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A note on nets

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A NOTE ON NETS

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For an accumulation point of a topological space we define a class of directed sets which can be domains of non-constant nets having this point as a cluster point. For certain accumulation points we also study the existence of a convergent net with prescribed domain.

We adopt the terminology and notation used in [2]. Throughout, unless otherwise specified, P will denote a general topological space.

If x is an accumulation point of P, then there is a net $\{x_\alpha\}_{\alpha \in D}$ in $P-(x)$ which converges to x (see [2], p. 66). However, the directed set D on which the net $\{x_\alpha\}_{\alpha \in D}$ is defined cannot be completely arbitrary. In this note we give a simple condition on D which is sufficient for the existence of a net $\{x_\alpha\}_{\alpha \in D}$ in $P-(x)$ having x as a cluster point (see [2], p. 71). Loosely speaking, this condition requires that the set D must be "longer" than a local base at x (see [2], p. 50). In the special case when the point x has a nested local base (see [1], p. 90, def. 4.12.4 or later in this paper), we give also a sufficient condition for the existence of a net $\{x_\alpha\}_{\alpha \in D}$ in $P-(x)$ converging to x.

We begin with three simple examples which will illustrate the problem.

1. Example. Let P be the set of all ordinals less than or equal to the first uncountable ordinal Ω and let D be a countable directed set. In the order topology Ω is an accumulation point of P. However, every net defined on D which is in $P-(\Omega)$ is already in $P-U$, where U is a neighborhood of Ω. Apparently, in this case, the set D was "too short."

2. Example. Let $P=[0, 1]$ and let D be the well ordered set of all countable ordinals. Suppose $\{x_\alpha\}_{\alpha \in D}$ is a net in $P-(0)$ converging to 0. Then there are ordinals $\alpha_n \in D$, $n=1, 2, \cdots$, such that $x_\alpha < 1/n$ for all ordinals $\alpha \in D$ for which $\alpha > \alpha_n$. Choose an ordinal $\alpha_0 \in D$ larger than all α_n. It follows that $x_{\alpha_0}=0$; a contradiction. Here the set D was "to long". However, we can still define a net $\{x_\alpha\}_{\alpha \in D}$ in $P-(0)$ which has 0 as its cluster point. Indeed, it suffices to let $x_\alpha=1/(n+1)$, where $\alpha \in D$ and n is the number of isolated ordinals between α and the largest limit ordinal less than or equal to α.

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3. Example. Let P and D be the same as in example 2 and let N be the well ordered set of all finite ordinals. The product directed set $D \times N$ (see [2], p. 68) is "quite long," e. g., it has no countable cofinal subset. Yet, letting $x_{(\alpha, n)} = 1/(n+1)$ for all $(\alpha, n) \in D \times N$, we have defined a net in $P-(0)$ which converges to 0.

If $(D, >)$ is a directed set and if $\alpha \in D$, we put $D_\alpha = \{\beta \in D : \beta > \alpha\}$. Obviously, D_α is a cofinal subset of D. We recall that every local base Γ at $x \in P$ is directed by the inclusion $\subset$.

4. Definition. A directed set D is said to be *sufficiently long* for a point $x \in P$ if and only if for every $\alpha \in D$ there is a local base Γ_α at x which can be isotonically embedded into D_α.

Let x be an accumulation point of P which has a countable local base. Then a directed set D is sufficiently long for x if and only if D has no maximal element.

If x is an accumulation point of P, we denote by ψ_x a choice function which to every neighborhood U of x associates a point $\psi_x(U)$ from $U-(x)$. If a directed set $(D, >)$ is sufficiently long for x, let $\varphi_\alpha : (\Gamma_\alpha, \subset) \rightarrow (D_\alpha, >)$, $\alpha \in D$, be isotonic embeddings whose existence is required by definition 4.

5. Theorem. *Let x be an accumulation point of P and let $(D, >)$ be a directed set which is sufficiently long for x. Then there is a net $\{x_\alpha, \alpha \in D, >\}$ in $P-(x)$ having x as its cluster point.*

Proof. By $\mathfrak{M}$ we denote the family of graphs of all maps $f : D^f \rightarrow P-(x)$, $D^f \subset D$, which are frequently in every neighborhood U of x (this has meaning, for D^f is a partially ordered set). Since D is sufficiently long for x, we can take $\alpha \in D$ and define $D^g = \varphi_\alpha(\Gamma_\alpha)$ and $g = \psi_x \varphi_\alpha^{-1} : D^g \rightarrow P-(x)$. Given a neighborhood U of x and $\beta \in D^g$, there is $V \in \Gamma_\alpha$ such that $V \subset U \cap \varphi_\alpha^{-1}(\beta)$. Letting $\gamma = \varphi_\alpha(V)$ we obtain $\gamma > \beta$ and $g(\gamma) \in U$. Therefore, the graph of g belongs to $\mathfrak{M}$. It can be readily verified that the union of every non-empty nest $\mathfrak{M}_0 \subset \mathfrak{M}$ belongs to $\mathfrak{M}$. By [2], p. 33, thm. 25 (a) there is a maximal element ϕ in $\mathfrak{M}$. Let ϕ be the graph of the map $f : D^f \rightarrow P-(x)$.

Suppose there is an $\alpha \in D-D^f$. If $\alpha < \beta$ for some $\beta \in D^f$, we let $D^h = D^f \cup (\alpha)$, $h(\gamma) = f(\gamma)$ for $\gamma \in D^f$, and $h(\alpha) = f(\beta)$. Then the graph of h belongs to $\mathfrak{M}$ which contradicts the maximality of ϕ. Therefore, $D^f \cap D_\alpha = \emptyset$. Since D is sufficiently long for x, we can define $D^h = D^f \cup \varphi_\alpha(\Gamma_\alpha)$,

$h(\beta)=f(\beta)$ for $\beta \in D^{f}$ and $h(\beta)=\psi_{x}[\varphi_{\alpha}^{-1}(\beta)]$ for $\beta \in \varphi_{\alpha}(\Gamma_{\alpha})$. As in the first part of the proof it can be shown that the graph of h belongs to $\mathfrak{M}$. This again contradicts the maximality of ϕ.

We conclude that $D^{f}=D$ and the proposition is proved; for it suffices to let $x_{\alpha}=f(\alpha)$ for all $\alpha \in D$.

6. Corollary. *Let x be an accumulation point of P and let D be a directed set. Suppose that P is a T_{1}-space and that x has a countable local base. Then a net $\{x_{\alpha}\}_{\alpha \in D}$ in $P-(x)$ having x as its cluster point exists if and only if D has no maximal element.*

The next example shows that, in general, a converse of theorem 5 is not correct.

7. Example.[1] Let N be the set of all finite ordinals with the discrete topology and let $\beta(N)$ be the Stone-Čech compactification of N (see [2], p. 153). Choose $x \in \beta(N)-N$ and introduce the relative topology into $P=N \cup(x)$. Then P is a Tychonoff space (see [2], p. 117) and x is an accumulation point of P which has no countable local base. Therefore, naturally ordered, the set N is not sufficiently long for x. Yet, the net $\{n\}_{n \in N}$ is in $P-(x)$ and has x as its cluster point.

A local base for a point $x \in P$ is called *nested* if it is completely ordered by inclusion. A point $x \in P$ is called nested whenever it has a nested local base. Obviously, every point which has a countable local base is nested. Some basic properties of nested points are discussed in [1], sec. 4. 12.

8. Definition. A directed set $(D, >)$ is said to be *suitably long* for a nested point $x \in P$ if and only if there is a nested local base Γ at x and an isotonic embedding $\varphi:(\Gamma, \subset) \rightarrow (D, >)$ such that $\varphi(\Gamma)$ has no upper bound in D.

Let x be an accumulation point of P which has a countable local base. Then a directed set D is suitably long for x if and only if it contains a countable unbounded chain (see [2], p. 32). In this case D is also sufficiently long for x. However, in general, a directed set which is suitably long for some point need not be sufficiently long for this point. We give an example.

9. Example. Let P be the same as in example 1 and let $\mathscr{D}$ be the

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1) This example was suggested by C. R. Borges.

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family of all finite subsets of $P-(\Omega)$. The symbol $>$ is used to denote the ordering in P. For $A, B \in \mathcal{D}$ we set $A \succ B$ if and only if $B \subset A$ or $B=(\beta)$ is a singleton and $\beta < \alpha$ for some $\alpha \in A$. It is easy to see that $\succ$ is a direction on $\mathcal{D}$. The family $\Gamma = \{P_\alpha : \alpha \in P-(\Omega)\}$ is a local base at Ω and the map $\varphi : (\Gamma, \subset) \to (\mathcal{D}, \succ)$, defined by the rule $\varphi(P_\alpha)=(\alpha)$ for all $\alpha \in P$, is an isotonic embedding. Since $\varphi(\Gamma)$ has no upper bound in $\mathcal{D}$, the directed set $(\mathcal{D}, \succ)$ is suitably long for Ω. On the other hand, if $A \in \mathcal{D}$ is not a singleton, then $\mathcal{D}_A$ contains only countable chains. Hence $(\mathcal{D}, \succ)$ is not sufficiently long for Ω.

10. Proposition. *Let x be a nested accumulation point of P and let $(D, >)$ be a directed set which is suitably long for x. Then there is a net $\{x_\alpha, \alpha \in D, >\}$ in $P-(x)$ converging to x.*

Proof. Let Γ and φ be as in definition 7. Since $\varphi(\Gamma)$ has no upper bound in D, the sets $\Gamma_\alpha = \{U \in \Gamma : \alpha \notin D_{\varphi(U)}\}$ are non-empty for all $\alpha \in D$. Denote by f a choice function which to every $\alpha \in D$ associate an element $f(\alpha)$ from Γ_α and put $x_\alpha = \psi_x[f(\alpha)]$ for all $\alpha \in D$. Given a neighborhood U of x, there is a $V \in \Gamma$ such that $V \subset U$. Let $\alpha \in D$ and $\alpha > \varphi(V)$. If $f(\alpha) \supset V$, then $\varphi[f(\alpha)] < \varphi(V) < \alpha$ which contradicts the definition of f. Therefore $f(\alpha) \subset V$ and hence $x_\alpha \in U$.

11. Corollary. *Let x be an accumulation point of P and let $(D, >)$ be a directed set. Suppose that P is a T_1-space and that x has a countable local base. Then a net $\{x_\alpha, \alpha \in D, >\}$ in $P-(x)$ converging to x exists if and only if D contains a countable unbounded chain.*

If f is a real-valued function of one real variable and if t is a real number then the set of all numbers $s=\lim f(t_n)$ where $\lim t_n=t$ is closed. The following generalizes this fact.

Let $f: P \to Q$ be a mapping of P into a topological space Q, let x be an accumulation point of P, and let Γ be a local base at x. We denote by Q^x the set of all cluster points of all nets $\{f(x_\alpha)\}$ where $\{x_\alpha\}$ is an arbitrary net in P converging to x. We denote by Q^Γ the set of all cluster points of all nets $\{f(x_U)\}_{U \in \Gamma}$ where $\{x_U\}_{U \in \Gamma}$ is a net in P defined on Γ and converging to x.

12. Proposition. *The set Q^x is always closed. If Γ is sufficiently long for every $y \in Q$, then also the set Q^Γ is closed.*

Proof. Let y be an accumulation point of Q^x and let $\varLambda$ be a local base at y. Choose $U \in \varGamma$ and $V \in \varLambda$. Since $V \cap Q^x \neq \varnothing$, there is a net $\{x_a\}$ in P which is eventually in U and such that $\{f(x_a)\}$ is frequently in V. Hence for some α, $x_a \in U$ and $f(x_a) \in V$. We let $x_{(U,V)} = x_a$ and denote by $>$ the product ordering in $\varGamma \times \varLambda$ (see [2], p. 68). Then the nets $\{x_{(U,V)}, (U, V) \in \varGamma \times \varLambda, >\}$ and $\{f(x_{(U,V)}), (U, V) \in \varGamma \times \varLambda, >\}$ converge to x and y, respectively.

Let $\varGamma$ be sufficiently long for every $y \in Q$ and let y be an accumulation point of Q^r. By the same method as in the proof of Theorem 5 we can construct a net $\{V_U, U \in \varGamma, \subset\}$ of neighborhoods of y which is frequently contained in every neighborhood of y. Choose $U \in \varGamma$. Since $Q^r \cap V_U \neq \varnothing$, there is a net $\{x_W\}_{W \in \varGamma}$ in P which is eventually in U and such that $\{f(x_W)\}_{W \in \varGamma}$ is frequently in V_U. Hence for some $W \in \varGamma$, $x_W \in U$ and $f(x_W) \in V_U$. If we let $x_U = x_W$, then $\{x_U\}_{U \in \varGamma}$ converges to x and $\{f(x_U)\}_{U \in \varGamma}$ has y as its cluster point.

13. Corollary. *If Q is first countable then Q^r is closed.*

The next example shows that the assumption of the second part of proposition 12 cannot be omitted.

14. Example. Let N be the set of all finite ordinals, Q the set of all ordinals less than or equal to the first uncountable ordinal Ω, and let $Q' = Q - (\Omega)$. In Q we consider the order topology. In $P = (Q' \times N) \cup (\infty)$ the points from $Q' \times N$ are isolated and $\varGamma = \{(Q' \times N_n) \cup (\infty)\}_{n=0}^{\infty}$ is a local base at ∞. Thus $\varGamma$ is not sufficiently long for Ω. If we set $f(\alpha, n) = \alpha$ for every $(\alpha, n) \in Q' \times N$ and $f(\infty) = 0$, then $Q^r = Q'$ which is not closed.

REFERENCES

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