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On restricted anti-Hopfian modules

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ON RESTRICTED ANTI-HOPFIAN MODULES

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1. Introduction. In the previous paper [3], we investigated the structure of anti-Hopfian modules (non-simple modules all of whose non-zero factor modules are isomorphic). In connection with the previous investigation, in the present paper, we shall study the structure of non-simple modules all of whose non-zero proper factor modules are isomorphic. We call such a module restricted anti-Hopfian. A restricted anti-Hopfian module has the striking property that every non-zero proper factor module is subdirectly irreducible. Non-simple modules with such property will be called restricted subdirectly irreducible, and will be studied in Section 2. Section 3 is devoted to the study of the structure of restricted anti-Hopfian modules, and in the final theorem (Theorem 14) we shall explicitly describe the structure of restricted anti-Hopfian modules over a commutative ring.

Throughout this paper, R will represent an associative ring with identity and all modules will be unitary right R-modules. For any module M, we denote the *Jacobson radical* and the *socle* of M by $\mathrm{Rad}(M)$ and $\mathrm{Soc}(M)$, respectively. Given a non-empty subset N of an R-module M, we put $\mathrm{Ann}_R(N) = \{r \in R \mid xr = 0 \text{ for all } x \in N\}$.

2. Restricted subdirectly irreducible modules.

Definitions. (a) A module M is said to be *uniserial* if the set of submodules of M is linearly ordered by inclusion.

(b) A non-zero module M is said to be *subdirectly irreducible* if the intersection H of all its non-zero submodules is not 0. In this case, the submodule H is called the *heart* of M.

(c) A module M is called *completely subdirectly irreducible* if every non-zero factor module of M is subdirectly irreducible.

(d) A non-simple module M is called *restricted subdirectly irreducible* (resp. *restricted Artinian*) if each proper non-zero factor module of M is subdirectly irreducible (resp. Artinian).

In this section, we shall study the structure of the restricted subdirectly irreducible modules.

First, we need the following

Lemma 1 (cf. [3, Proposition 1]). *An R-module M is completely sub-*

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directly irreducible if and only if M is Artinian and uniserial.

Proof. It suffices to prove the only if part. Clearly, the set of submodules of M is linearly ordered. Suppose that there exists a countably infinite strictly descending chain

$$M_1 \supsetneqq M_2 \supsetneqq M_3 \supsetneqq \cdots,$$

of submodules of M. If we set $N = \bigcap_{i \in \mathbf{N}} M_i$, then each $\bar{M}_i = M_i/N$ is a non-zero submodule of M/N, but $\bigcap_{i \in \mathbf{N}} \bar{M}_i = 0$. This is contrary to our assumption.

The *quasi-cyclic* (*p-Prüfer*) *group* will be denoted by $\mathbf{Z}(p^\infty)$, and a *cyclic group* of order n by $\mathbf{Z}(n)$.

Example 2. $\mathbf{Z}(p^\infty)$ is completely subdirectly irreducible. In fact, every non-zero factor group of $\mathbf{Z}(p^\infty)$ is isomorphic to $\mathbf{Z}(p^\infty)$. But $\mathbf{Z}(p^\infty)$ is not Noetherian.

We shall now prove the following theorem which plays an important role in this paper.

Theorem 3. *Let M be an R-module. Then, M is restricted subdirectly irreducible if and only if one of the following holds:*

(1) *M is a direct sum of two simple modules*;

(2) *M is restricted Artinian and uniserial*;

(3) *M is Artinian, $M/\mathrm{Soc}(M)$ is non-zero uniserial, $\mathrm{Soc}(M)$ is a direct sum of two simple modules and $\mathrm{Soc}(M)$ is a waist of M (that is, every submodule is comparable with $\mathrm{Soc}(M)$).*

Moreover, if $M \neq \mathrm{Rad}(M)$ and M satisfies (2) or (3), then M is local.

Proof. It suffices to prove the only if part. Let N be a non-zero proper submodule of M. Since M is restricted subdirectly irreducible, every non-zero factor submodule of M/N is subdirectly irreducible. Therefore, by Lemma 1, M/N is Artinian and uniserial. This proves that M is restricted Artinian and M/N is uniserial for every non-zero proper submodule N of M. If M is uniserial, then (2) in this theorem holds. Suppose M is not uniserial. Then there exist two submodules M_1 and M_2 which are not comparable. If $M_1 \cap M_2 \neq 0$, then $M/(M_1 \cap M_2)$ is not subdirectly irreducible. This contradiction implies that $M_1 \cap M_2 = 0$. Then M is embeded in the Artinian module $M/M_1 \oplus M/M_2$, and so M is also Artinian. We shall prove that M_1

and M_2 are simple. If M_1 is not simple, then M_1 contains a simple submodule $M' \neq M_1$. Then $\mathrm{Soc}(M/M')$ isomorphically contains $\mathrm{Soc}(M_1/M') \oplus \mathrm{Soc}(M_2)$. This contradicts the hypothesis that M/M' is subdirectly irreducible. Therefore M_1 is simple. Similarly, we can prove that M_2 is also simple. Hence every submodule of M is comparable with $\mathrm{Soc}(M)$. By the same reason as above, $\mathrm{Soc}(M)$ is a direct sum of two simple modules. Hence, in this case, (1) or (3) in our assertion holds.

Next, we assume that $M \neq \mathrm{Rad}(M)$ and M satisfies (2) or (3), then M does not satisfy (1). If there exist two distinct maximal submodules M_1 and M_2, then $M_1 \cap M_2 = 0$. In this case, M satisfies (1). This is a contradiction. Therefore, if $M \neq \mathrm{Rad}(M)$ and M satisfies (2) or (3), then M is local. This completes the proof.

In case R is commutative, we can prove the following

Theorem 4. *Let R be a commutative ring, and M an R-module such that $M \neq \mathrm{Rad}(M)$. Then, M is restricted subdirectly irreducible if and only if one of the following holds :*

(1) *M is a direct sum of two simple modules ;*

(2) *M is local, Noetherian and uniserial ;*

(3) $\mathrm{Soc}(M)$ *is a unique maximal submodule of M, and is a direct sum of two simple modules.*

Proof. If M satisfies (1) or (3), then clearly M is restricted subdirectly irreducible. Suppose that M satisfies (2). For any $m \in M \backslash \mathrm{Rad}(M)$, we have that $M = mR \cong R/\mathrm{Ann}_R(m)$. Let J be the Jacobson radical of $R/\mathrm{Ann}_R(m)$. Then we can easily see that if $MJ^n \neq 0$ for some positive integer n, then MJ^{n+1} is a unique maximal submodule of MJ^n. By the Krull intersection theorem, $\bigcap_{n=1}^{\infty} MJ^n = 0$. Therefore, $0, M, MJ, MJ^2, \dots$ are the only submodules of M. Hence M is restricted subdirectly irreducible.

Conversely, suppose that M is restricted subdirectly irreducible. First, we consider the case when M satisfies (2) in Theorem 3. Then M is local and $M = mR$ for any $m \in M \backslash \mathrm{Rad}(M)$. Let N be a non-zero submodule of M. Then M/N is Artinian, and so is $\overline{R} = R/\mathrm{Ann}_R(m+N)$ ($\cong M/N$ as R-modules). Clearly, $\overline{R}$ is Noetherian and hence the cyclic module M/N over $\overline{R}$ is also Noetherian. This shows that M is Noetherian. Next, we consider the case when M satisfies (3) of Theorem 3. Suppose, to the contrary, that $\mathrm{Soc}(M)$ is not maximal. Then $M/\mathrm{Soc}(M)$ is not simple. Let $N'/\mathrm{Soc}(M)$ be the heart of $M/\mathrm{Soc}(M)$, and N/N' the heart of M/N'. Then we have a chain of submodules

$$\mathrm{Soc}(M) \subsetneqq N' \subsetneqq N,$$

where both N/N' and $N'/\mathrm{Soc}(M)$ are simple, and both N and N' are local Artinian. If we take $x \in N \backslash N'$, then $N = xR$ and $N' = xaR$ for some $a \in R$. Since $\widetilde{R} = R/\mathrm{Ann}_R(x) \cong xR = N$, $\widetilde{R}$ is local and Artinian. Clearly, $\mathrm{Rad}(\widetilde{R}) = \tilde{a}\widetilde{R}$, where $\tilde{a} = a + \mathrm{Ann}_R(x)$. Therefore, we conclude that

$$\widetilde{R} \supsetneqq \tilde{a}\widetilde{R} \supsetneqq \tilde{a}^2\widetilde{R} \supsetneqq \cdots$$

is a unique composition series of $\widetilde{R}$. Hence N has also a unique composition series. This is a contradiction. Therefore $\mathrm{Soc}(M)$ is a unique maximal submodule of M. This completes the proof.

Example 5. Let K be a field, and $R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| \, a \in K,\ b \in K \oplus K \right\}$. Then the right R-module R_R satisfies (3) in Theorem 4.

Let R be a Dedekind domain, K the field of fractions of R, and P a prime ideal of R. We denote by $R(P^\infty)$ the P-primary part of K/R and, following Kaplansky [4, p. 335], we call this the *module of type* P^∞. It is easily seen that $R(P^\infty)$ is isomorphic to K/R_P, where R_P is the localization of R at P.

When R is a Dedekind domain, we can completely classify the restricted subdirectly irreducible R-modules as follows :

Theorem 6. *Let R be a Dedekind domain, and M an R-module. Then, M is restricted subdirectly irreducible if and only if one of the following holds :*

(1) *$M \cong R/P \oplus R/Q$ for some prime ideals P and Q ;*

(2) *$M \cong R/P^n$ for some prime ideal P and some positive integer n ;*

(3) *M is isomorphic to $R(P^\infty)$ for some prime ideal P ;*

(4) *R is a discrete valuation ring and M is isomorphic to the field of fractions K of R.*

Proof. "If" : This follows from Theorem 3.

"Only if" : First, suppose that $M \neq \mathrm{Rad}(M)$. If M satisfies (3) in Theorem 4, then M is isomorphic to R/I for some non-zero ideal I. Since R is a Dedekind domain, we have a decomposition

$$I = P_1^{n_1} P_2^{n_2} \cdots P_k^{n_k}$$

with some prime ideals P_i and positive integers n_i. Hence

$$R/I \cong R/P_1^{n_1} \oplus R/P_2^{n_2} \oplus \cdots \oplus R/P_k^{n_k}.$$

Since Soc(M) is a direct sum of two simple modules, we conclude $k = 2$. But, in this case, R/I is not local. Hence, this case cannot occur. If M satisfies (2) in Theorem 4, M is also a cyclic R-module. Since M is local, M is isomorphic to R/P^n for some prime ideal P and some positive integer n. Clearly, if M satisfies (1) in Theorem 4, then (1) in this theorem holds. Next, suppose that $M = \text{Rad}(M)$. In this case, we claim that M is divisible. Suppose, to the contrary, that M is not divisible. Then there exists a non-zero element p in R such that $Mp \neq M$. Since R is a Dedekind domain, we have a decomposition

$$(p) = P_1^{n_1} P_2^{n_2} \cdots P_t^{n_t}$$

with some prime ideals P_i and positive integers n_i. Then $MP_i \neq M$ for some i, and thus M/MP_i is a non-zero vector space over the field R/P_i. Therefore, there exists a maximal submodule N of M containing MP_i. This is contrary to the assumption that $M = \text{Rad}(M)$, and so we conclude that M is a divisible R-module. Then by Kaplansky [4, Theorem 7], M is the direct sum of a vector space over K and modules of type P^∞ for various prime ideals P. Since M is restricted subdirectly irreducible, we conclude that either M is isomorphic to $R(P^\infty)$ for some prime ideal P or M is isomorphic to K. In the latter case, since K is a uniserial R-module (by Theorem 3), it is easy to see that R has exactly one non-zero prime ideal, that is, R is a discrete valuation ring. This completes the proof.

As a particular case of Theorem 6, we have

Corollary 7. *An abelian group M is restricted subdirectly irreducible if and only if one of the following holds* :

(1) $M \cong \mathbf{Z}(p) \oplus \mathbf{Z}(q)$ *for some primes p and q* ;
(2) $M \cong \mathbf{Z}(p^n)$ *for some prime p and some positive integer n* ;
(3) $M \cong \mathbf{Z}(p^\infty)$ *for some prime p.*

3. Restricted anti-Hopfian modules.

Definitions. (e) A module M is said to be *Hopfian* if every surjective endomorphism of M is an isomorphism.

(f) A submodule N of M is said to be a *non-Hopf kernel* (*for* M) if there exists an isomorphism of M/N to M.

(g) A non-simple module M is said to be *anti-Hopfian* if every proper submodule of M is a non-Hopf kernel.

(h) A non-simple module M is said to be *restricted anti-Hopfian* if any two non-zero proper factor modules of M are isomorphic. Clearly, every anti-Hopfian module is restricted anti-Hopfian.

As is well known, every module has a subdirectly irreducible factor module (see, e.g., Anderson and Fuller [1, p. 95]). Hence every restricted anti-Hopfian module is restricted subdirectly irreducible. The purpose of this section is to study about the structure of restricted anti-Hopfian modules and their endomorphism rings.

First, we shall consider the case when M has at least one maximal submodule.

Theorem 8. *Let M be an R-module such that $M \neq$ Rad(M). Then, M is restricted anti-Hopfian if and only if one of the following holds :*

(1) *M has exactly one non-zero proper submodule ;*

(2) *M is a direct sum of two isomorphic simple modules.*

Proof. The if part is clear. We shall prove the only if part. Since M is restricted subdirectly irreducible, we can apply Theorem 3. At first, we consider the case when M satisfies (2) in Theorem 3. Then we claim that M has exactly one non-zero proper submodule. Suppose, to the contrary, that

$$0 \subsetneqq J_1 \subsetneqq J \subsetneqq M$$

is a chain of submodules of M. Then M/J and M/J_1 are not isomorphic, because M/J is simple and M/J_1 is not simple. This contradicts our hypothesis on M. Therefore, M has exactly one non-zero proper submodule, that is, (1) in this theorem holds. If M satisfies (1) in Theorem 3, then (2) in this theorem holds, clearly. Finally, we consider the case that M satisfies (3) in Theorem 3. Let J be the unique maximal submodule of M and Soc$(M) = S_1 \oplus S_2$, where S_1 and S_2 are simple. Then M/J and M/S_1 are not isomorphic. Hence, this case cannot occur, completing the proof.

Corollary 9. *Let R be a Dedekind domain, and M an R-module such that $M \neq$ Rad(M). Then, M is restricted anti-Hopfian if and only if one of the following holds :*

(1) $M \cong R/P^2$;
(2) $M \cong R/P \oplus R/P$, *where* P *is a non-zero prime ideal of* R.

Proof. This is immediate from Theorems 6 and 8.

A ring R is said to be a (*right*) *CH-ring* if every cyclic right R-module is Hopfian. Clearly, every right Noetherian ring is a CH-ring. As is well known, every finitely generated module over a commutative ring R is Hopfian (see, e.g., Armendariz, Fisher and Snider [2]). Hence, every commutative ring is a CH-ring.

Next, we shall consider a restricted anti-Hopfian module M with $M = \mathrm{Rad}(M)$. When this is the case, for any non-zero proper submodule N of M, M/N is a non-simple R-module all of whose factor modules are isomorphic. Hence, M is a restricted anti-Hopfian module with $M = \mathrm{Rad}(M)$ if and only if M/N is anti-Hopfian for every non-zero proper submodule N of M.

Now, by making use of Theorem 3 and [3, Theorem 2], we shall characterize restricted anti-Hopfian modules M over a CH-ring with $M = \mathrm{Rad}(M)$.

Theorem 10. *Let* R *be a CH-ring, and* M *an R-module such that* $M = \mathrm{Rad}(M)$. *Then,* M *is restricted anti-Hopfian if and only if one of the following holds :*

(1) 1a) *The set of proper submodules of* M *forms a chain*

$$0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq M_3 \subsetneqq \cdots$$

such that

$$\bigcup_{i \in \mathbf{N}} M_i = M, \text{ and}$$

1b) M_2/M_1 *is a non-Hopf kernel for* M/M_1.

(2) 2a) *The set of proper submodules of* M *forms a chain*

$$\cdots \subsetneqq M_{-2} \subsetneqq M_{-1} \subsetneqq M_0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq \cdots$$

such that

$$\bigcap_{i \in \mathbf{Z}} M_i = 0, \ \bigcup_{i \in \mathbf{Z}} M_i = M, \text{ and}$$

2b) *for each* i, M_{i+1}/M_i *is a non-Hopf kernel for* M/M_i.

(3) 3a) $\mathrm{Soc}(M)$ *is a waist of* M, *and is a direct sum of two isomorphic simple modules and the set of proper submodules of* M *containing* $\mathrm{Soc}(M)$ *forms a chain*

$$M_1 = \mathrm{Soc}(M) \subsetneqq M_2 \subsetneqq M_3 \subsetneqq \cdots$$

such that

$$\bigcup_{i \in \mathbf{N}} M_i = M, \text{ and}$$

3b) *for every simple submodule S of M, M_1/S is a non-Hopf kernel for M/S.*

Proof. "Only if" : First, suppose that M satisfies (2) in Theorem 3, namely M is restricted Artinian and uniserial. If $\mathrm{Soc}(M) = M_1 \neq 0$, M/M_1 is anti-Hopfian and hence, by [3, Theorem 2], (1) in our assertion holds. Next, we shall show that if $\mathrm{Soc}(M) = 0$ then (2) in this theorem holds. Let M_1 be a non-zero proper submodule of M. By [3, Theorem 2], since M/M_1 is anti-Hopfian, the set of proper submodules of M containing M_1 forms a chain

$$M_1 \subsetneqq M_2 \subsetneqq M_3 \subsetneqq \cdots$$

such that

$$\bigcup_{i \in \mathbf{N}} M_i = M.$$

Since M_1 has a non-zero proper submodule M_0' and M/M_0' is anti-Hopfian, again by [3, Theorem 2] M_1 has the unique maximal submodule M_0. Continuing this procedure, we have a chain of the submodules of M

$$\cdots \subsetneqq M_{-2} \subsetneqq M_{-1} \subsetneqq M_0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq M_3 \subsetneqq \cdots.$$

It is easy to see that those are the only non-zero proper submodules of M, $\bigcap_{i \in \mathbf{Z}} M_i = 0$ and $\bigcup_{i \in \mathbf{Z}} M_i = M$. The assertion 2b) is obvious.

Finally, suppose that M satisfies (3) in Theorem 3. Then, by hypothesis, $\mathrm{Soc}(M)$ is a waist of M and the set of proper submodules of M containing $\mathrm{Soc}(M)$ forms a chain

$$M_1 = \mathrm{Soc}(M) \subsetneqq M_2 \subsetneqq M_3 \subsetneqq \cdots$$

such that

$$\bigcup_{i \in \mathbf{N}} M_i = M.$$

It is easy to see that $\mathrm{Soc}(M)$ is a direct sum of two isomorphic simple modules. Again by [3, Theorem 2], M_1/S is a non-Hopf kernel for M/S for every simple submodule S of M.

"If" : Assume (1). Since the factor module M/M_1 is anti-Hopfian by [3, Theorem 2], we see that

$$M/M_1 \cong (M/M_1)/(M_i/M_1) \cong M/M_i$$

for all $i \in \mathbb{N}$.

Assume (2). Let M_i be an arbitrary non-zero proper submodule of M. Since the factor module M/M_i is anti-Hopfian by [3, Theorem 2], M is restricted anti-Hopfian.

Finally, assume (3). Let S be an arbitrary simple submodule of M. Again by [3, Theorem 2], the factor module M/S is anti-Hopfian, and so we obtain $M/S \cong M/N$ for every proper submodule N of M containing S. This shows that M is restricted anti-Hopfian, completing the proof.

Corollary 11. *Let R be a commutative ring, and M an R-module such that $M = \mathrm{Rad}(M)$. Then, M is restricted anti-Hopfian if and only if one of the following holds :*

(1) *The set of proper submodules of M forms a chain*

$$0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq M_3 \subsetneqq \cdots$$

such that

$$\bigcup_{i \in \mathbb{N}} M_i = M,$$

that is, M is anti-Hopfian.

(2) *The set of proper submodules of M forms a chain*

$$\cdots \subsetneqq M_{-2} \subsetneqq M_{-1} \subsetneqq M_0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq \cdots$$

such that

$$\bigcap_{i \in \mathbb{Z}} M_i = 0, \ \bigcup_{i \in \mathbb{Z}} M_i = M.$$

Proof. In view of Theorem 10 and [3, Theorem 8], it suffices to show that M does not satisfy (3) in Theorem 10. Suppose, to the contrary, that M satisfies (3) in Theorem 10, and choose a simple submodule S of M. Then, $\mathrm{Soc}(M)$ is a waist of M and the set of proper submodules of M containing $\mathrm{Soc}(M)$ forms a chain

$$M_1 = \mathrm{Soc}(M) = S \oplus S' \subsetneqq M_2 \subsetneqq M_3 \subsetneqq \cdots$$

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such that

$$\bigcup_{i \in \mathbf{N}} M_i = M$$

with some simple submodule S' of M. And so, there exist m_1 and m_2 in M such that $S = m_1R$, $M_2 = m_2R$. Since $S \subsetneqq M_2$, there exists r_0 in R such that $m_1 = m_2r_0$. Now we define $f \in \mathrm{End}_R(M)$ by $f(x) = xr_0$ $(x \in M)$. Since $f(S') \subset f(M_2)$ and $0 \neq f(M_2) = S$, we see that $S' \subset \mathrm{Ker}(f)$. Hence $\mathrm{Ker}(f)$ is a non-zero proper submodule of M. Since every non-zero proper submodule is finitely generated, f must be an epimorphism, because M is not finitely generated. Hence $M/\mathrm{Ker}(f) \cong M$. This shows that M is anti-Hopfian, which contradicts [3, Theorem 8].

We shall describe here some properties of restricted anti-Hopfian modules M, and the structure of their endomorphism rings $\mathrm{End}_R(M)$.

Proposition 12. *Let R be a CH-ring, and M an R-module such that $M = \mathrm{Rad}(M)$. If M is not anti-Hopfian but restricted anti-Hopfian, then*

(1) *every proper submodule of M is finitely generated*;

(2) *$S = \mathrm{End}_R(M)$ is a division ring.*

Proof. (1). In case M satisfies (1) or (2) in Theorem 10, every proper submodule of M has a unique maximal submodule, so that it is cyclic. On the other hand, in case M satisfies (3) in Theorem 10, $\mathrm{Soc}(M)$ is generated by two elements and other proper submodules are cyclic.

(2). Let g be an arbitrary non-zero element of S. Then $g(M) \cong M/\mathrm{Kef}(g)$. If $g(M)$ is a proper submodule of M, then M is finitely generated by (1). This contradicts the assumption $M = \mathrm{Rad}(M)$. Thus we have $g(M) = M$ and hence $M \cong M/\mathrm{Ker}(g)$. Since M is not anti-Hopfian, $\mathrm{Ker}(g) = 0$. Therefore S is a division ring.

Lemma 13. *Let R be a commutative ring, and M an R-module such that $M = \mathrm{Rad}(M)$. If M is not anti-Hopfian but restricted anti-Hopfian, then*

(1) *every proper submodule of M is cyclic*;

(2) *any two non-zero proper submodules are isomorphic*;

(3) *$\bar{R} = R/\mathrm{Ann}_R(M)$ is a discrete valuation ring*;

(4) *M is an injective $\bar{R}$-module (so that M is a quasi-injective R-module).*

Proof. By Corollary 11, the set of non-zero proper submodules of M

forms a chain

$$\cdots \subsetneqq M_{-2} \subsetneqq M_{-1} \subsetneqq M_0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq \cdots$$

such that

$$\bigcap_{i \in Z} M_i = 0, \ \bigcup_{i \in Z} M_i = M.$$

(1). Since each M_i has the unique maximal submodule M_{i-1}, we obtain $M_i = m_i R$ for any $m_i \in M_i \backslash M_{i-1}$.

(2) and (3). Let m_i be a generator of M_i for each i, namely $M_i = m_i R$. Then there exists $r_0 \in R$ such that $m_i = m_{i+1} r_0$. We now define $f \in \mathrm{End}_R(M)$ by $f(x) = x r_0$ $(x \in M)$. Since $f(m_{i+1}) = m_{i+1} r_0 = m_i$, f is an isomorphism by Proposition 12. Then $M_{i+1} \cong f(M_{i+1}) = M_i$; furthermore $f(M_j) = M_{j-1}$ for any j. Hence $M_i \cong M_t$ for any t, so that $\mathrm{Ann}_R(M) = \mathrm{Ann}_R(M_i)$. Therefore $M_i = m_i R \cong R/\mathrm{Ann}_R(M) = \bar{R}$. Taking the structure of the module M_i into consideration, we conclude that $\bar{R}$ is a discrete valuation ring.

(4). Let a be an arbitrary non-zero element of $\bar{R}$. We define an R-epimorphism $h : M \to Ma$ by $h(x) = xa$ $(x \in M)$. By Proposition 12, $M \cong Ma$. Since M is not finitely generated, we conclude that $M = Ma$. Therefore M is a divisible $\bar{R}$-module. As is well known, over a Dedekind domain, divisibility is the same with injectivity (see, e.g., Rotman [5, Theorem 4.27]). Therefore M is an injective $\bar{R}$-module. This completes the proof.

We denote the lattice of the R-submodules of M by $\mathscr{L}_R(M)$. $\mathrm{Q}(U)$ denotes the field of fractions of an integral domain U. When R is a commutative ring, we can explicitly describe the class of restricted anti-Hopfian R-modules.

Theorem 14. *Let R be a commutative ring, and M an R-module. Then, M is restricted anti-Hopfian if and only if one of the following holds :*

(1) *M has exactly one non-zero proper submodule ;*

(2) *M is a direct sum of two isomorphic simple modules ;*

(3) *$S = \mathrm{End}_R(M)$ is a discrete valuation ring, $M \cong \mathrm{Q}(S)/S$ and $\mathscr{L}_S(M) = \mathscr{L}_R(M)$;*

(4) *$\bar{R} = R/\mathrm{Ann}_R(M)$ is a discrete valuation ring and M is isomorphic to $\mathrm{Q}(\bar{R})$.*

Proof. To prove this theorem, it suffices to show that the following three statements hold :

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(I) M is a restricted anti-Hopfian module with $M \neq \text{Rad}(M)$ if and only if (1) or (2) holds.

(II) M is an anti-Hopfian module if and only if (3) holds.

(III) M is not an anti-Hopfian module, but a restricted anti-Hopfian module with $M = \text{Rad}(M)$ if and only if (4) holds.

Proof of (I). This follows from Theorem 8.

Proof of (II). "Only if" : This follows from [3, Theorem 10] and its proof.

"If" : Let P be the unique maximal ideal of S. Since $M \cong Q(S)/S$ $(\cong S(P^\infty))$, M is anti-Hopfian by [3, Theorem 9]. This together with $\mathscr{L}_S(M) = \mathscr{L}_R(M)$ implies that M is an anti-Hopfian R-module.

Proof of (III). "Only if" : By Lemma 13 (3), $\bar{R}$ is a discrete valuation ring. Since $M = \text{Rad}(M)$ and M is not anti-Hopfian, none of (1), (2) and (3) in Theorem 6 can occur. Therefore M is isomorphic to $Q(\bar{R})$.

"If" : Let P be the unique maximal ideal of $\bar{R}$. Then the set of proper submodules of $Q(\bar{R})$ forms a chain

$$\cdots \subsetneqq P^2 \subsetneqq P \subsetneqq \bar{R} = P^0 \subsetneqq P^{-1} \subsetneqq P^{-2} \subsetneqq \cdots,$$

where P^{-n} denotes the inverse of P^n in the ideal group of $\bar{R}$. It is easy to see that

$$\bigcap_{i \in Z} P^{-i} = 0 \text{ and } \bigcup_{i \in Z} P^{-i} = Q(\bar{R}).$$

Now, our assertion follows from the conditions (2) in Corollary 11.

Combining Theorem 6 with Corollary 9 and Theorem 14 we readily obtain the following

Corollary 15. *Let R be a Dedekind domain, and M an R-module. Then, M is restricted anti-Hopfian if and only if one of the following holds :*

(1) $M \cong R/P^2$;

(2) $M \cong R/P \oplus R/P$, *where P is a non-zero prime ideal of R* ;

(3) *M is isomorphic to $R(P^\infty)$ for some prime ideal P* ;

(4) *R is a discrete valuation ring and M is isomorphic to the field of fractions K of R.*

In particular, if $M = \text{Rad}(M)$, the following statements are equivalent :

1) *M is a restricted anti-Hopfian module.*

2) *M is a restricted subdirectly irreducible module.*

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REFERENCES

[1] F. W. ANDERSON and K. R. FULLER: Rings and Categories of Modules, Springer-Verlag, Berlin and New York, 1974.

[2] E. P. ARMENDARIZ, J. W. FISHER and R. L. SNIDER: On injective and surjective endomorphisms of finitely generated modules, Comm. Algebra **6** (7) (1978), 659–672.

[3] Y. HIRANO and I. MOGAMI: Modules whose proper submodules are non-Hopf kernels, to appear in Comm. Algebra.

[4] I. KAPLANSKY: Modules over Dedekind rings and valuation rings, Trans. Amer. Math. Soc. 72 (1952), 327–340.

[5] J. J. ROTMAN: An Introduction to Homological Algebra, Academic Press, 1979.

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