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Throughout the present note, R will represent a ring with 1, and $\bar{R}$ the residue class ring of R modulo its (Jacobson) radical $J(R)$. Further, G will represent a group, and RG the group ring of G over R. Occasionally, we consider the following ring epimorphisms:

$$\phi_G : RG \longrightarrow R \quad (\textstyle\sum_{\sigma\in G} r_\sigma\sigma \longmapsto \sum_{\sigma\in G} r_\sigma),$$
$$\psi_G : RG \longrightarrow \bar{R} \quad (\textstyle\sum_{\sigma\in G} r_\sigma\sigma \longmapsto \sum_{\sigma\in G} \bar{r}_\sigma).$$

More generally, if H is a normal subgroup and $G^* = G/H$ then we can consider the following ring epimorphisms:

$$\phi_G^H : RG \longrightarrow RG^* \quad (\textstyle\sum_{\sigma\in G} r_\sigma\sigma \longmapsto \sum_{\sigma\in G} r_\sigma\sigma^*),$$
$$\psi_G^H : RG \longrightarrow \bar{R}G^* \quad (\textstyle\sum_{\sigma\in G} r_\sigma\sigma \longmapsto \sum_{\sigma\in G} \bar{r}_\sigma\sigma^*),$$

where σ^* is the residue class of σ modulo H.

In what follows, we shall show that all the results in [3] are still valid without assuming that R is semi-primary. As to notations and terminologies used here without mention, we follow [3].

First, we shall prove a slight modification of [2; Lemma 2].

Lemma 1. *Let G be a finite p-group: $|G| = p^n$. If $J(R)=0$ and $pR=0$ then $J(RG)=\operatorname{Ker}\phi_G$ and $(J(RG))^{p^n}=0$.*

Proof. Evidently, $\operatorname{Ker}\phi_G \supseteq J(RG)$. It remains therefore to prove $(\operatorname{Ker}\phi_G)^{p^n}=0$. We shall prove this by making use of the induction with respect to n. In case $n=1$, it is easy to see that $(\operatorname{Ker}\phi_G)^p = (\sum_{\sigma\in G} R(1-\sigma))^p = 0$. Next, suppose $n>1$ and that our assertion is true for $n-1$. Choose a normal subgroup H of G such that $(G:H)=p$, and set $G^* = G/H$. Then, by the case $n=1$, we have $(\phi_G^H(\operatorname{Ker}\phi_G))^p = (\operatorname{Ker}\phi_{G^*})^p = 0$, whence it follows $(\operatorname{Ker}\phi_G)^p \subseteq \operatorname{Ker}\phi_G^H = (\operatorname{Ker}\phi_H)G$. Accordingly, by the induction hypothesis, $(\operatorname{Ker}\phi_G)^{p^n} = ((\operatorname{Ker}\phi_H)G)^{p^{n-1}} = (\operatorname{Ker}\phi_H)^{p^{n-1}}G = 0$.

Now, by the light of Lemma 1, we can prove [3; Th. 2] without assuming that R is semi-primary.

Theorem 1. *Let G be a locally finite p-group. If $p\bar{R}=0$ then $J(RG)=\operatorname{Ker}\psi_G$.*

Proof. It is enough to prove that Ker $\psi_G \subseteq J(RG)$. Let $x=\sum_1^n r_i\sigma_i$ be an arbitrary element of Ker ψ_G, where $r_i \in R$ and $\sigma_i \in G$. We set $K=<\sigma_1, \cdots, \sigma_n>$. Then, K is a finite p-group and x is in Ker ψ_K. By Lemma 1, we have ψ_K^1(Ker $\psi_K) \subseteq J(\overline{R}K)$. Hence, (Ker $\psi_K)^{|K|} \subseteq$ Ker $\psi_K^1 = J(R)K \subseteq J(RK)$, whence it follows that x is quasi-regular.

Finally, we shall present the following which contains [3; Ths. 1 and 3] and [2; Th. 2].

Theorem 2. *Let $p\overline{R}=0$. If H is a normal subgroup of G such that G/H is a locally finite p'-group then $J(RG)=J(RH)G$.*

Proof. As $J(RH)G \subseteq J(RG)$ by [3; Lemma 1], it remains only to prove the converse inclusion. Let $x=\sum_1^n a_i\sigma_i$ be an arbitrary element of $J(RG)$, where $a_i \in RH$ and $\sigma_i \in G$. Let $K=<H, \sigma_1, \cdots, \sigma_n>$. Then, it is evident that K/H is a finite p'-group. Now, patterning after the proof of [4; Prop. 1.5], one will easily see that $J(RK)=J(RH)K$. Combining this with [1; Prop. 9], we obtain

$$x \in J(RG) \cap RK \subseteq J(RK) = J(RH)K \subseteq J(RH)G,$$

completing the proof.

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