Mathematical Journal of Okayama University

Volume 10, *Issue* 1 1960 *Article* 6

OCTOBER 1960

On conformal collineations

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ON CONFORMAL COLLINEATIONS

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This paper is devoted to the study of the decomposition of a conformal collineation relative to the reducibility of a manifold.

§ 1. Conformal collineation on an irreducible Riemannian manifold.

We consider an n-dimensional Riemannian manifold M with metric tensor $g_{\mu\lambda}$. The Christoffel symbol, the curvature tensor and the Ricci tensor are denoted by $\{{}^{\kappa}_{\mu\lambda}\}$, $K_{\nu\mu\lambda}{}^{\kappa}$ and $K_{\mu\lambda}$ respectively.

An infinitesimal transformation v^{κ} is called a *conformal collineation*[1] if it satisfies the equation

$$\pounds_v\{{}^{\kappa}_{\mu\lambda}\} = \nabla_\mu \nabla_\lambda v^\kappa + v^\nu K_{\nu\mu\lambda}{}^{\kappa} = A^{\kappa}_{\mu}\sigma_\lambda + A^{\kappa}_{\lambda}\sigma_\mu - g_{\mu\lambda}\sigma^\kappa, \tag{1.1}$$

where $\pounds_v$ indicates the Lie differentiation with respect to v^κ, ∇ the covariant differentiation, A^{κ}_{λ} is the unity tensor and σ_λ is a vector field. The class of conformal collineations contains affine and conformal transformations. Since we have

$$\nabla_\mu \nabla_\alpha v^\alpha = n\sigma_\mu, \tag{1.2}$$

σ_λ is the gradient vector field of a scalar function σ:

$$\sigma_\lambda = \partial_\lambda \sigma. \tag{1.3}$$

Substituting (1.1) into the well-known formula [5, p. 17]

$$\pounds_v K_{\nu\mu\lambda}{}^{\kappa} = \nabla_\nu \pounds_v\{{}^{\kappa}_{\mu\lambda}\} - \nabla_\mu \pounds_v\{{}^{\kappa}_{\nu\lambda}\}, \tag{1.4}$$

we obtain the equation

$$\begin{aligned}(\pounds_v K_{\nu\mu\lambda}{}^{\alpha})g_{\alpha\kappa} &= v^\alpha \nabla_\alpha K_{\nu\mu\lambda\kappa} + K_{\alpha\mu\lambda\kappa}\nabla_\nu v^\alpha + K_{\nu\alpha\lambda\kappa}\nabla_\mu v^\alpha \\ &\qquad + K_{\nu\mu\alpha\kappa}\nabla_\lambda v^\alpha - K_{\nu\mu\lambda}{}^{\alpha}\nabla_\alpha v_\kappa \\ &= -g_{\nu\kappa}\nabla_\mu\sigma_\lambda + g_{\mu\kappa}\nabla_\nu\sigma_\lambda - g_{\mu\lambda}\nabla_\nu\sigma_\kappa + g_{\nu\lambda}\nabla_\mu\sigma_\kappa.\end{aligned} \tag{1.5}$$

Now, from (1.1), we have

$$\nabla_\mu(\nabla_\lambda v_\kappa + \nabla_\kappa v_\lambda) = 2\sigma_\mu g_{\lambda\kappa}, \tag{1.6}$$

or

$$\nabla_\mu(\nabla_\lambda v_\kappa + \nabla_\kappa v_\lambda - 2\sigma g_{\lambda\kappa}) = 0. \tag{1.7}$$

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1) All transformations appearing in this paper are infinitesimal, so we shall omit the modifier "infinitesimal".

If the Riemannian manifold M is irreducible, we have therefore

(1.8) $$\nabla_\lambda v_\kappa + \nabla_\kappa v_\lambda - 2\sigma g_{\lambda\kappa} = 2c g_{\lambda\kappa},$$

c being a constant. Thus the vector field v^κ satisfies the equation

(1.9) $$\pounds_v g_{\mu\lambda} = 2(\sigma + c) g_{\mu\lambda},$$

and we obtain the following

Theorem 1. *If a Riemannian manifold M is irreducible, then a conformal collineation on M is a conformal transformation.*

§ 2. Conformal collineation on a locally reducible Riemannian manifold.

Let a Riemannian manifold M be locally a product

(2.1) $$M_0 \times M_1 \times \cdots \times M_r,$$

where M_0 is the euclidean part and $M_1, \cdots, M_r$ are the irreducible parts. Let each part M_t be of dimension n_t $(t = 0, 1, \cdots, r)$; $n_0 + n_1 + \cdots + n_r = n$. There exists then a local coordinate system $(x^{a_0}, x^{a_1}, \cdots, x^{a_r})$, called a *separated coordinate system*, where the metric tensor field $g_{\mu\lambda}$ is given by a reduced matrix

(2.2) $$(g_{\mu\lambda}) \overset{*}{=} \begin{pmatrix} \delta_{j_0 i_0} & & & 0 \\ & g_{j_1 i_1} & & \\ & & \ddots & \\ 0 & & & g_{j_r i_r} \end{pmatrix},$$

$\delta_{j_0 i_0}$ being the Kronecker delta and the notation $\overset{*}{=}$ meaning that the equation holds in a separated coordinate system. In such a system, the non-vanishing components of $\left\{ {\kappa \atop \mu\lambda} \right\}$ and $K_{\nu\mu\lambda}{}^{\kappa}$ are only $\left\{ {h_t \atop j_t i_t} \right\}$ and $K_{k_t j_t i_t}{}^{h_t}$ respectively, which are dependent only of the variables x^{a_t} belonging to M_t $(t = 1, 2, \cdots, r)$. If we define tensor fields $\overset{t}{g}_{\mu\lambda}$ $(t = 0, 1, \cdots, r)$ by

$$(\overset{t}{g}_{\mu\lambda}) \overset{*}{=} \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & g_{j_t i_t} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{pmatrix},$$

then they have obviously the properties

(2.3) $$\nabla_\nu \overset{t}{g}_{\mu\lambda} = 0$$

and

(2.4) $$g_{\mu\lambda} = \overset{0}{g}_{\mu\lambda} + \overset{1}{g}_{\mu\lambda} + \cdots + \overset{r}{g}_{\mu\lambda}.$$

Now, referring the equation (1. 5) to a separated coordinate system and putting $\kappa = h_s$, $\lambda = i_t$, $\mu = j_t$, $\nu = k_s$ $(s \neq t)$, we have

$$(2.5) \qquad -g_{k_s h_s} \Gamma_{j_t} \sigma_{i_t} - g_{j_t i_t} \Gamma_{k_s} \sigma_{h_s} \overset{*}{=} 0$$

and consequently

$$(2.6) \qquad \Gamma_{j_t} \sigma_{i_t} \overset{*}{=} \alpha_t g_{j_t i_t} \qquad (t = 0, 1, \cdots, r)$$

and the proportional factors $\alpha_0, \alpha_1, \cdots, \alpha_r$ satisfy the relations

$$(2.7) \qquad \alpha_s + \alpha_t = 0 \qquad (s \neq t).$$

If M has at least three parts, then the proportional factors α_t all vanish and we have

$$(2.8) \qquad \Gamma_{j_t} \sigma_{i_t} \overset{*}{=} 0 \qquad (t = 0, 1, \cdots, r).$$

Moreover, putting $\kappa = h_u$, $\lambda = i_s$, $\mu = j_t$, $\nu = k_u$ $(s, t, u \neq)$ in (1. 5), we have also

$$(2.9) \qquad \Gamma_{j_t} \sigma_{i_s} \overset{*}{=} 0 \qquad (s \neq t).$$

The equations (2. 8) and (2. 9) together make up the tensor equation

$$(2.10) \qquad \Gamma_\mu \sigma_\lambda = 0.$$

The equations (2. 9) imply that σ may be written in the form

$$(2.11) \qquad \sigma = \sigma_0 + \sigma_1 + \cdots + \sigma_r ,$$

where each σ_t is a function depending only on the variables x^{a_t} belonging to M_t in a separated coordinate system. However, by (2. 8), $\partial_{i_t} \sigma_t$ is a parallel vector field on the part M_t and hence σ_t for $t = 1, \cdots, r$ are constants in virtue of the irreducibility of M_t and σ_0 is a linear function of the variables x^{a_0} belonging to the euclidean part M_0. Thus σ may be written as

$$(2.12) \qquad \sigma \overset{*}{=} a_{i_0} x^{i_0} + a,$$

a_{i_0} and a being constants.

On the other hand, putting $\kappa = h_s$, $\lambda = i_t$, $\mu = j_u$ $(s, t, u \neq)$ in (1. 6), we have

$$(2.13) \qquad \partial_{j_u} (\Gamma_{i_t} v_{h_s} + \Gamma_{h_s} v_{i_t}) \overset{*}{=} 0,$$

and therefore the expressions in the parentheses are dependent only of x^{a_s} and x^{a_t}. Putting also $\kappa = h_s$, $\lambda = i_t$, $\mu = j_s$ $(s \neq 0)$, we have

$$(2.14) \qquad \Gamma_{j_s} (\Gamma_{i_t} v_{h_s} + \Gamma_{h_s} v_{i_t}) \overset{*}{=} 0.$$

The expressions in the parentheses for each value of i_t are regarded as the components of a parallel vector field on the irreducible part M_s and we have hence

(2. 15) $$\nabla_{i_t} v_{h_s} + \nabla_{h_s} v_{i_t} \overset{*}{=} 0$$

for any pair of h_s and i_t $(s \neq t)$. Moreover, from (1. 1), we have

(2. 16) $$\partial_{j_t} \nabla_{i_s} v^{h_s} = \sigma_{j_t} \delta_{i_s}^{h_s},$$

and consequently the equations

(2. 17) $$\nabla_{i_s} v_{h_s} \overset{*}{=} \sigma g_{j_s h_s} + f_{i_s h_s},$$

where $f_{i_s h_s}$ are functions dependent only of x^{a_s}. Substituting (2. 17) into (1. 6) referred to M_s, we have

(2. 18) $$\nabla_{j_s}(f_{i_s h_s} + f_{h_s i_s}) \overset{*}{=} 0.$$

Therefore we see that for $s = 0$

(2. 19) $$2\beta_{i_0 h_0} \overset{*}{=} f_{i_0 h_0} + f_{h_0 i_0}$$

are constants and for $s \neq 0$

(2. 20) $$f_{i_s h_s} + f_{h_s i_s} \overset{*}{=} 2c_s g_{i_s h_s},$$

c_s being constants. Thus we have

(2. 21) $$\nabla_{i_0} v_{h_0} + \nabla_{h_0} v_{i_0} \overset{*}{=} 2\sigma \delta_{i_0 h_0} + 2\beta_{i_0 h_0}$$

and

(2. 22) $$\nabla_{i_s} v_{h_s} + \nabla_{h_s} v_{i_s} \overset{*}{=} 2\sigma g_{i_s h_s} + 2c_s g_{i_s h_s} \qquad (s \neq 0).$$

If we define a tensor field $\beta_{\mu\lambda}$ by

(2. 23) $$(\beta_{\mu\lambda}) \overset{*}{=} \begin{pmatrix} \beta_{i_0 h_0} & 0 \\ 0 & 0 \end{pmatrix},$$

then $\beta_{\mu\lambda}$ is a symmetric parallel tensor field. The equations (2. 15), (2. 21) and (2. 22) together make up the tensor equation

(2. 24) $$\pounds_v g_{\mu\lambda} = 2\sigma g_{\mu\lambda} + 2\beta_{\mu\lambda} + 2\sum_{s=1}^{r} c_s \overset{s}{g}_{\mu\lambda}.$$

Conversely, if a vector field v^κ satisfies the equation (2. 24), then we substitute (2. 24) into the well-known equation

(2. 25) $$\pounds_v \left\{ {\kappa \atop \mu\lambda} \right\} = \frac{1}{2} g^{\kappa\alpha} (\nabla_\mu \pounds_v g_{\lambda\alpha} + \nabla_\lambda \pounds_v g_{\mu\alpha} - \nabla_\alpha \pounds_v g_{\mu\lambda}),$$

and obtain the equation (1. 1). Thus we have established

Theorem 3. *In order that a vector field v^κ be a conformal collineation, it is necessary and sufficient that v^κ satisfy the equation* (2. 24).

From (2. 6) and (2. 22), we notice here that the vector field given by v^{h_t} on each irreducible part M_t, which we call the *restriction* of v^κ on M_t, defines a concircular transformation [7].

§ 3. Conformal collineation in a locally euclidean manifold.

A locally euclidean manifold M of dimension $n \geqq 2$ may be regarded locally as a product of n straight lines. Accordingly, in a local orthogonal coordinate system (x_h), the function σ is given by[2)]

$$\sigma = \sum a_i x_i + a. \tag{3.1}$$

The equation (1. 1) is reduced to

$$\partial_j \partial_i v^h = \delta_{ih} a_j + \delta_{jh} a_i - \delta_{ji} a_h. \tag{3.2}$$

We seek for the general solution of this equation, cf. [3]. First, from (3. 2) with $h, i, j \neq$, we see that $\partial_i v^h$ are dependent only of the variables x_h and x_i. If $h \neq i = j$ in (3. 2), we have

$$\partial_i \partial_i v^h = -a_h \qquad (h \neq i), \tag{3.3}$$

from which

$$\partial_i v^h = -a_h x_i + \phi_{ih} \qquad (h \neq i), \tag{3.4}$$

ϕ_{ih} being a function of x_h. For $h = j \neq i$ in (3. 2), we have

$$\partial_h \partial_i v^h = \frac{d\phi_{ih}}{dx_h} = a_i \qquad (h \neq i) \tag{3.5}$$

and hence

$$\phi_{ih} = a_i x_h + b_{ih} \qquad (h \neq i), \tag{3.6}$$

b_{ih} being constants. Therefore, from (3. 4), we see that the components v^h are written in the form

$$v^h = -\frac{1}{2} a_h \sum_{i \neq h} x_i^2 + x_h \sum_{i \neq h} a_i x_i + \sum_{i \neq h} b_{ih} x_i + \psi_h, \tag{3.7}$$

where, for each value of h, ψ_h is a function of x_h. From (3. 2) we have also

$$\partial_h \partial_h v^h = a_h \tag{3.8}$$

and, substituting (3. 7) into these equations,

$$\frac{d^2 \psi_h}{dx_h^2} = a_h, \tag{3.9}$$

from which

$$\psi_h = \frac{1}{2} a_h x_h^2 + b_{hh} x_h + b_h, \tag{3.10}$$

b_{hh} and b_h being constants. Thus the vector field v^h is expressed as

2) In this paragraph we do not adopt the summation convension and omit the notation $\overset{*}{=}$ for equations in an orthogonal coordinate system.

(3.11) $$v^h = -\frac{1}{2}a_h \sum_i x_i^2 + x_h \sum_i a_i x_i + \sum_i b_{ih} x_i + b_h.$$

If we define vector fields u^h and w^h by

(3.12) $$u^h = \sum_i b_{ih} x_i + b_h,$$

(3.13) $$w^h = -\frac{1}{2}a_h \sum_i x_i^2 + x_h \sum_i a_i x_i,$$

then u^h defines an affine transformation and w^h a conformal transformation in the locally euclidean manifold. Thus we have

Theorem 3. *A conformal collineation v^κ in a locally euclidean manifold is decomposed into*

(3.14) $$v^\kappa = u^\kappa + w^\kappa,$$

where u^κ is an affine transformation and w^κ a conformal transformation. As it can be easily proved, the decomposition (3.14) *is unique to within a homothetic transformation.*

Since the conformal homeomorphism of a euclidean space onto itself is only a homothety, we can obtain

Theorem 4. *If a conformal collineation v^κ on a euclidean space generates a global one-parameter group of transformations, then the collineation is affine.*

§ 4. The case where M has at least three parts.

By means of the notice at the biginning of § 3, the case where the euclidean part is of dimension $\geqq 2$ is one of the present cases.

If no part of M is locally euclidean in this case, then, by the argument proceding (2.12), the function σ is constant and we have

(4.1) $$\pounds_v \left\{ {}^{\kappa}_{\mu\lambda} \right\} = 0,$$

that is

Theorem 5. *If a Riemannian manifold M has at least three parts and no part is locally euclidean, then a conformal collineation on M is an affine transformation.*

By use of a theorem due to S. Ishihara and M. Obata [1] and S. Kobayashi [2], we can further say

Theorem 6. *If, in addition to the assumption of the above theorem, the manifold M is complete, then a conformal collineation on M is an isometry.*

If there exists a euclidean part M_0, then σ is given by (2.12) and we have

(4.2) $$\sigma_{\iota_0} \overset{*}{=} a_{\iota_0}, \quad \sigma_{\iota_s} \overset{*}{=} 0 \qquad (s \neq 0).$$

The equation (1. 1) with $\kappa = h_0$ is separated into the following equations :

$$
\begin{aligned}
&\nabla_{j_t}\nabla_{i_s} v_{h_0} \overset{*}{=} -g_{j_t i_s} a_{h_0}, \\
(4.3) \qquad &\partial_{j_s}\partial_{i_0} v_{h_0} \overset{*}{=} 0, \qquad (s, t \neq 0) \\
&\partial_{j_0}\partial_{i_0} v_{h_0} \overset{*}{=} \delta_{j_0 h_0} a_{i_0} + \delta_{i_0 h_0} a_{j_0} - \delta_{j_0 i_0} a_{h_0}.
\end{aligned}
$$

By the second equations $\partial_{i_0} v_{h_0}$ are independent of x^{a_s} $(s \neq 0)$, and by the third equations we have the expressions

$$
(4.4) \qquad v_{h_0} \overset{*}{=} -\frac{1}{2} a_{h_0} \sum x_{i_0}^2 + x_{h_0} \sum a_{i_0} x_{i_0} + \sum b_{i_0 h_0} x_{i_0} + \gamma_{h_0},
$$

γ_{h_0} being the functions independent of x^{a_0}. Substituting (4. 4) into the first of (4. 3), the functions γ_{h_0} are solutions of the equations

$$
(4.5) \qquad \nabla_{j_t}\nabla_{i_s} \gamma_{h_0} \overset{*}{=} -g_{j_t i_s} a_{h_0} \qquad (s, t \neq 0).
$$

Now we define a vector field w^κ by the equations

$$
(4.6) \qquad
\begin{aligned}
w_{h_0} &\overset{*}{=} -\frac{1}{2} a_{h_0} \sum x_{i_0}^2 + x_{h_0} \sum a_{i_0} x_{i_0} + \gamma_{h_0}, \\
w_{i_s} &\overset{*}{=} -\sum x_{h_0} \partial_{i_s} \gamma_{h_0}
\end{aligned}
\qquad (s \neq 0)
$$

in the separated system. We can easily verify that the vector field w^κ satisfies the equation

$$
(4.7) \qquad \pounds_w g_{\mu\lambda} = 2\sigma g_{\mu\lambda},
$$

that is, w^κ is a conformal transformation. Since the equation (1. 1) holds also for w^κ, if we put

$$
(4.8) \qquad u^\kappa = v^\kappa - w^\kappa,
$$

then we have

$$
(4.9) \qquad \pounds_u \{^{\,\kappa}_{\mu\lambda}\} = 0,
$$

that is, the vector field u^κ is an affine transformation. Thus

Theorem 7. *If a locally reducible Riemannian manifold M has at least three parts, one of which is euclidean, then a conformal collineation v^κ on M is decomposed into*

$$
(4.10) \qquad v^\kappa = u^\kappa + w^\kappa,
$$

where u^κ is an affine transformation and w^κ a conformal transformation.

Since, in the present case, the function σ depends only on the points of M_0, the equations (2. 22) means that the restriction of v^κ on each part M_s $(s \neq 0)$ defines a homothetic transformation on M_s. If M is complete and simply connected, then M_s $(s \neq 0)$ are complete, simply connected and irreducible. By means of a well-known theorem [1], the homothetic transformation should be an isometry on each M_s $(s \neq 0)$. Hence

(4.11) $$c_s = -\sigma \qquad (s \neq 0)$$

and σ is constant. Then the equation (2.24) is reduced to

(4.12) $$\pounds_v g_{\mu\lambda} = 2\sigma \overset{0}{g}_{\mu\lambda} + 2{}_{f}\beta_{\mu\lambda},$$

and the collineation is affine. The simple connectedness can be removed and we obtain the following

Theorem 8. *If, in addition to the assumption of Theorem 7, the manifold M is complete, then a conformal collineation on M is an affine transformation.*

§ 5. The case where M has two irreducible parts.

We can not go on with the discussions in this general case as yet, but proceed in the case of a manifold of constant scalar curvature, to which we shall confine ourselves in this paragraph. We call here $K = K_{\mu\lambda} g^{\mu\lambda}$ and $k = K/n(n-1)$ the contracted curvature and the scalar curvature of an n-dimensional manifold M respectively.

Let M be locally the product of two parts :

(5.1) $$M = M_1 \times M_2.$$

There occur the two following cases:

i) The two parts are both irreducible.

ii) One part is irreducible and the other is a straight line.

First we consider Case i). Denote the contracted and scalar curvatures of the part M_s by K_s and k_s :

(5.2) $$K_s \overset{*}{=} K_{j_s i_s} g^{j_s i_s}, \quad k_s = \frac{K_s}{n_s(n_s-1)} \qquad (s = 1, 2).$$

We have clearly

(5.3) $$K = K_1 + K_2$$

and K_1 and K_2 are constant, and consequently so are k_1 and k_2. Since the restrictions on M_1 and M_2, denoted here by v_1 and v_2, of a conformal collineation v^κ define concircular transformations on M_1 and M_2 respectively, we can derive the equations

(5.4) $$\begin{aligned} \pounds_{v_1} K_1 &= -2(\sigma + c_1)K_1 - 2(n_1-1)n_1\alpha_1 = 0, \\ \pounds_{v_2} K_2 &= -2(\sigma + c_2)K_2 - 2(n_2-1)n_2\alpha_2 = 0 \end{aligned}$$

from the equations similar to (1.5) for the restrictions v_1 and v_2 by taking account of (2.6) and (2.22). By (2.7) we may put

(5.5) $$\alpha_1 = -\alpha_2 = -\alpha$$

and then from (5. 4) follow the equations

(5. 6) $$\alpha = (\sigma + c_1)k_1 = -(\sigma + c_2)k_2$$

or

(5. 7) $$(k_1 + k_2)\sigma = -(c_1 k_1 + c_2 k_2).$$

If $k_1 + k_2 \neq 0$, we see that σ is a constant and the collineation is affine. If $k_1 = -k_2 \neq 0$, we have $c_1 = c_2$ and the collineation is a conformal transformation. If $k_1 = k_2 = 0$, then α vanishes identically and we have

(5. 8) $$\Gamma_{j_1}\sigma_{i_1} \overset{*}{=} \Gamma_{j_2}\sigma_{i_2} \overset{*}{=} 0.$$

In virtue of the irreducibility of M_1 and M_2, we have $\sigma_\lambda = 0$ and the collineation is affine. Combining these results with Theorem 5, we obtain the following

Theorem 9. *Let a Riemannian manifold M be of constant scalar curvature and have no euclidean part. If M itself is irreducible or M is the product of two irreducible parts whose scalar curvatures are signed oppositely to each other, then a conformal collineation on M is a conformal transformation. Otherwise it is an affine transformation.*

Next we consider Case ii). We suppose that in (5. 1) M_1 is the irreducible part and M_2 the straight line. Then the indices belonging to M_2 take only the number n. Clearly K_1 satisfies the first equation of (5. 4), and K_2 and k_2 vanish. Thus we have

(5. 9) $$\alpha = (\sigma + c)k_1$$

and, from (2. 6), (2. 7) and (2. 22), the equations

(5. 10) $$\Gamma_{j_1}\sigma_{i_1} \overset{*}{=} -(\sigma + c_1)k_1 g_{j_1 i_1}, \qquad \Gamma_n \sigma_n \overset{*}{=} (\sigma + c_1)k_1.$$

If we define a vector field w_κ by

(5. 11) $$w_{h_1} \overset{*}{=} -\sigma_{h_1}, \qquad w_n \overset{*}{=} \sigma_n,$$

then it is verified that the vector field w^κ satisfies the equation

(5. 12) $$\pounds_w g_{\mu\lambda} = \Gamma_\mu w_\lambda + \Gamma_\lambda w_\mu = 2(\sigma + c_1)k_1 g_{\mu\lambda}.$$

Hence w^ι is a conformal transformation. On the other hand, putting

(5. 13) $$c_2 \overset{*}{=} \beta_{nn},$$

the equation (2. 24) is written as

(5. 14) $$\pounds_v g_{\mu\lambda} = 2(\sigma g_{\mu\lambda} + c_1 \overset{1}{g}_{\mu\lambda} + c_2 \overset{2}{g}_{\mu\lambda}).$$

If $k_1 \neq 0$ and we put

(5. 15) $$u^\kappa = v^\kappa - \frac{1}{k_1} w^\kappa,$$

then, from (5. 12) and (5. 15), we have

(5. 16) $$\pounds_u g_{\mu\lambda} = 2(c_2 - c_1)\overset{2}{g}_{\mu\lambda}.$$

Substituting (5. 16) into (2. 25), we can see that the vector field u^κ defines an affine transformation.

If $k_1 = 0$, then we have

(5. 17) $$\nabla_{j_1}\sigma_{i_1} \overset{*}{=} \nabla_n \sigma_n \overset{*}{=} 0$$

and, by the irreducibility of M_1,

(5. 18) $$\sigma_{i_1} \overset{*}{=} 0, \quad \sigma_n \overset{*}{=} a_n$$

and hence

(5. 19) $$\sigma \overset{*}{=} a_n x^n + a,$$

where a_n and a are constants. By the same argument as that in § 4, the n-th component of v^κ is given by

(5. 20) $$v^n \overset{*}{=} \frac{1}{2} a_n x_n^2 + c_2 x_n + \gamma,$$

where γ is a function of the variables x^{a_1} belonging to M_1 and satisfies the equation

(5. 21) $$\nabla_{j_1}\nabla_{i_1}\gamma \overset{*}{=} - g_{j_1 i_1} a_n.$$

If we define a vector field w^κ by the equations

(5. 22) $$w_{i_1} \overset{*}{=} - x_n \partial_{i_1}\gamma, \qquad w_n \overset{*}{=} \frac{1}{2} a_n x_n^2 + \gamma$$

in the separated coordinate system, then the vector field w^κ is a conformal transformation satisfying the equation

(5. 23) $$\pounds_w g_{\mu\lambda} = 2\sigma g_{\mu\lambda}.$$

Moreover we can see that the vector field u^κ given by

(2. 24) $$u^\kappa = v^\kappa - w^\kappa$$

is an affine transformation. Combining these results with Theorem 7, we establish the following

Theorem 10. *Suppose that a Riemannian manifold M is of constant* *scalar curvature and has a euclidean part. Then a conformal collineation v^κ on M is decomposed into*

$$v^\kappa = u^\kappa + w^\kappa, \tag{5.25}$$

where u^κ is an affine transformation and w^κ a conformal one.

Thus the further discussions on conformal collineations, in particular, of a complete and reducible Riemannian manifold, are connected with K. Yano and T. Nagano's study [6] as for the part of affine transformation and with the author's recent work [4] as for the part of conformal transformation.

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(Received July 15, 1960)