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ON NIL AND NILPOTENT DERIVATIONS

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One of the purposes of this note is to answer the following problem raised by L. O. Chung [2] : Let R be a prime ring, I a non-zero ideal of R, and d a derivation of R. If d is nil on I, that is, for each $x \in I$ there is a positive integer n (depending on x) such that $d^n(x) = 0$, then is d also nil on R? Stating the conclusion first, the answer is negative (see Example 1 below), but is positive for left (or right) strongly prime rings (Theorem 1). Next, we shall consider nil derivations of PI-rings. We shall prove that every derivation of a semiprime PI-ring R which is trivial on the center of R can be extended to an inner derivation of the maximal right quotient ring Q of R (Theorem 3). Using this result, we shall prove that every nil derivation of a semiprime PI-ring R which is trivial on the center of R is nilpotent.

A ring R is said to be *left* (resp. *right*) *strongly prime* (abbr. *SP*) if every non-zero left (resp. right) ideal of R contains a finite set whose left (resp. right) annihilator ideal is zero ([4]). Completely prime rings (domains) and prime right Goldie rings are examples of left SP rings. More generally, every prime ring with DCC on left annihilators is left SP. For other examples of left SP rings, see [4].

In preparation for proving Theorem 1, we state the following

Lemma 1. *Let d be a derivation of a ring R. Then, for each positive integer k there exist integers c_{ki} $(0 \leq i \leq k)$ such that $d^k(x)y = \sum_{i=0}^{k} c_{ki}d^{k-i}(x \cdot d^i(y))$ for all $x, y \in R$.*

Proof. We proceed by induction on k. Obviously $d(x) \cdot y = d(xy) - x \cdot d(y)$. Hence, we can put $c_{10} = 1$ and $c_{11} = -1$. Since

$$d^k(x) \cdot y = d(d^{k-1}(x)) \cdot y = d(d^{k-1}(x) \cdot y) - d^{k-1}(x)d(y),$$

we can choose c_{ki} as follows : $c_{k0} = c_{k-1,0}$ $(= 1)$, $c_{kk} = -c_{k-1,k-1}$ $(= (-1)^k)$, and $c_{ki} = c_{k-1,i} - c_{k-1,i-1}$ $(0 < i < k)$.

Theorem 1. *Let R be a left (resp. right) strongly prime ring, and d a derivation of R. If d is nil on a non-zero left (resp. right) ideal I of R, then so is d on R.*

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Proof. Note that $J = I + d(I) + d^2(I) + \cdots$ is a non-zero left ideal of R such that $d(J) \subseteq J$ and d is nil on J. Thus, without loss of generality, we may assume that $d(I) \subseteq I$. By hypothesis, I contains a finite set F with $l(F) = 0$, where $l(F)$ denotes the left annihilator of F in R. There exists a positive integer m such that $d^m(x) = 0$ for all $x \in F$. Let r be an arbitrary element of R. Then we can choose a positive integer n such that $d^n(r \cdot d^i(x)) = 0$ for all $i\,(0 \leq i \leq m-1)$ and all $x \in F$. Then, by Lemma 1, we have $d^{m+n-1}(r)x = 0$ for all $x \in F$, and therefore $d^{m+n-1}(r)F = 0$. Since $l(F) = 0$, we conclude that $d^{m+n-1}(r) = 0$, which proves that d is nil on R.

The following example answers the problem of Chung in the negative.

Example 1. Let K be a field of characteristic zero, and R the ring of both row and column finite matrices over $K[x]$ (of countably infinite degree). Let I be the set of matrices with only finite nonzero entries. Then we can easily check that I is a nonzero ideal of R. We define a mapping δ of R to R by $\delta((a_{ij})) = \left(\dfrac{da_{ij}}{dx}\right)$. Then δ is a derivation of R which is nil on I, but not on R.

Next we shall show that some type of nil derivations of semiprime PI-rings are nilpotent. We shall begin with

Theorem 2. *Let R be a prime PI-ring with center C. If d is a nil derivation of R such that $d(C) = 0$, then d is nilpotent.*

Proof. We set $S = C \setminus \{0\}$. Then it is well known (see, e.g., [6]) that the algebra $Q\,(= R_S)$ of central quotients of R is simple and finite dimensional over its center C_S. If we define $\tilde{d}(xy^{-1}) = d(x)y^{-1}$ for all $xy^{-1} \in Q$, $\tilde{d}$ is a nil derivation of Q and $\tilde{d}$ is an extension of d. Since $\tilde{d}(C_S) = 0$ and Q has a finite basis over C_S, by Leibniz's rule we can conclude that $\tilde{d}$ is nilpotent on Q (and hence on R).

We would like to generalize Theorem 2 to semiprime PI-rings. In preparation for this, we require some results about inner derivations on semiprime PI-rings. We also generalize the following well-known fact to semiprime PI-rings : Any derivation of a central simple algebra of finite rank, which is trivial on the center, is inner.

Let x, y be two elements of a ring R. We set $[x,y]_1 = [x,y]$, and inductively $[x,y]_k = [[x,y]_{k-1},y]$ for any integer $k>1$.

Lemma 2. *Let R be the matrix algebra $(K)_n$ over a field K. Let x, y be elements of R. If $[x,y]_k = 0$ for some k, then $[x,y]_{n^2} = 0$.*

Proof. Considering $\mathrm{ad}(y) : r \to [r,y]$ as an element of $(K)_{n^2}$, by Cayley-Hamilton Theorem, ad(y) satisfies its characteristic polynomial

$$(*) \qquad x^{n^2}+c_{n^2-1}X^{n^2-1}+\cdots+c_iX^i = 0,\ c_i \neq 0\ (i \geq 0).$$

Let m be the least positive integer satisfying $[x,y]_m = 0$. If $m > n^2$, then by (*) we have

$$[x,y]_{m-1} = -c_i^{-1}([x,y]_{m+n^2-i-1}+\cdots+c_{i+1}[x,y]_m) = 0.$$

This contradiction shows that $m \leq n^2$.

Corollary 1. *Let R be a semiprime ring satisfying a polynomial identity of degree d, and $x,y \in R$. If $[x,y]_k = 0$ for some k, then $[x,y]_{[d/2]^2} = 0$, where $[d/2]$ is the largest integer in $d/2$.*

Proof. Since R is semiprime, R is a subdirect sum of prime rings R_j. By Kaplansky's Theorem [6, Theorem 1.5.16], the algebra $Q(R_j)$ of central quotients of R_j is a central simple algebra of dimension m_j^2 with $m_j \leq [d/2]$. Therefore each R_j can be viewed as a subring of $(K_j)_{m_j}$ for some field K_j, and hence R is a subring of $\prod_j (K_j)_{m_j}$. Then, by Lemma 2, we conclude that $[x,y]_{[d/2]^2} = 0$.

If R is a finite dimensional simple algebra over its center C, then it is well known that any derivation d of R with $d(C)=0$ is inner. We generalize this result as follows :

Theorem 3. *Let R be a semiprime PI-ring with 1, and C the center of R. If d is a derivation of R with $d(C) = 0$, then d can be extended to an inner derivation of the maximal right quotient ring Q of R.*

Proof. First, note that R is right nonsingular ([5, Proposition 1]). Furthermore, in view of [5, Lemma 2], a right ideal J of R is essential in R if and only if $J \cap C$ is essential in C. We set $F = \{IR \mid I$ is an essential ideal of $C\}$. Then, by the fact mentioned just above, every element f of Q can be represented by an R-homomorphism $f : J_R \to R_R$ with some $J \in F$.

Now, d can be extended as follows : Let $f: J_R \to R_R$ be an element of Q ($J \in F$). Since $d(C) = 0$ and $J \in F$, we can easily see that $d(J) \subseteq J$. Therefore we can define $d(f)$ by $d(f)(x) = d(f(x)) - f(d(x))$ for all $x \in J$. Let q be an arbitrary element of the center $C(Q)$ of Q. Then q can be represented by an R-R-homomorphism $q: J \to R$ with some $J \in F$ ([5, Proposition 2 and Theorem 2]). We write $J = IR$ with some essential ideal I of C. Let x be an arbitrary element of I. Then for any $r \in R$ we have $rq(x) = q(rx) = q(xr) = q(x)r$, and so $q(x) \in C$. By the definition of $d(q)$, we have $d(q)(x) = d(q(x)) - q(d(x)) = 0$, because $d(C) = 0$. From this we have $d(q)(J) = 0$, that is, $d(q) = 0$. Consequently, we have shown that d is trivial on $C(Q)$. Since the maximal right quotient ring Q is regular self-injective and satisfies a polynomial identity ([5, Corollary 2]), [1, Theorem 3.5] shows that Q is a finite direct sum of Azumaya algebras : $Q = Q_1 \oplus Q_2 \oplus \dots \oplus Q_k$. Clearly, the restriction of d to each Q_i is a derivation with $d(C(Q_i)) = 0$. Therefore, by [3, Theorem, p.76], d is inner on each Q_i, and hence d is inner on Q.

Now, we can prove the following

Theorem 4. *Let R be a semiprime PI-ring with center C. If d is a nil derivation of R with $d(C) = 0$, then d is nilpotent.*

Proof. If R is a ring without 1, we define the ring $R' = \mathbf{Z} \oplus R$ with multiplication given by $(a, r)(b, s) = (ab, as + br + rs)$. We can extend d to R' by $d((a, r)) = (0, d(r))$. Since the left annihilator A of R in R' is stable under d, d induces a derivation $\tilde{d}$ of $R'' = R'/A$. It is easy to check that R'' and $\tilde{d}$ satisfy the hypotheses of our theorem. Since R'' contains R as a subring and $\tilde{d}$ is an extension of d, it suffices to prove our assertion for R'' with $\tilde{d}$. Consequently, without loss of generality, we may assume that R has 1. Now, by Theorem 3 and its proof, d can be extended to an inner derivation of the maximal right quotient ring Q of R and Q is also a semiprime PI-ring. Hence, d is nilpotent by Corollary 1.

Remark. Recently, in the middle of preparing this paper, the authors received from Prof. Y. Kobayashi a copy of his joint paper with L. O. Chung and entitled "Nil derivations and chain conditions in prime rings" (submitted to Proc. Amer. Math. Soc.), where they have proved Theorem 1 for prime rings with DCC on left annihilators.

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REFERENCES

[1] E. P. ARMENDARIZ and S. A. STEINBERG: Regular self-injective rings with a polynomial identity, Trans. Amer. Math. Soc. **190** (1974), 417−425.

[2] L. O. CHUNG: Nil derivations, to appear.

[3] F. DEMEYER and E. INGRAHAM: Separable Algebras over Commutative Rings, Lecture Notes in Math. **181**, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[4] D. HANDELMAN and J. LAWRENCE: Strongly prime rings, Trans. Amer. Math. Soc. **211** (1975), 209−223.

[5] L. H. ROWEN: Maximal quotients of semiprime PI-algebras, Trans. Amer. Math. Soc. **196** (1974), 127−135.

[6] L. H. ROWEN: Polynomial Identities in Ring Theory, Academic Press, New York, 1980.

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