modules* are defined for ${}_RM$ with $RM \neq 0$. Then, we can adopt the definitions of *SP rings, endoprimitive rings* and *weakly primitive rings* given in §0, and R is called an *STP ring* if ${}_RR$ is a faithful STP module. For a ring R without identity, we can easily see that the equivalence of 1) and 3) in Theorem 2.1 (1) and that of (2) and (3) in [7, Theorem 2.1] are still valid. The latter equivalence shows that our definition of an endoprimitive ring agrees with the one given in [7, p. 557].

In what follows, R and S will represent rings, V an R-S-bimodule, and W an S-R-bimodule ; all the classes of rings considered are assumed to be non-empty and closed under isomorphisms. A Morita context (R, V, W, S) is said to be *S-faithful* if $S \neq 0$ and $VsW \neq 0$ for every non-zero $s \in S$. A class $\mathcal{N}$ of rings is called a *normal class* if $R \in \mathcal{N}$ implies $S \in \mathcal{N}$ whenever (R, V, W, S) is an S-faithful Morita context. It is known that the class of all prime rings are normal [1]. The first aim of this section is to deduce [7, Theorem 4.2] together with others as a consequence of an easy proposition.

Given a Morita context (R, V, W, S) and a non-zero element m in a left R-module M, we set $M_0 = R^1m$, $U = \{w \in W \mid VwM = 0\}$, $U_0 = \{w \in W \mid Vwm = 0\}$, $J = \{s \in S \mid VsW \subseteq l_R(M)\}$, and $J_0 = \{s \in S \mid VsW \subseteq l_R(M_0)\}$. Obviously, U and U_0 are S-submodules of W, and J and J_0 are ideals of S. Under the above notations, we prove the next.

Proposition 3.1. (1) *If M is sP and $VSW \nsubseteq l_R(M)$, then ${}_{S/J}W/U$ is a faithful sP module.*

(2) *If M_0 is SP and $VSW \nsubseteq l_R(M_0)$, then ${}_{S/J_0}W/U_0$ is a faithful SP module.*

Proof. (1) Obviously, $(S/J)(W/U) \neq 0$ and ${}_{S/J}W/U$ is faithful. Now, suppose that $sSsw \subseteq U$ for some $s \in S$ and $w \in W$. Then $VswRVswM \subseteq VsSswM = 0$. Since M is sP, we have $VswM = 0$, which implies $sw \in U$. Thus, ${}_{S/J}W/U$ is an sP module.

(2) It is obvious that $(S/J_0)(W/U_0) \neq 0$ and ${}_{S/J_0}W/U_0$ is faithful. Let w_0 be an arbitrary element in $W \setminus U_0$. Then $Vw_0m \neq 0$, and $v_0w_0m \neq 0$ for some $v_0 \in V$. Since M_0 is SP, there exists a finite set $\{a_1, \cdots, a_n\} \subseteq R$ such

that $\bigcap_{i=1}^{n} l_R(a_i v_0 w_0 m) \subseteq l_R(m)$. Now, for any $w \in W$ we have $\bigcap_{i=1}^{n} U_0(wa_i v_0 w_0)^{-1} \subseteq U_0 w^{-1}$, which proves that ${}_{S/J_0}W/U_0$ is an SP module.

We can now prove the following that includes [7, Theorem 4.2].

Theorem 3.1. (1) *The class of all semiprime rings is normal.*

(2) *The class of all endoprimitive rings is normal.*

(3) *The class of all rings with faithful uniform SP modules is normal.*

(4) *The class of all rings with faithful monoform SP modules is normal.*

Proof. (1) Since R is a semiprime ring if and only if there exists a faithful sP R-module, (1) is immediate by Proposition 3.1 (1).

(2) By Proposition 3.1 (2).

(3) We can take a faithful uniform SP (cyclic) module as M_0 in Proposition 3.1 (2). Then ${}_SW/U_0$ is a faithful uniform SP module by Proposition 3.1 (2) and [18, Proposition II.7].

(4) We can take a faithful monoform SP (cyclic) module as M_0 in Proposition 3.1 (2). Then ${}_SW/U_0$ is a faithful monoform SP module by Proposition 3.1 (2) and [18, Propositioh II.8].

Remark 3.1. Theorem 3.1 (1) has been given in [1, Corollary 21] together with the normality of the classes of all rings with no non-zero nil left (right or two-sided) ideals and of all Levitzki semisimple rings.

We shall now prove further properties of normal classes.

Proposition 3.2. *Let $\mathcal{N}$ be a normal class of rings.*

(1) *If $R \in \mathcal{N}$, then $eRe \in \mathcal{N}$ for each non-zero idempotent e of R.*

(2) *Let R be a faithful ring ($r_R(R) = l_R(R) = 0$). If $R \in \mathcal{N}$, then the $n \times n$ matrix ring $(R)_n \in \mathcal{N}$ for each positive integer n.*

(3) *Let R be a faithful ring. If $(R)_n \in \mathcal{N}$ for some positive integer n, then $R \in \mathcal{N}$.*

(4) *Let e be a non-zero idempotent of a ring R such that $eRe \cap RaR \neq 0$ for any non-zero $a \in R$. If $eRe \in \mathcal{N}$, then $R \in \mathcal{N}$.*

(5) *Let e be a non-zero idempotent of a semiprime ring R. If $eRe \in \mathcal{N}$ then $R/(l_R(Re) \cap r_R(eR)) \in \mathcal{N}$.*

Proof. Consider the following contexts: (1) (R, Re, eR, eRe).

(2) $(R, R^{(n)}, {}^{(n)}R, (R)_n)$, where ${}^{(n)}R = \{{}^t(a_1, \cdots, a_n) \mid a_i \in R\}$.

(3) $((R)_n, {}^{(n)}R, R^{(n)}, R)$.

(4) (eRe, eR, Re, R).
(5) $(eRe, eR, Re, R/(l_R(Re) \cap r_R(eR))$.

As an immediate consequence of Proposition 3.2 (1) and (5), we obtain [19, Corollary 2 to Proposition 5].

Corollary 3.1. *Let $\mathcal{N}$ be a normal class of prime ring, and e a non-zero idempotent of a prime ring R. Then, $eRe \in \mathcal{N}$ if and only if $R \in \mathcal{N}$.*

The next generalizes [18, Proposition II.10].

Proposition 3.3. *Let $\mathcal{N}$ be a normal class of semiprime rings, and $R \in \mathcal{N}$. If I is a right ideal of R and L is a left ideal of R, then the following are equivalent:*
1) $I \cap L \in \mathcal{N}$.
2) *$I \cap L$ is a semiprime ring.*
3) *For any non-zero $x \in I \cap L$, $x(I \cap L) \neq 0$ and $(I \cap L)x \neq 0$.*
4) *For any non-zero $x \in I \cap L$, $LxI \neq 0$.*

Proof. Obviously, 1)$\Rightarrow$ 2)$\Rightarrow$ 3).

3)$\Rightarrow$ 4). Suppose that $LxI=0$ for some $x \in I \cap L$. Since $xLx \subseteq I \cap L$ and $xLx(I \cap L)=0$, we have $xLx=0$, and hence $Lx=0$ by the semiprimeness of R. Thus, $x=0$.

4)$\Rightarrow$ 1). Obviously, $(R, L, I, I \cap L)$ is an $(I \cap L)$-faithful Morita context. Thus, $I \cap L \in \mathcal{N}$.

Corollary 3.2 (cf. [18, Corollary II.8]). *Let $\mathcal{N}$ be a normal class of semiprime rings, and $R \in \mathcal{N}$. Then, a left (resp. right) ideal L (resp. I) of R is in $\mathcal{N}$ if and only if $r_R(L) \cap L=0$ (resp. $l_R(I) \cap I=0$).*

Proposition 3.4. *Let $\mathcal{N}$ be a normal class of semiprime rings, and L a left ideal of a semiprime ring R with $r_R(L)=0$. Then the following are equivalent:*
1) $R \in \mathcal{N}$.
2) $L \in \mathcal{N}$.
3) $LR \in \mathcal{N}$.
4) *Every subring of R containing L is in $\mathcal{N}$.*
5) *Some subring of R containing L is in $\mathcal{N}$.*

Proof. Obviously, 4) implies 1) and 5).

5) (or 1))$\Rightarrow$ 2). Suppose that a subring T of R containing L is in

$\mathcal{N}$. Then L is in $\mathcal{N}$ by Corollary 3.2.

2) ⇒ 3). As is easily seen, (L,R,L,LR) is an LR-faithful Morita context. Thus, $LR \in \mathcal{N}$.

3) ⇒ 1). It is easy to see that (LR,LR,R,R) is an R-faithful Morita context. Hence, we have $R \in \mathcal{N}$.

2) ⇒ 4). Let S be an arbitrary subring of R containing L. Since (L,S,L,LS) is an LS-faithful Morita context and (LS,LS,S,S) is an S-faithful Morita context, we see that S is in $\mathcal{N}$.

According to [1, Theorem 27], the class of (left) primitive rings is a normal class. Now, the next is an immediate consequence of Proposition 3.4.

Corollary 3.3 ([15, Corollary]). *Let R be a semiprime ring, and L a left ideal of R with $r_R(L)=0$. Then the following are equivalent*:

1) *L is a primitive ring.*
2) *Every subring of R containing L is primitive.*
3) *Some subring of R containing L is primitive.*

Let $\mathcal{P}$ be a class of semiprime rings. We consider the following conditions:

i) Every non-zero ideal of R is in $\mathcal{P}$ whenever R is in $\mathcal{P}$.

ii) Let A be a non-zero ideal of R. Then $R/A^{\perp}$ belongs to $\mathcal{P}$ whenever A is in $\mathcal{P}$, where $A^{\perp}=l_R(A) \cap r_R(A)$.

iii) Let A be a non-zero ideal of a semiprime ring R such that A is essential in ${}_RR$ (and in R_R). Then, R belongs to $\mathcal{P}$ whenever A is in $\mathcal{P}$.

iv) Let A be a non-zero ideal of a semiprime ring R such that $A^{\perp}=0$. Then, R belongs to $\mathcal{P}$ whenever A is in $\mathcal{P}$.

Proposition 3.5 ([11, Lemma 6]). *Let $\mathcal{P}$ be a class of semiprime rings. Then the following are equivalent*:

1) *$\mathcal{P}$ satisfies* i) *and* ii).
2) *$\mathcal{P}$ satisfies* i) *and* iii).
3) *$\mathcal{P}$ satisfies* i) *and* iv).

Following Ju. Rjabuhin (see [11]), a class $\mathcal{P}$ of semiprime rings is called a *weakly special class* if $\mathcal{P}$ satisfies one of the equivalent conditions 1)− 3) in Proposition 3.5.

Theorem 3.2. *Every normal class $\mathcal{P}$ of semiprime rings is a weakly special class.*

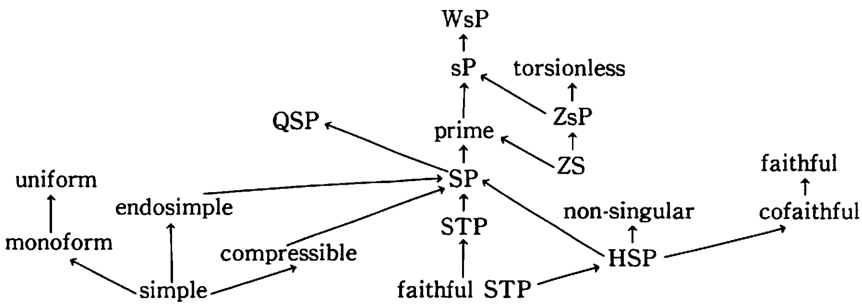
Proof. First, we prove i). Let A be a non-zero ideal of R in $\mathcal{P}$. Considering the A-faithful Morita context (R,R,A,A), we get $A \in \mathcal{P}$. Next, in order to show ii), suppose that a non-zero ideal A of a ring R belongs to $\mathcal{P}$. Since A is a semiprime ring, it is easy to see that the Morita context $(A,A,R/A^{\perp},R/A^{\perp})$ is $R/A^{\perp}$-faithful. Thus, $R/A^{\perp}$ is in $\mathcal{P}$.

Needless to say, a special class defined in [2] is nothing but a weakly special class of prime rings. And so, every normal class of prime rings is a special class.

Remark 3.2. By [8, Proposition 3.3], the class of all SP rings is special. Similarly, we can prove that the class of all STP rings is special. Furthermore the classes of all completely prime rings and of all reduced rings are easily seen to be special. However, these classes are not normal: Let V be an infinite dimensional vector space over a field F, and $E=\mathrm{Hom}_F(V,V)$. Then, (F,V,V^*,E) is an E-faithful Morita context. But, since E is not simple but regular, E can not be an SP ring by Theorem 1.4.

Remark 3.3. Let $\mathcal{P}$ be a weakly special class of semiprime rings. As a direct consequence of Proposition 3.5 and [11, Theorem 1], we see that for each $R \in \mathcal{P}$, there exists a ring S with identity in $\mathcal{P}$ such that R is isomorphic to an ideal of S. In particular, if R is a semiprime ring (resp. reduced ring) then there exists a semiprime ring (resp. reduced ring) with identity such that R is isomorphic to its ideal. Every SP (resp. STP) ring can be embedded in an SP (resp. STP) ring with identity as an ideal. And every completely prime ring can be embedded in a completely prime ring with identity as an ideal (cf. [5, p. 101] and [16, p. 518]).

A table of modules



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(*Received February 3, 1982*)

Math. J. Okayama Univ. 25 (1983), 195

CORRIGENDUM TO "ON STRONGLY PRIME MODULES AND RELATED TOPICS" (This Journal, Vol. 24, pp. 117—132)

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An error in the proof of 2)⇒3) in Theorem 1.3 has been kindly pointed out to the present author by Professor H. Katayama. Since the preradical P defined by the trace need not be LE in general, the proof 2)⇒3) should be changed slightly as follows: Let N be a non-zero proper submodule of M. We set $P(X)=(y)\mathrm{Hom}_R(Ry,X)$ ($y\in N$ and $X\in R$-Mod) and $\bar{P}(X)=X\cap P(\widehat{X})$, where $\widehat{X}$ denotes the injective hull of X. Let $\bar{P}^*(X)=\cap\, Y$, where Y runs through all the submodules of X with $\bar{P}(X/Y)=0$. Then $\bar{P}^*$ is the least LE radical larger than P. By 2), we have $\bar{P}^*(M)=M$. Hence we get $\bar{P}^*(M/N)=M/N$. By the definition of $\bar{P}^*$, we have $\bar{P}(M/N)\neq 0$, and therefore $P(\widehat{M/N})\neq 0$. That is, $\mathrm{Hom}_R(Ry,\widehat{M/N})\neq 0$ for some non-zero element y of N. Then we can choose a non-zero element $\bar{f}$ in $\mathrm{Hom}_R(Ry,\widehat{M/N})$. Since $\widehat{M/N}$ is the injective hull of M/N, there exists $a\in R$ such that $0\neq a(y)\bar{f}\in M/N$. Now, let $f=\bar{f}|_{Ray}$. Then f is a non-zero element of $\mathrm{Hom}_R(Ray,M/N)$, which proves 3).

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(*Received April 28, 1983*)