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Genta Kawahara\*

Shigeo Kida<sup>†</sup>

Mitsuru Tanaka<sup>‡</sup>

Shinichiro Yanase\*\*

\*Ehime University

<sup>†</sup>National Institute for Fusion Science

<sup>‡</sup>Kyoto Institute of Technology

\*\*Okayama University

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## Wrap, tilt and stretch of vorticity lines around a strong thin straight vortex tube in a simple shear flow

By GENTA KAWAHARA<sup>1</sup>, SHIGEO KIDA<sup>2</sup>,  
MITSURU TANAKA<sup>3</sup> AND SHINICHIRO YANASE<sup>4</sup>

<sup>1</sup>Department of Mechanical Engineering, Ehime University, Matsuyama 790–77, Japan

<sup>2</sup>Theory and Computer Simulation Center, National Institute for Fusion Science, Toki 509–52, Japan

<sup>3</sup>Department of Mechanical and System Engineering, Kyoto Institute of Technology, Kyoto 606, Japan

<sup>4</sup>Department of Engineering Sciences, Okayama University, Okayama 700, Japan

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The mechanism of wrap, tilt and stretch of vorticity lines around a strong thin straight vortex tube of circulation  $\Gamma$  starting with a vortex filament in a simple shear flow ( $\mathbf{U} = S X_2 \hat{\mathbf{X}}_1$ ,  $S$  being a shear rate) is investigated analytically. An asymptotic expression for the vorticity field is obtained at a large Reynolds number  $\Gamma/\nu \gg 1$ ,  $\nu$  being the kinematic viscosity of fluid, and during the initial time  $S t \ll 1$  of evolution as well as  $S t \ll (\Gamma/\nu)^{1/2}$ . The vortex tube, which is inclined from the streamwise ( $X_1$ ) direction both in the vertical ( $X_2$ ) and spanwise ( $X_3$ ) directions, is tilted, stretched and diffused under the action of the uniform shear and viscosity. The simple shear vorticity is on the other hand, wrapped and stretched around the vortex tube by a swirling motion, induced by it to form double spiral vortex layers of high azimuthal vorticity of alternating sign. The magnitude of the azimuthal vorticity increases up to  $O((\Gamma/\nu)^{1/3} S)$  at distance  $r = O((\Gamma/\nu)^{1/3} (\nu t)^{1/2})$  from the vortex tube. The spirals induce axial flows of the same spiral shape with alternate sign in adjacent spirals which in turn tilt the simple shear vorticity toward the axial direction. As a result, the vorticity lines wind helically around the vortex tube accompanied by conversion of vorticity of the simple shear to the axial direction. The axial vorticity increases in time as  $S^2 t$ , the direction of which is opposite to that of the vortex tube at  $r = O((\Gamma/\nu)^{1/2} (\nu t)^{1/2})$  where the vorticity magnitude is strongest. In the near region  $r \ll (\Gamma/\nu)^{1/3} (\nu t)^{1/2}$ , on the other hand, a viscous cancellation takes place in tightly wrapped vorticity of alternate sign, which leads to the disappearance of the vorticity normal to the vortex tube. Only the axial component of the simple shear vorticity is left there, which is stretched by the simple shear flow itself. As a consequence, the vortex tube inclined toward the direction of the simple shear vorticity (a cyclonic vortex) is intensified, while the one oriented in the opposite direction (an anticyclonic vortex) is weakened. The growth rate of vorticity due to this effect attains a maximum (or minimum) value of  $\pm S^2/3^{3/2}$  when the vortex tube is oriented in the direction of  $\hat{\mathbf{X}}_1 + \hat{\mathbf{X}}_2 \mp \hat{\mathbf{X}}_3$ . The present asymptotic solutions are expected to be closely related to the flow structures around intense vortex tubes observed in various kinds of turbulence such as helical winding of vorticity lines around a vortex tube, the dominance of cyclonic vortex tubes, the appearance of opposite-signed vorticity around streamwise vortices and a zig-zag arrangement of streamwise

vortices in homogeneous isotropic turbulence, homogeneous shear turbulence and near-wall turbulence.

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## 1. Introduction

Tube-like vortical structures of concentrated high vorticity have been commonly observed in many turbulent flow fields. In homogeneous isotropic turbulence, there exist strong coherent elongated vortices in a weaker background vorticity, and a relatively large portion of turbulence kinetic energy is dissipated around them (Siggia 1981; Kerr 1985; Hosokawa & Yamamoto 1989; She, Jackson & Orszag 1990; Ruetsch & Maxey 1991; Vincent & Meneguzzi 1991; Douady, Couder & Brachet 1991; Kida & Ohkitani 1992; Jiménez *et al.* 1993; Kida 1993). In homogeneous shear turbulence, Kida & Tanaka (1992, 1994) showed the presence of longitudinal vortex tubes which induce an intense Reynolds shear stress, and clarified their generation and development processes. In near-wall turbulence, it was found that streamwise vortex tubes play a central role in the production of turbulence kinetic energy (Robinson, Kline & Spalart 1988; Brooke & Hanratty 1993; Bernard, Thomas & Handler 1993). In near-wall turbulence streamwise vortices are closely related to the generation of high skin friction (Choi, Moin & Kim 1993; Kravchenko, Choi & Moin 1993). Another example of tube-like concentrated vortices is the ribs observed in a turbulent mixing layer (see Hussain 1986). These observations lead us to believe that tube-like vortices may be one of the key ingredients of coherent structures which make a significant contribution to the production and dissipation of turbulence kinetic energy. They are also expected to control heat, mass and momentum transfers. Clarification of the dynamics of vortex tubes would lead to a new concept useful for understanding and controlling turbulence phenomena.

In the time evolution of tube-like structures their interactions with a background turbulence field are considered to play a significant role. It is understood at least conceptually that a background turbulence stretches and rotates vortex tubes as well as deforms their shape and that the vortex tubes, on the other hand, wrap and stretch the background vorticity lines. We must admit, however, that the knowledge of the actual dynamical process in these interactions is still poor. There has been much effort devoted to this subject. Moore (1985) investigated the dynamics of a diffusing straight vortex tube perfectly aligned with a simple shear flow. He derived a large-Reynolds-number asymptotic solution to show that excessive vorticity wrapping enhances viscous cancellation to expell the shear flow vorticity near the vortex tube. In their asymptotic analysis of a strong vortex tube subjected to a uniform non-axisymmetric irrotational strain, Moffatt, Kida & Ohkitani (1994) found that at large Reynolds numbers, a stretched vortex tube can survive for a long time even when two of the principal rates of strain are positive. Recently, Jiménez, Moffatt & Vasco (1996) applied Moffatt *et al.* (1994) asymptotic theory to the dynamics of a two-dimensional diffusing vortex tube in an imposed weak strain. They showed a good agreement between the results of their theory and a numerical simulation of two-dimensional turbulence.

In this paper, we study vorticity dynamics, especially vortex wrapping, tilting and stretching, around a strong thin straight vortex tube starting with a vortex filament in a simple shear flow ( $\mathbf{U} = SX_2\hat{\mathbf{X}}_1$ ,  $S$  being a shear rate). A straight vortex filament of circulation  $\Gamma$  is set at an initial instant, being inclined away from the streamwise

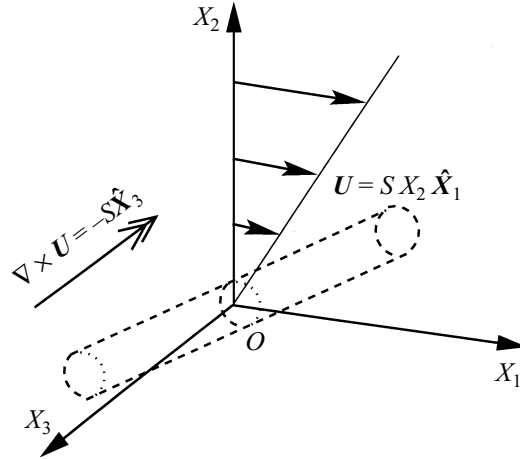


FIGURE 1. A straight vortex tube in a simple shear flow.

( $X_1$ ) direction both in the vertical ( $X_2$ ) and spanwise ( $X_3$ ) directions. The vortex filament is tilted, stretched and diffused under the action of the uniform shear and viscosity. The strength of a vortex tube may be measured by the vortex Reynolds number  $\Gamma/\nu$ , where  $\nu$  is the kinematic viscosity of fluid. We are particularly interested in a strong vortex tube ( $\Gamma/\nu \gg 1$ ) since the vortex Reynolds number often takes large values in typical turbulence. For example, Robinson (1991) observed  $\Gamma/\nu \approx 140$  in boundary-layer turbulence at momentum-thickness Reynolds number  $Re_\theta = 670$ , while Jiménez *et al.* (1993) found that in homogeneous isotropic turbulence  $\Gamma/\nu$  increases with Taylor-microscale Reynolds number  $Re_\lambda$  as  $\Gamma/\nu \sim Re_\lambda^{1/2}$ . An asymptotic analysis is performed at a large Reynolds number  $\Gamma/\nu \gg 1$  and at the initial time  $St \ll 1$  of evolution. The problem to be considered here includes the ones treated by Moore (1985) and by Jiménez *et al.* (1996) as special cases.

In §2, we derive the equations of motion of a vortex tube in a simple shear flow in a coordinate system rotating with the central axis of the vortex tube under the assumption that the vorticity and induced velocity of the vortex tube are uniform along its axis. Asymptotic solutions starting with a vortex filament are presented for  $\Gamma/\nu \gg 1$  and  $St \ll 1$  by extending Moore's (1985) and Moffatt *et al.*'s (1994) methods in §3 (details of the analysis for higher orders are described in Appendices A and B). In §4, we provide a physical interpretation of the asymptotic solutions to explore structures of the vorticity field. Section 5 is devoted to concluding remarks.

## 2. Formulation

We consider the motion of a straight vortex tube in a simple shear flow with uniform pressure  $P$  (see figure 1). Let the coordinate system  $OX_1X_2X_3$  be at rest, the  $X_1$ -axis being aligned with the shear flow direction. The uniform shear velocity  $\mathbf{U}$  is taken to depend only on  $X_2$ , i.e.  $\mathbf{U} = SX_2\hat{\mathbf{X}}_1$ , where  $S (> 0)$  denotes the shear rate, which is constant in time, and  $\hat{\mathbf{X}}_i$  is the unit vector in the  $X_i$ -direction ( $i = 1, 2, 3$ ). In this configuration the uniform shear vorticity is given by  $\nabla \times \mathbf{U} = -S\hat{\mathbf{X}}_3$ , which is anti-parallel to the  $X_3$ -axis. Hereafter, we call  $X_1$ ,  $X_2$  and  $X_3$  the streamwise, the vertical and the spanwise coordinates, respectively.

The vortex tube is inclined both vertically and horizontally away from the stream-

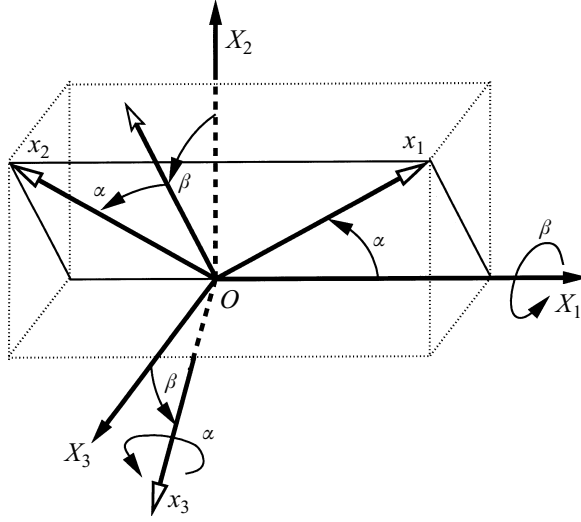


FIGURE 2. Structural coordinate system  $Ox_1x_2x_3$  and the original stationary coordinate system  $OX_1X_2X_3$ .

wise direction. It will be tilted and stretched by the uniform shear. The origin  $O$  is located on the central axis of the vortex tube, so that it is a stagnation point of the flow. We suppose that the vortex tube is of infinite extent, and its vorticity and induced velocity are uniform along its axis.

### 2.1. Structural coordinate system

We formulate the problem in a rotating coordinate system  $Ox_1x_2x_3$  as shown in figure 2. Rotating the stationary coordinate system  $OX_1X_2X_3$  by an angle  $\beta$  around the  $X_1$ -axis, we set the new  $X_3$ -direction as the  $x_3$ -axis. Next, we further rotate  $OX_1X_2X_3$  by an angle  $\alpha$  around the  $x_3$ -axis (new  $X_3$ -axis), and then the new  $X_1$ - and  $X_2$ -directions are set as the  $x_1$ - and  $x_2$ -coordinates, respectively. Rotation angles,  $\alpha$  and  $\beta$ , are taken so that the resulting  $x_1$ -axis may coincide with the central axis of the vortex tube. The vorticity of the vortex tube is taken to be pointed in the positive  $x_1$ -direction. Hereafter, we call  $Ox_1x_2x_3$  the structural coordinate system,  $x_1$  the axial coordinate and  $(x_2, x_3)$  the normal plane. Flow symmetry allows us to take  $\alpha$  and  $\beta$  in the range  $0 \leq \alpha < \pi$  and  $-\frac{1}{2}\pi \leq \beta \leq \frac{1}{2}\pi$  without loss of generality. In the case of  $\alpha = 0$ , the vortex tube is aligned with the streamwise direction. When  $\alpha < (or >) \frac{1}{2}\pi$ , it is inclined downstream (or upstream). In the cases of  $\beta = \pm\frac{1}{2}\pi$ , the tube axis is located on the horizontal plane  $X_2 = 0$ . When  $\beta < (or >) 0$ , the spanwise vorticity component of the vortex tube is negative (or positive). Hereafter, a vortex tube for  $\beta < (or >) 0$  is referred to as a cyclonic (or anticyclonic) vortex.

Two vectors,  $(V_1, V_2, V_3)$  in  $OX_1X_2X_3$  and  $(v_1, v_2, v_3)$  in  $Ox_1x_2x_3$ , are connected by the relation

$$V_i = M_{ij}v_j \quad (i = 1, 2, 3), \quad (2.1)$$

where

$$\{M_{ij}\} = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha \cos \beta & \cos \alpha \cos \beta & -\sin \beta \\ \sin \alpha \sin \beta & \cos \alpha \sin \beta & \cos \beta \end{pmatrix} \quad (i, j = 1, 2, 3) \quad (2.2)$$

is a transformation matrix which represents a system rotation. Here and subsequently,

the summation convention is employed for repeated subscripts. Similarly, the unit vectors representing the axes in the two coordinate systems are related by

$$\widehat{X}_i = M_{ij}\hat{x}_j \quad (i = 1, 2, 3). \quad (2.3)$$

As the vortex tube evolves, the structural coordinate system  $Ox_1x_2x_3$  rotates around some axis which passes through the origin  $O$ . It follows from the definition of  $\alpha$  and  $\beta$  that the angular velocity of the system rotation  $\boldsymbol{\Omega}$  is given by

$$\boldsymbol{\Omega} = (d_t\beta)\widehat{X}_1 + (d_t\alpha)\hat{x}_3, \quad (2.4)$$

where  $d_t \equiv d/dt$ . By making use of (2.2) and (2.3), we can express each component of the angular velocity vector in  $Ox_1x_2x_3$  as

$$\Omega_1 = (d_t\beta)\cos\alpha, \quad \Omega_2 = -(d_t\beta)\sin\alpha, \quad \Omega_3 = d_t\alpha. \quad (2.5a-c)$$

## 2.2. Angular velocity of structural coordinate system

The motion of an incompressible viscous fluid of uniform mass density (taken as unity) is described by the Navier–Stokes equation, or equivalently the vorticity equation, which are respectively written in the structural coordinate system  $Ox_1x_2x_3$  as †

$$\partial_t \mathbf{u} + [(\mathbf{u} - \boldsymbol{\Omega} \times \mathbf{x}) \cdot \nabla] \mathbf{u} = \mathbf{u} \times \boldsymbol{\Omega} - \nabla p + \nu \nabla^2 \mathbf{u}, \quad (2.6)$$

$$\partial_t \boldsymbol{\omega} + [(\mathbf{u} - \boldsymbol{\Omega} \times \mathbf{x}) \cdot \nabla] \boldsymbol{\omega} = \boldsymbol{\omega} \times \boldsymbol{\Omega} + (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}, \quad (2.7)$$

where  $\mathbf{u}(x_1, x_2, x_3, t)$  is the velocity field relative to the stationary coordinate system,  $\boldsymbol{\omega} = \nabla \times \mathbf{u}$  is the vorticity,  $p$  is the pressure and  $\nabla$  is the gradient operator in the structural coordinate system. The continuity equation is written as

$$\nabla \cdot \mathbf{u} = 0. \quad (2.8)$$

Now let us decompose the velocity, the vorticity and the pressure fields into contributions from the simple shear flow and the fluctuation field as

$$\mathbf{u} = \mathbf{U} + \mathbf{u}', \quad \boldsymbol{\omega} = \nabla \times \mathbf{U} + \boldsymbol{\omega}', \quad p = P + p'. \quad (2.9)$$

Then, the time evolutions of the fluctuation velocity and vorticity are described by

$$\partial_t \mathbf{u}' + [(\mathbf{u}' + \bar{\mathbf{u}}) \cdot \nabla] \mathbf{u}' = \mathbf{u}' \times \boldsymbol{\Omega} - (\mathbf{u}' \cdot \nabla) \mathbf{U} - \nabla p' + \nu \nabla^2 \mathbf{u}', \quad (2.10)$$

$$\partial_t \boldsymbol{\omega}' + [(\mathbf{u}' + \bar{\mathbf{u}}) \cdot \nabla] \boldsymbol{\omega}' = \boldsymbol{\omega}' \times \boldsymbol{\Omega} + (\boldsymbol{\omega}' \cdot \nabla) \mathbf{U} + [(\boldsymbol{\omega}' + \nabla \times \mathbf{U}) \cdot \nabla] \mathbf{u}' + \nu \nabla^2 \boldsymbol{\omega}', \quad (2.11)$$

$$\nabla \cdot \mathbf{u}' = 0, \quad (2.12)$$

$$\boldsymbol{\omega}' = \nabla \times \mathbf{u}', \quad (2.13)$$

where

$$\bar{\mathbf{u}} = \mathbf{U} - \boldsymbol{\Omega} \times \mathbf{x} \quad (2.14)$$

is the simple shear velocity relative to the structural coordinate system. The simple shear velocity and vorticity are respectively written as

$$\mathbf{U} = S X_2 \widehat{X}_1 = S M_{1i} M_{2j} x_j \hat{x}_i, \quad (2.15)$$

$$\nabla \times \mathbf{U} = -S \widehat{X}_3 = -S M_{3i} \hat{x}_i. \quad (2.16)$$

Notice that in general the coordinate  $x_1$  appears explicitly in  $\bar{u}_2$  and  $\bar{u}_3$ . If we

† Recall that a time derivative of a vector field  $\mathbf{A}$  in a stationary coordinate system is replaced by  $\partial_t \mathbf{A} \rightarrow \partial_t \mathbf{A} - [(\boldsymbol{\Omega} \times \mathbf{x}) \cdot \nabla] \mathbf{A} + \boldsymbol{\Omega} \times \mathbf{A}$  in the structural coordinate system.

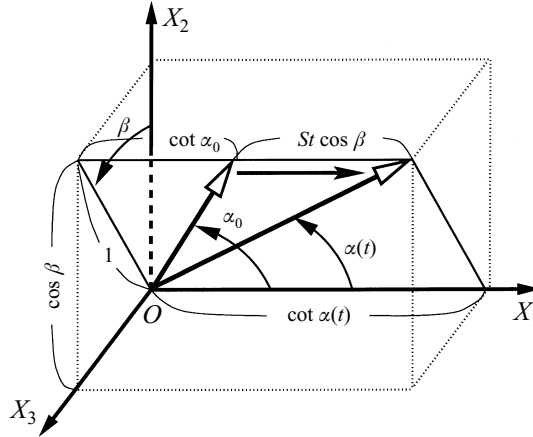


FIGURE 3. Movement of a vortex tube (i.e. the  $x_1$ -axis) which is shown by a white-head arrow.

require, however, that  $\mathbf{u}'$  and  $\boldsymbol{\omega}'$  are uniform in the  $x_1$ -direction, it follows from (2.10) and (2.11) that  $\bar{u}_2$  and  $\bar{u}_3$  are independent of  $x_1$ . Then we have

$$\Omega_2 = 0, \quad \Omega_3 = -S \sin^2 \alpha \cos \beta. \quad (2.17)$$

Equations (2.5b,c) then give

$$(d_t \beta) \sin \alpha = 0, \quad (2.18)$$

$$d_t \alpha = -S \sin^2 \alpha \cos \beta. \quad (2.19)$$

Equations (2.18) and (2.19) have a trivial solution  $\alpha \equiv 0$  for any arbitrary  $\beta$ . Except for this trivial case, (2.18) requires that

$$d_t \beta = 0. \quad (2.20)$$

In the case of  $\alpha \equiv 0$ , the vortex axis ( $x_1$ -axis) is identical with the  $X_1$ -axis, and any rotation around this axis does not change the orientation of it, so that we can take  $\beta$  to be constant in time  $t$ . Hence, we can assume that  $\beta$  is constant in any case. Equation (2.5a) then yields

$$\Omega_1 = 0. \quad (2.21)$$

Equations (2.17) and (2.21) tell us that  $\boldsymbol{\Omega}$  has only the  $x_3$ -component. By integrating (2.19), we obtain

$$\cot \alpha = \cot \alpha_0 + S t \cos \beta \quad (2.22)$$

with  $\alpha_0$  denoting the initial value of  $\alpha$  at  $t = 0$ . It follows from (2.22) that  $\alpha \rightarrow 0$  as  $\alpha_0 \rightarrow 0$ . Thus, the trivial solution ( $\alpha \equiv 0$ ) is included in (2.22). These considerations lead us to the conclusion that a vortex tube rotates on a plane inclined to the spanwise direction at an angle of  $\beta$  which is invariant in time, and angle  $\alpha$  from the streamwise direction approaches zero according to (2.22) as time progresses. This implies that the central axis of the vortex tube, the velocity and vorticity of which are uniform along it, must be passively convected by the uniform shear flow (see figure 3). Note that in the special cases of  $\alpha = 0$  or  $\beta = \pm \frac{1}{2}\pi$ , the vortex tube is not inclined vertically and is stationary.

## 2.3. Basic equations

Suppose now that the fluctuation fields  $\omega'$ ,  $\mathbf{u}'$  and  $p'$  are independent of  $x_1$ , i.e.  $\partial_1 = 0$ . We then obtain closed equations for  $\omega'_1$  and  $u'_1$  from (2.10) and (2.11) as

$$\partial_t \omega'_1 - \frac{\partial(\psi, \omega'_1)}{\partial(x_2, x_3)} - S(\gamma(t)x_2 + \lambda(t)x_3)\partial_2 \omega'_1 = S\gamma(t)\omega'_1 + S\xi(t)\partial_3 u'_1 + v\nabla_{\perp}^2 \omega'_1, \quad (2.23)$$

$$\begin{aligned} \partial_t u'_1 - \frac{\partial(\psi, u'_1)}{\partial(x_2, x_3)} - S(\gamma(t)x_2 + \lambda(t)x_3)\partial_2 u'_1 \\ = -S\gamma(t)u'_1 - S(\cos \alpha \sin \beta \partial_2 + \cos \beta \partial_3)\psi + v\nabla_{\perp}^2 u'_1, \end{aligned} \quad (2.24)$$

where  $\psi$  ( $u'_2 = \partial_3 \psi$ ,  $u'_3 = -\partial_2 \psi$ ) is the streamfunction, which is related to  $\omega'_1$  via

$$\nabla_{\perp}^2 \psi = -\omega'_1, \quad (2.25)$$

and

$$\gamma(t) = \frac{\partial_1 U_1}{S} = \cos \alpha \sin \alpha \cos \beta, \quad (2.26)$$

$$\lambda(t) = \frac{(\nabla \times \mathbf{U}) \cdot \hat{\mathbf{x}}_1}{S} = -\sin \alpha \sin \beta, \quad (2.27)$$

$$\xi(t) = \frac{2\Omega_3}{S} = -2 \sin^2 \alpha \cos \beta \ (\leq 0) \quad (2.28)$$

(cf. (2.15)–(2.17)). Here,  $\nabla_{\perp}^2 = \partial_2^2 + \partial_3^2$  is a two-dimensional Laplacian operator. Note that  $\gamma(t)$  represents the axial rate of strain of the simple shear flow,  $\lambda(t)$  the axial component of the simple shear vorticity, and  $\xi(t)$  the vorticity corresponding to the angular velocity of the structural coordinate system, all of which are normalized by the simple shear rate. Note also that the nonlinear stretching-and-tilting terms  $\omega'_j \partial_j u'_1$  have disappeared from (2.23) because the flow field is uniform along the vortex tube. Once (2.23) and (2.24) are solved, we can calculate the other two fluctuation vorticity components through

$$\omega'_2 = \partial_3 u'_1, \quad \omega'_3 = -\partial_2 u'_1. \quad (2.29)$$

The second and third terms on the left-hand sides of (2.23) and (2.24) represent the advection by the fluctuation velocity and the simple shear, respectively. On the right-hand side of (2.23), the first term represents the vorticity stretching via the simple shear, while the second is the production of the axial ( $x_1$ ) component of the fluctuation vorticity via the tilting by the velocity fluctuation of the vorticity associated with the system rotation which has only an  $x_3$ -component. This second term is also interpreted as a sum of three contributions: the tilting of the  $x_2$ -component of the fluctuation vorticity through the simple shear,  $\omega'_2 \partial_2 U_1 = S \cos^2 \alpha \cos \beta \partial_3 u'_1$ ; the tilting of the  $x_3$ -component of the simple shear vorticity via the velocity fluctuation,  $-SM_{33} \partial_3 u'_1 = -S \cos \beta \partial_3 u'_1$ ; and the effect of frame rotation,  $(\omega' \times \mathbf{\Omega}) \cdot \hat{\mathbf{x}}_1 = -S \sin^2 \alpha \cos \beta \partial_3 u'_1$ . If  $\beta = \pm \frac{1}{2}\pi$ , all of these three contributions vanish. If  $\alpha = 0$ , the tiltings of the fluctuation vorticity and the simple shear vorticity cancel out, and the effect of frame rotation vanishes. Thus, in these two special cases, the production term on the right-hand side of (2.23) disappears. Except for these cases, the effect of the tilting of the simple shear vorticity is important in production of the axial vorticity. Note that this term is negative (or positive) according as  $\omega'_2 = \partial_3 u'_1 >$  (or  $<$ ) 0. On the right-hand side of (2.24), the first two terms originate from the advection of the simple shear velocity by the velocity fluctuation and the frame rotation.



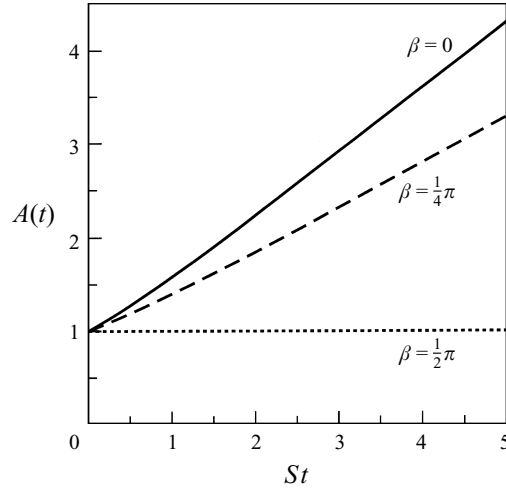


FIGURE 4. Time-variation of stretch factor  $A(t)$  for  $\alpha_0 = \frac{1}{4}\pi$  and for three values of  $\beta$ .

#### 2.4. Transformed equations

For convenience of analytical treatment, we introduce plane polar coordinates  $(r, \theta)$  with  $x_2 = r \cos \theta$  and  $x_3 = r \sin \theta$ , and employ Lundgren's (1982) transformation of radial coordinate and time as

$$R = A(t)^{1/2}r, \quad T = \int_0^t A(s) ds, \quad (2.30)$$

where

$$A(t) = \exp\left(S \int_0^t \gamma(s) ds\right) \quad (2.31)$$

represents the stretch factor along the vortex tube. In the present case, it follows from (2.22) and (2.26) that

$$A(t) = \frac{\sin \alpha_0}{\sin \alpha} \quad (2.32)$$

for  $\alpha \neq 0$  and  $\beta \neq \pm \frac{1}{2}\pi$ , and then we have

$$T = \frac{\sin \alpha_0}{2S \cos \beta} \left[ \cot \alpha \operatorname{cosec} \alpha - \cot \alpha_0 \operatorname{cosec} \alpha_0 + \ln \left( \frac{\cot \alpha + \operatorname{cosec} \alpha}{\cot \alpha_0 + \operatorname{cosec} \alpha_0} \right) \right]. \quad (2.33)$$

If  $\alpha = 0$  or  $\beta = \pm \frac{1}{2}\pi$ , then  $\gamma(t) \equiv 0$  and  $A(t) \equiv 1$ , and thus we have  $R = r$  and  $T = t$ . Since  $A(t) > 0$  for  $t \geq 0$ ,  $T$  increases monotonically with time  $t$ . For  $t \ll 1$  it changes as

$$T = t + \frac{1}{2}S \cos \alpha_0 \sin \alpha_0 \cos \beta t^2 + \dots, \quad (2.34)$$

and for  $t \gg 1$  it behaves asymptotically as

$$T = t^2 \left[ \frac{1}{2}S \sin \alpha_0 \cos \beta + \cos \alpha_0 \frac{1}{t} + O\left(\frac{\ln t}{t^2}\right) \right]. \quad (2.35)$$

The variations of  $A$  and  $T$  are plotted against time  $t$  for  $\alpha_0 = \frac{1}{4}\pi$  and for three values of  $\beta$  in figures 4 and 5, respectively.

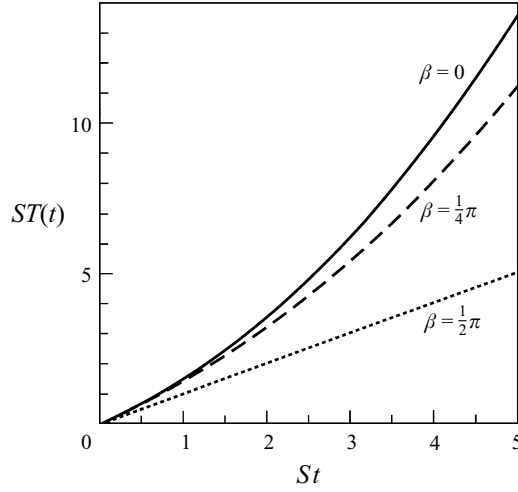


FIGURE 5. Time-variation of modified time  $T(t)$  for  $\alpha_0 = \frac{1}{4}\pi$  and for three values of  $\beta$ .

Equation (2.24) has a particular solution

$$u'_{1p} = -S \cos \beta x_2 + S \cos \alpha \sin \beta x_3 (\equiv -Sr \operatorname{Re} [if_\infty(t)e^{-i\theta}], \text{ say}), \quad (2.36)$$

which will turn out to play a key role in vorticity dynamics near the vortex core (see §3.4), where

$$f_\infty(t) = -\cos \alpha \sin \beta - i \cos \beta = -D(t)e^{i\varphi(t)} \quad (2.37)$$

with

$$D(t) = (\cos^2 \alpha \sin^2 \beta + \cos^2 \beta)^{1/2}, \quad \varphi(t) = \arctan \left( \frac{\cos \beta}{\cos \alpha \sin \beta} \right) \quad (0 \leq \varphi(t) \leq \pi). \quad (2.38)$$

Note that  $\partial_3 u'_{1p} = S \cos \alpha \sin \beta = SM_{32}$  and  $-\partial_2 u'_{1p} = S \cos \beta = SM_{33}$ , i.e. the vorticity associated with this particular solution is equal to minus the component normal to the vortex tube of the simple shear vorticity. If we introduce a new dependent variable  $u'_1$  by

$$u'_1 = u'_{1p} + u''_1 \quad (2.39)$$

and substitute it into (2.24), we can eliminate the inhomogeneous term on the right-hand side of (2.24). Then,  $\partial_3 u''_1$  and  $-\partial_2 u''_1$  are equal to the  $x_2$ - and  $x_3$ -components of the total vorticity, respectively, i.e.  $\omega_2 = \partial_3 u''_1$  and  $\omega_3 = -\partial_2 u''_1$ .

Equations (2.23) and (2.24) are now transformed into closed equations for new dependent variables†

$$\omega(\mathbf{R}, \theta, T) = \omega'_1(r, \theta, t)/A(t) = -\nabla_R^2 \psi, \quad Ru(\mathbf{R}, \theta, T) = A(t)u''_1(r, \theta, t) \quad (2.40)$$

as

$$-\frac{1}{R} \frac{\partial(\psi, \omega)}{\partial(\mathbf{R}, \theta)} + (\partial_T - v\nabla_R^2)\omega = SL_1\omega + SL_2u + \frac{2S^2\gamma(t)\lambda(t)}{A(t)^2}, \quad (2.41)$$

$$-\frac{1}{R} \frac{\partial(\psi, Ru)}{\partial(\mathbf{R}, \theta)} + (\partial_T - v\nabla_R^2)Ru = SL_1Ru, \quad (2.42)$$

† Notice that the axial velocity is expressed by  $Ru$  not by  $u$ .

where

$$\nabla_R^2 = \partial_R^2 + \frac{1}{R} \partial_R + \frac{1}{R^2} \partial_\theta^2 \quad (2.43)$$

is the two-dimensional Laplacian operator, and

$$L_1 = \frac{1}{2A(t)} [\gamma(t)(-\sin 2\theta \partial_\theta + R \cos 2\theta \partial_R) + \lambda(t)(\cos 2\theta \partial_\theta + R \sin 2\theta \partial_R - \partial_\theta)], \quad (2.44)$$

$$L_2 = \frac{\xi(t)}{A(t)^{5/2}} [\cos \theta \partial_\theta + \sin \theta (R \partial_R + 1)] \quad (2.45)$$

are first-order differential operators. The components of the total vorticity are expressed in terms of  $\omega$  and  $u$  as

$$\omega_1 = S\lambda(t) + A(t)\omega, \quad (2.46)$$

$$\omega_2 = A(t)^{-1/2} [\cos \theta \partial_\theta + \sin \theta (R \partial_R + 1)]u, \quad (2.47)$$

$$\omega_3 = A(t)^{-1/2} [\sin \theta \partial_\theta - \cos \theta (R \partial_R + 1)]u. \quad (2.48)$$

The right-hand sides of (2.41) and (2.42) represent the effects of the simple shear on the fluctuation fields. The first terms,  $SL_1\omega$  and  $SL_1Ru$ , represent respectively the deformation of the spatial distribution of  $\omega$  and  $u$  in the normal  $(x_2, x_3)$ -plane by the simple shear. The last two terms on the right-hand side of (2.41) represent the coupling effect of the axial vorticity and velocity, that is, the second term on the right-hand side of (2.23), which is composed of the tilting of the  $x_2$ -component of the fluctuation vorticity by the simple shear, the tilting of the  $x_3$ -component of the simple shear vorticity via the velocity fluctuation, and the effect of the frame rotation. The last term is the contribution from particular solution (2.36). Note that if a vortex tube was not inclined vertically ( $\alpha = 0$  or  $\beta = \pm \frac{1}{2}\pi$ ), the second and third terms would vanish, so that  $\omega$  would be decoupled from  $u$ . In these special cases the problem is much simplified. Pearson & Abernathy (1984) and Moore (1985) studied the time evolution of a diffusing vortex tube perfectly aligned with a simple shear ( $\alpha = 0$ ), and recently Jiménez *et al.* (1996) examined the structure of a two-dimensional diffusing vortex tube in an imposed weak strain ( $\alpha = \frac{1}{2}\pi$  and  $\beta = \pm \frac{1}{2}\pi$ ). The present analysis includes both of them.

### 3. Asymptotic analysis at $Re \gg 1$ and $ST \ll 1$

In this section, we consider an early stage of time evolution of a strong thin straight vortex tube starting with a vortex filament. A straight vortex filament with circulation  $\Gamma$  is put in a simple shear flow at an initial instant  $T = 0$ . That is, the fluctuation vorticity is concentrated on a straight line  $R = 0$ , i.e.

$$\omega|_{T=0} = \frac{\Gamma \delta(R)}{\pi R}, \quad (3.1)$$

and the fluctuation axial velocity along the filament is null,  $u'_1 = 0$ , so that, from (2.36)–(2.40),

$$u|_{T=0} = S \operatorname{Re} [if_0 e^{-i\theta}], \quad (3.2)$$

where

$$f_0 = -\cos \alpha_0 \sin \beta - i \cos \beta = -D_0 e^{i\varphi_0} \quad (3.3)$$

Variables	$T$	$R$	$\omega$	$\psi$	$Ru$
Units	$1/S$	$(\nu/S)^{1/2}$	$\epsilon^{-1}S (= \Gamma S/\nu)$	$\epsilon^{-1}\nu (= \Gamma)$	$(\nu S)^{1/2}$

TABLE 1. Units for variables

with

$$D_0 = (\cos^2 \alpha_0 \sin^2 \beta + \cos^2 \beta)^{1/2}, \quad \varphi_0 = \arctan \left( \frac{\cos \beta}{\cos \alpha_0 \sin \beta} \right) \quad (0 \leq \varphi_0 \leq \pi). \quad (3.4)$$

Note that  $\varphi_0$  represents an initial angle from the  $x_2$ -axis to a projection of the  $X_3$ -axis on the normal  $(x_2, x_3)$ -plane (see (2.2) and (2.3)).<sup>†</sup> In the case of  $\alpha_0 < \frac{1}{2}\pi$ ,  $\varphi_0$  is greater than, equal to or less than  $\frac{1}{2}\pi$  according as the vortex tube is cyclonic, neutral or anticyclonic.

Here, we define Reynolds number by

$$Re = \frac{\Gamma}{2\pi\nu}, \quad (3.5)$$

and denote the reciprocal of it as

$$\epsilon = \frac{1}{2\pi Re} = \frac{\nu}{\Gamma}. \quad (3.6)$$

In the following, an asymptotic analysis will be performed at a large Reynolds number ( $Re \gg 1$ ,  $\epsilon \ll 1$ ) and at an early time of evolution ( $ST \ll 1$ ).

### 3.1. Non-dimensionalization

We use shear rate  $S$  and kinematic viscosity  $\nu$  in order to non-dimensionalize the variables in (2.41) and (2.42). A characteristic time scale is then taken to be  $1/S$ , and a length scale is  $(\nu/S)^{1/2}$ . Therefore, the axial velocity  $Ru$  is scaled by  $(\nu S)^{1/2}$ , and  $u$  itself is scaled by  $S$ . The vorticity  $\omega$  and the streamfunction  $\psi$  are scaled respectively by  $\epsilon^{-1}S (= \Gamma S/\nu)$  and by  $\epsilon^{-1}\nu (= \Gamma)$  so that the dimensionless vortex strength and streamfunction may be independent of  $\Gamma$  at  $T = 0$ . The scaling units employed here are tabulated in Table 1.

By rewriting (2.41) and (2.42) with the dimensionless variables using the same notation for them as before, we obtain

$$-\frac{1}{R} \frac{\partial(\psi, \omega)}{\partial(R, \theta)} + \epsilon(\partial_T - \nabla_R^2)\omega = \epsilon L_1 \omega + \epsilon^2 L_2 u + \epsilon^2 \frac{2\gamma(t)\lambda(t)}{A(t)^2}, \quad (3.7)$$

$$-\frac{1}{R} \frac{\partial(\psi, Ru)}{\partial(R, \theta)} + \epsilon(\partial_T - \nabla_R^2)Ru = \epsilon L_1 Ru, \quad (3.8)$$

where  $L_1, L_2, \gamma(t), \lambda(t), \xi(t)$  and  $A(t)$  are given by the same expressions as before.<sup>‡</sup>

<sup>†</sup> When  $\alpha_0 = \frac{1}{2}\pi$  and  $\beta = \pm\frac{1}{2}\pi$ , the  $X_3$ -axis is normal to the  $(x_2, x_3)$ -plane, so that the  $x_1$ -axis (central axis of the vortex tube) is anti-parallel or parallel to the simple shear vorticity. In this case,  $f_0 = 0$  ( $u'_{1p} = 0$ ), and thus  $f(\eta) \equiv 0$  (see §3.4). This implies that  $u'_1 \equiv 0$ .

<sup>‡</sup> Dimensionless variables are used only in §3 except for §3.1.

## 3.2. Early-time approximation

Consider the early period of time evolution of a strong thin vortex tube which starts with a straight filament. We anticipate that viscous diffusion (i.e. the left-hand sides of (3.7) and (3.8)) has the primary effect on dynamics of the vortex tube and that the simple shear (i.e. the right-hand sides of (3.7) and (3.8)) plays a secondary role. We then seek solutions to (3.7) and (3.8) in the form

$$\omega = \omega^{(0)} + \omega^{(1)} + \omega^{(2)} + \cdots, \quad (3.9)$$

$$\psi = \psi^{(0)} + \psi^{(1)} + \psi^{(2)} + \cdots, \quad (3.10)$$

$$u = u^{(0)} + u^{(1)} + u^{(2)} + \cdots, \quad (3.11)$$

where

$$\omega^{(j)} = -\nabla_R^2 \psi^{(j)} \quad (j = 0, 1, 2, \cdots). \quad (3.12)$$

It is assumed that  $\omega^{(0)}$  and  $\psi^{(0)}$  represent a diffusing strong vortex tube, and that  $Ru^{(0)}$  represents the deformation of the velocity field from the simple shear flow by the vortex tube. Then,  $\omega^{(j)}$  and  $\psi^{(j)}$  ( $j = 1, 2, \cdots$ ) describe successively the higher-order interactions between the vortex tube and the simple shear. (It turns out that expansions (3.9)–(3.11) are equivalent to a power series in  $T$  of  $\omega$ ,  $\psi$  and  $u$  when they are regarded as functions of  $T$  and a similarity variable  $\eta$  defined by (3.23).) We shall take account of the effects of the simple shear one by one via  $\omega^{(j)}$  and  $\psi^{(j)}$  ( $j = 1, 2, \cdots$ ). Substituting (3.9)–(3.11) into (3.7) and (3.8), we have, at the leading order,

$$-\frac{1}{R} \frac{\partial(\psi^{(0)}, \omega^{(0)})}{\partial(R, \theta)} + \epsilon(\partial_T - \nabla_R^2)\omega^{(0)} = 0, \quad (3.13)$$

$$-\frac{1}{R} \frac{\partial(\psi^{(0)}, Ru^{(0)})}{\partial(R, \theta)} + \epsilon(\partial_T - \nabla_R^2)Ru^{(0)} = 0. \quad (3.14)$$

The next higher-order equations for vorticity are written as

$$-\frac{1}{R} \left[ \frac{\partial(\psi^{(0)}, \omega^{(1)})}{\partial(R, \theta)} + \frac{\partial(\psi^{(1)}, \omega^{(0)})}{\partial(R, \theta)} \right] + \epsilon(\partial_T - \nabla_R^2)\omega^{(1)} = \epsilon L_1 \omega^{(0)} + \epsilon^2 L_2 u^{(0)} + \epsilon^2 \frac{2\gamma(t)\lambda(t)}{A(t)^2}, \quad (3.15)$$

and so on. These equations are supplemented by the initial and boundary conditions as

$$\omega^{(0)}|_{T=0} = \frac{\delta(R)}{\pi R}, \quad \omega^{(0)}|_{R=\infty} = 0, \quad (3.16)$$

$$\omega^{(1)}|_{T=0} = \omega^{(2)}|_{T=0} = \cdots = 0, \quad \omega^{(1)}|_{R=\infty} = \omega^{(2)}|_{R=\infty} = \cdots = 0, \quad (3.17)$$

$$\partial_R \psi^{(0)}|_{R=\infty} = \partial_R \psi^{(1)}|_{R=\infty} = \partial_R \psi^{(2)}|_{R=\infty} = \cdots = 0, \quad (3.18)$$

$$u^{(0)}|_{T=0} = u^{(0)}|_{R=\infty} = \text{Re} [if_0 e^{-i\theta}], \quad (3.19)$$

$$u^{(1)}|_{T=0} = u^{(2)}|_{T=0} = \cdots = 0, \quad (3.20)$$

$$\left. \begin{aligned} u^{(1)}|_{R=\infty} &= T \text{Re} [d_T (A(t)^{1/2} f_\infty(t)) |_{T=0} i e^{-i\theta}], \\ u^{(2)}|_{R=\infty} &= \frac{1}{2} T^2 \text{Re} [d_T^2 (A(t)^{1/2} f_\infty(t)) |_{T=0} i e^{-i\theta}], \end{aligned} \right\} \quad (3.21)$$

and so on, where the conditions for  $u^{(j)}|_{R=\infty}$  ( $j = 0, 1, 2, \cdots$ ) have been obtained by an expansion of (scaled) particular solution (2.36),  $-A(t)u'_{1p}/R$ . In addition,  $\omega^{(j)}$  ( $j = 1, 2, \cdots$ ),  $\psi^{(k)}$  and  $Ru^{(k)}$  ( $k = 0, 1, 2, \cdots$ ) are assumed to be regular at  $R = 0$ . The initial condition, on the other hand, has been derived from (3.1) and (3.2). It has been

also assumed that the fluctuation parts of the velocity and the axial vorticity may decay at infinity. An additive constant in the streamfunction will be taken to be zero since it does not affect the flow. Solutions are determined successively starting from leading-order equation (3.13), which will be done in the following three subsections.

### 3.3. Axial vorticity

We first consider the leading-order solutions. Under initial and boundary conditions (3.16), the solution of (3.13) is uniquely determined as

$$\omega^{(0)} = \frac{1}{4\pi T} e^{-\eta^2}, \quad (3.22)$$

where

$$\eta = \frac{R}{2T^{1/2}} \quad (3.23)$$

is a similarity variable. Substitution of (3.22) into (3.12) for  $j = 0$  leads to

$$\psi^{(0)} = -\frac{1}{2\pi} \int_0^\eta \frac{1 - e^{-s^2}}{s} ds, \quad (3.24)$$

which is regular at  $\eta = 0$  and satisfies (3.18).

It follows that for  $\alpha < \frac{1}{2}\pi$  ( $\gamma(t) > 0$ ) the leading-order axial vorticity  $A(t)\omega^{(0)}$  represents a diffusing and stretching vortex tube under the action of viscosity and the axial stress of the simple shear. For  $\alpha > \frac{1}{2}\pi$  ( $\gamma(t) < 0$ ), on the other hand, it represents a diffusing and compressing vortex tube.

### 3.4. Axial velocity and normal vorticity

Next we consider the axial velocity deformed by the vortex tube. We seek a solution to (3.14) written in a separation-of-variable form in similarity variable  $\eta$  and angular coordinate  $\theta$  as

$$u^{(0)} = \text{Re} [if(\eta)e^{-i\theta}]. \quad (3.25)$$

By substituting (3.25) into (3.14), we obtain

$$f'' + \left(2\eta + \frac{3}{\eta}\right) f' + iRe \frac{1 - e^{-\eta^2}}{\eta^2} f = 0. \quad (3.26)$$

Hereafter in this subsection, the prime is used to denote differentiation with respect to  $\eta$ . Boundary conditions to be imposed are that  $Rf(\eta)$  is regular at  $\eta = 0$  and that  $f(\infty) = f_0$  ( $= -D_0 e^{i\varphi_0}$ ) (see (3.19)). The asymptotic expansion of the solution to (3.26) for large and small values of  $\eta$  can be easily calculated. For  $\eta \gg Re^{1/2}$ , we have

$$f(\eta) = -D_0 e^{i\varphi_0} \left[ 1 + \frac{iRe}{4\eta^2} - \frac{Re^2}{32\eta^4} - \frac{(iRe + 8)Re^2}{384\eta^6} + \dots \right] + O(e^{-\eta^2}), \quad (3.27)$$

while, for  $\eta \ll Re^{-1/2}$ , we have

$$f(\eta) = c_0 \left[ 1 - \frac{iRe}{8} \eta^2 + \left( \frac{iRe}{24} - \frac{Re^2}{192} \right) \eta^4 + \dots \right], \quad (3.28)$$

where  $c_0$  is a constant, which will be determined by the asymptotic condition  $f(\infty) = -D_0 e^{i\varphi_0}$  (see below).

Equation (3.26) is identical with the one obtained by Moore (1985) who analysed the dynamics of a diffusing vortex tube perfectly aligned with a simple shear flow, which corresponds to the present case of  $\alpha = 0$ . He has presented the asymptotic

solution to (3.26) for  $Re \gg 1$  using the WKB (Wentzel–Kramers–Brillouin) method. Here, following his method, we derive an asymptotic solution to our problem for  $Re \gg 1$  ( $\epsilon \ll 1$ ).

In order to apply the WKB method, it is convenient to eliminate the first-order-derivative terms in (3.26). To do so we introduce a new dependent variable  $g(\eta)$  by

$$f(\eta) = \eta^{-3/2} e^{-\eta^2/2} g(\eta). \quad (3.29)$$

Substitution of (3.29) into (3.26) leads to

$$g'' + \left[ iReH(\eta) - \eta^2 - 4 - \frac{3}{4\eta^2} \right] g = 0, \quad (3.30)$$

where

$$H(\eta) = \frac{1 - e^{-\eta^2}}{\eta^2}. \quad (3.31)$$

In the following we consider three regions of values of  $\eta$  separately, that is,  $\eta = O(Re^{-1/2})$ ,  $O(1)$  and  $O(Re^{1/4})$ .

First, suppose that  $\eta = O(Re^{-1/2})$  and put  $\eta = Re^{-1/2}\zeta$ . Then (3.30) is written as

$$g'' + \left[ i - \frac{3}{4\zeta^2} + O(Re^{-1}) \right] g = 0, \quad (3.32)$$

which is valid for  $\zeta \ll Re^{1/2}$  (i.e. for  $\eta \ll 1$ ). This equation has a solution

$$g = c_1 \zeta^{1/2} J_1(e^{\pi i/4} \zeta) + O(Re^{-1}), \quad (3.33)$$

which is regular at  $\zeta = 0$ . Here,  $c_1$  is a constant and  $J_1$  is the Bessel function of the first kind. For  $\zeta \ll 1$  solution (3.33) is expanded as

$$g = \frac{1}{2} c_1 e^{\pi i/4} \zeta^{3/2} \left( 1 - \frac{i}{8} \zeta^2 - \frac{1}{192} \zeta^4 + \dots \right), \quad (3.34)$$

and for  $\zeta \gg 1$  it is written, in the leading order, as

$$g \approx \frac{c_1}{(2\pi)^{1/2}} \left[ e^{5\pi i/8} \exp(e^{-\pi i/4} \zeta) + e^{-7\pi i/8} \exp(e^{3\pi i/4} \zeta) \right]. \quad (3.35)$$

By requiring that (3.34) may coincide with (3.28), we obtain, using definition (3.29) of  $g$ , that

$$c_0 = \frac{1}{2} c_1 e^{\pi i/4} Re^{3/4}. \quad (3.36)$$

Next, in region  $\eta = O(1)$ , equation (3.30) is written as

$$g'' + Re \left[ iH(\eta) + O(Re^{-1}) \right] g = 0, \quad (3.37)$$

which is valid for  $\eta \ll Re^{1/2}$ . We then apply the WKB approximation to obtain

$$g = H(\eta)^{-1/4} \left[ c_2 \exp(Re^{1/2}n(\eta)) + c_3 \exp(-Re^{1/2}n(\eta)) \right] + O(Re^{-1}), \quad (3.38)$$

where  $c_2$  and  $c_3$  are new constants, and

$$n(\eta) = e^{-\pi i/4} \int_0^\eta H(s)^{1/2} ds. \quad (3.39)$$

The asymptotic forms of (3.38) for small and large values of  $\eta$  are respectively written as

$$g \approx c_2 \exp(e^{-\pi i/4} Re^{1/2} \eta) + c_3 \exp(e^{3\pi i/4} Re^{1/2} \eta) \quad \text{for } \eta \ll 1 \quad (3.40)$$

and

$$g \approx \eta^{1/2} [c_2 \exp(e^{-\pi i/4} Re^{1/2}(\ln \eta + \mu)) + c_3 \exp(e^{3\pi i/4} Re^{1/2}(\ln \eta + \mu))] \quad \text{for } \eta \gg 1, \quad (3.41)$$

where

$$\mu = \int_0^1 H(s)^{1/2} ds + \int_1^\infty \left[ H(s)^{1/2} - \frac{1}{s} \right] ds. \quad (3.42)$$

Matching conditions of (3.40) with (3.35) give

$$c_1 = c_2(2\pi)^{1/2} e^{-5\pi i/8}, \quad (3.43)$$

$$c_3 = \frac{c_1}{(2\pi)^{1/2}} e^{-7\pi i/8}. \quad (3.44)$$

In the third region,  $\eta = O(Re^{1/4})$ , we put  $\eta = Re^{1/4}\chi$  to obtain

$$g'' + Re \left[ \frac{i}{\chi^2} - \chi^2 - 4Re^{-1/2} + O(Re^{-1}) \right] g = 0, \quad (3.45)$$

which is valid for  $Re^{-1/2} \ll \chi \ll Re^{1/2}$  (i.e. for  $Re^{-1/4} \ll \eta \ll Re^{3/4}$ ). We again apply the WKB approximation to (3.45) and find†

$$g = e^{-\pi i/4} \chi^{1/2} (\chi^4 - i)^{-1/4} [c_4 (\chi^2 + (\chi^4 - i)^{1/2}) \exp(Re^{1/2}\sigma(\chi)) + c_5 (\chi^2 + (\chi^4 - i)^{1/2})^{-1} \exp(-Re^{1/2}\sigma(\chi))] + O(Re^{-1}), \quad (3.46)$$

where

$$\sigma(\chi) = \frac{1}{2} e^{\pi i/4} [e^{-\pi i/4} (\chi^4 - i)^{1/2} - \arctan(e^{-\pi i/4} (\chi^4 - i)^{1/2}) + \frac{1}{2}\pi]. \quad (3.47)$$

For small values of  $\chi$ , the function  $\sigma$  can be expressed asymptotically as

$$\sigma = e^{-\pi i/4} \ln \chi + \rho + O(\chi^4), \quad (3.48)$$

where

$$\rho = \frac{1}{2} e^{3\pi i/4} \ln 2 + 2^{-3/2} \left[ \frac{1}{4}\pi + 1 + i \left( \frac{1}{4}\pi - 1 \right) \right]. \quad (3.49)$$

For large  $\chi$ , on the other hand,  $\sigma$  has the expansion

$$\sigma = \frac{1}{2}\chi^2 + \frac{i}{4\chi^2} - \frac{1}{48\chi^6} + \dots \quad (3.50)$$

Hence, (3.46) is written as

$$g \approx \chi^{1/2} [c_4 e^{-3\pi i/8} \exp(e^{-\pi i/4} Re^{1/2} \ln \chi + Re^{1/2} \rho) + c_5 e^{\pi i/8} \exp(e^{3\pi i/4} Re^{1/2} \ln \chi - Re^{1/2} \rho)] \quad \text{for } \chi \ll 1, \quad (3.51)$$

and

$$g \approx 2c_4 e^{-\pi i/4} \chi^{3/2} \exp\left(\frac{1}{2} Re^{1/2} \chi^2\right) \quad \text{for } \chi \gg 1. \quad (3.52)$$

By matching (3.51) with (3.41), we find

$$c_2 = c_4 e^{-3\pi i/8} Re^{-1/8} \kappa(Re), \quad (3.53)$$

$$c_3 = c_5 e^{-\pi i/8} Re^{1/8} \kappa(Re), \quad (3.54)$$

† There are typographic errors in the WKB solution given by Moore (1985) in his (3.12). The two linearly independent solutions constructed by the WKB method should be

$$\chi^{1/2} (\chi^4 - i)^{-1/4} (\chi^2 + (\chi^4 - i)^{1/2})^{\pm 1} e^{\pm Re^{1/2} \sigma}.$$



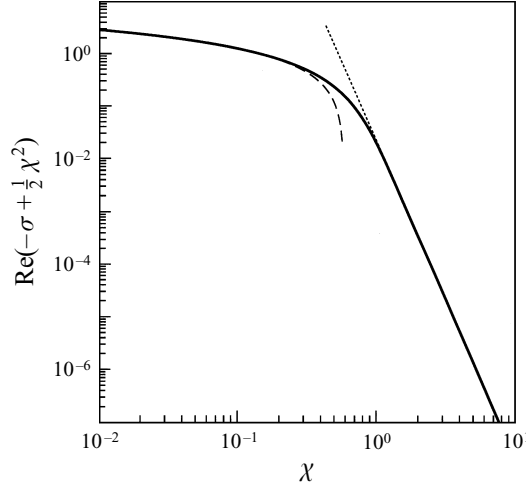


FIGURE 6. Real part of  $-\sigma + \frac{1}{2}\chi^2$  versus  $\chi$ . Dashed and dotted lines denote the asymptotic forms for small and large  $\chi$ , respectively (see equations (3.48)–(3.50)).

where

$$\kappa(Re) = \exp(Re^{1/2}(e^{3\pi i/4} \ln Re^{1/4} + e^{3\pi i/4} \mu + \rho)). \quad (3.55)$$

Finally, we extend the third region to infinity so that boundary condition  $f(\infty) = f_0$  ( $= -D_0 e^{i\varphi_0}$ ) can be applied to determine constant  $c_4$ . We compare (3.52) with the boundary condition using definition (3.29) of  $g$  to obtain

$$c_4 = -\frac{1}{2} D_0 e^{i(\varphi_0 + \pi/4)} Re^{3/8}. \quad (3.56)$$

Constants,  $c_2$ ,  $c_1$ ,  $c_0$ ,  $c_3$  and  $c_5$  are determined in turn through (3.53), (3.43), (3.36), (3.44) and (3.54). The results are that

$$c_0 = -\frac{1}{2} (\frac{1}{2}\pi)^{1/2} D_0 e^{i(\varphi_0 - \pi/2)} Re \kappa(Re), \quad (3.57)$$

$$c_1 = -(\frac{1}{2}\pi)^{1/2} D_0 e^{i(\varphi_0 - 3\pi/4)} Re^{1/4} \kappa(Re), \quad (3.58)$$

$$c_2 = -\frac{1}{2} D_0 e^{i(\varphi_0 - \pi/8)} Re^{1/4} \kappa(Re), \quad (3.59)$$

$$c_3 = -\frac{1}{2} D_0 e^{i(\varphi_0 - 13\pi/8)} Re^{1/4} \kappa(Re), \quad (3.60)$$

$$c_5 = \frac{1}{2} D_0 e^{i(\varphi_0 - 7\pi/4)} Re^{3/8} \kappa(Re)^2. \quad (3.61)$$

When  $\alpha_0 = \frac{1}{2}\pi$  and  $\beta = \pm\frac{1}{2}\pi$ , then  $D_0 = 0$  and  $u'_{1p} = 0$  (see first footnote on p. 125) and therefore all of the above constants vanish. Hence, in this case it is concluded that  $f(\eta) \equiv 0$ , and thus  $u''_1 \equiv 0$  and  $u'_1 \equiv 0$ . In this special situation the central axis of the vortex tube is parallel or anti-parallel to the simple shear vorticity. Except for this trivial case, (3.55) implies that  $|\kappa|$  is exponentially small as  $Re \rightarrow \infty$ , and so are  $|c_0|$ ,  $|c_1|$ ,  $|c_2|$ ,  $|c_3|$  and  $|c_5|$ .

Now we come back to consider the behaviour of  $f(\eta)$ . Since  $c_1$ ,  $c_2$  and  $c_3$  are exponentially small constants, solutions (3.33) and (3.38) become very small as  $Re \rightarrow \infty$ . Hence, in the region  $\eta \lesssim 1$ ,  $|f|$  is very small for  $Re \gg 1$ . Next, in the region  $\eta = Re^{1/4}\chi$  ( $\chi = O(1)$ ), the dominant contributor to solution (3.46) is the first term since  $c_5$  is an exponentially small constant. Then, (3.29), (3.46) and (3.56) give

$$f = -\frac{1}{2} D_0 e^{i\varphi_0} \chi^{-1} (\chi^4 - i)^{-1/4} (\chi^2 + (\chi^4 - i)^{1/2}) \exp(Re^{1/2}(\sigma - \frac{1}{2}\chi^2)). \quad (3.62)$$

Since the real part of the argument,  $\sigma - \frac{1}{2}\chi^2$ , in the exponential function is shown

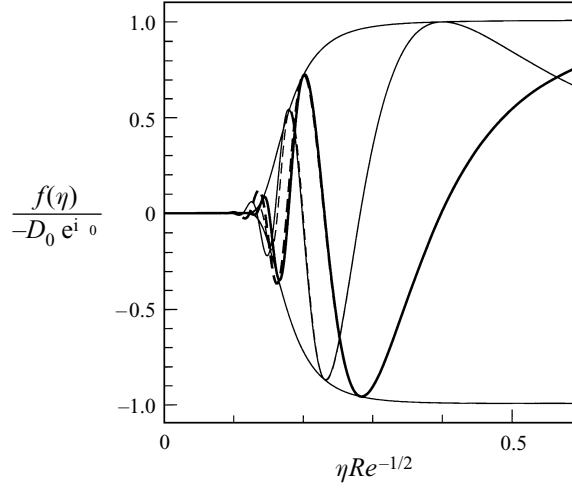


FIGURE 7. Solution  $f(\eta)$  to equation (3.26) at  $Re = 1000$ . Numerical solutions and asymptotic form (3.63) are represented by solid and dashed curves respectively. Thick and thin curves denote the real and imaginary parts respectively. The envelopes,  $\pm \exp(-Re^2/48\eta^6)$  are also drawn with thin solid curves.

numerically to be negative (see figure 6),  $|f|$  is also very small for  $Re \gg 1$  in the region  $\eta = O(Re^{1/4})$  ( $\chi = O(1)$ ). These considerations lead us to the conclusion that  $|f|$  (and so  $Ru$  and  $u'_1$ ) decreases to zero exponentially as  $Re \rightarrow \infty$  up to the region  $\eta = O(Re^{1/4})$ . This implies that  $u'_1 \approx u'_{1p}$  at  $\eta \lesssim Re^{1/4}$  (see (2.39)). In other words, the fluctuation axial velocity is well described in terms of particular solution (2.36). To examine the functional form of  $f(\eta)$  in the region  $\eta \gg Re^{1/4}$  ( $\chi \gg 1$ ) we expand (3.62) in a series of inverse powers of  $\chi$ , by making use of (3.50), to obtain, in terms of original variable  $\eta$ , that

$$f(\eta) \approx -D_0 e^{i\phi_0} \exp\left(\frac{iRe}{4\eta^2} - \frac{Re^2}{48\eta^6}\right), \quad (3.63)$$

and that

$$f'(\eta) \approx D_0 e^{i\phi_0} \frac{iRe}{2\eta^3} \exp\left(\frac{iRe}{4\eta^2} - \frac{Re^2}{48\eta^6}\right). \quad (3.64)$$

In figure 7 we plot  $f(\eta)/(-D_0 e^{i\phi_0})$  expressed by asymptotic solution (3.63) (dashed curve) at  $Re = 1000$  together with a numerical solution (solid curve) of (3.26) solved by a shooting method, where thick and thin curves denote the real and imaginary parts, respectively. The agreement between the two solutions is excellent except for relatively small values of  $\eta$ . The region of disagreement should shrink as  $Re$  increases. In order to see the Reynolds number dependence we plot  $f(\eta)$  for two different Reynolds numbers on two different scales in figure 8. It oscillates more and more frequently with increasing Reynolds number. The solution itself scales as  $Re^{1/2}$  at large  $\eta$ , while the envelope scales as  $Re^{1/3}$ .

We next consider the vorticity component normal to the vortex tube. By using (2.47), (2.48) and (3.25), the normal components of the total vorticity  $\boldsymbol{\omega} = \nabla \times \mathbf{U} + \boldsymbol{\omega}'$  can be expressed in terms of  $f(\eta)$  as

$$\omega_2 = A(t)^{-1/2} \text{Re} \left[ f + \frac{1}{2}\eta f' - \frac{1}{2}\eta f' e^{-i2\theta} \right], \quad (3.65)$$

$$\omega_3 = A(t)^{-1/2} \text{Im} \left[ f + \frac{1}{2}\eta f' + \frac{1}{2}\eta f' e^{-i2\theta} \right]. \quad (3.66)$$

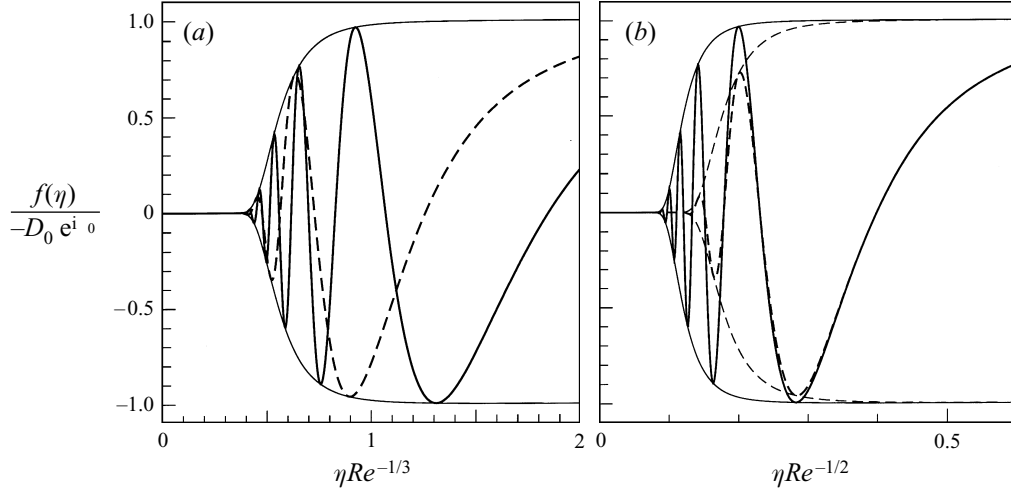


FIGURE 8. Reynolds number dependence of asymptotic solution (3.63). The real parts of solutions are plotted against (a)  $\eta Re^{-1/3}$  and (b)  $\eta Re^{-1/2}$ . Thick dashed and solid curves denote solutions at  $Re = 1000$  and  $10000$ , respectively. Thin dashed and solid lines represent their envelopes,  $\pm \exp(-Re^2/48\eta^6)$ .

If we use asymptotic forms (3.63) and (3.64), then (3.65) and (3.66) become

$$\omega_2 = -A(t)^{-1/2} D_0 \left[ \cos\left(\frac{Re}{4\eta^2} + \varphi_0\right) + \frac{Re}{2\eta^2} \cos\left(\frac{Re}{4\eta^2} + \varphi_0 - \theta\right) \sin\theta \right] \exp\left(-\frac{Re^2}{48\eta^6}\right), \quad (3.67)$$

$$\omega_3 = -A(t)^{-1/2} D_0 \left[ \sin\left(\frac{Re}{4\eta^2} + \varphi_0\right) - \frac{Re}{2\eta^2} \cos\left(\frac{Re}{4\eta^2} + \varphi_0 - \theta\right) \cos\theta \right] \exp\left(-\frac{Re^2}{48\eta^6}\right). \quad (3.68)$$

In figure 9 are shown the spatial distribution of normal vorticity  $(\omega_2^2 + \omega_3^2)^{1/2}$  and the projected vorticity lines on the normal  $(x_2, x_3)$ -plane, which were obtained from (3.67) and (3.68), at  $Re = 1000$ . Parts (a), (b) and (c) represent the cyclonic ( $\alpha_0 = \arctan \sqrt{2}$ ,  $\beta = -\frac{1}{4}\pi$ ), neutral ( $\alpha_0 = \frac{1}{4}\pi$ ,  $\beta = 0$ ), and anticyclonic ( $\alpha_0 = \arctan \sqrt{2}$ ,  $\beta = \frac{1}{4}\pi$ ) cases, in which the vortex tube is oriented in the direction of  $\hat{X}_1 + \hat{X}_2 - \hat{X}_3$ ,  $\hat{X}_1 + \hat{X}_2$ , and  $\hat{X}_1 + \hat{X}_2 + \hat{X}_3$ , respectively. Here, the relative magnitude of the normal vorticity is represented by colour: the red is the highest (7S) and the blue is the lowest (i.e. zero). It can be seen that the vortex tube wraps and stretches vorticity lines around it to form two spiral vortex layers of high azimuthal vorticity oriented alternately

FIGURE 9. Spatial distribution of magnitude  $(\omega_2^2 + \omega_3^2)^{1/2}$  of vorticity normal to a vortex tube at  $Re = 1000$  for (a) the cyclonic ( $\alpha_0 = \arctan \sqrt{2}$ ,  $\beta = -\frac{1}{4}\pi$ ), (b) neutral ( $\alpha_0 = \frac{1}{4}\pi$ ,  $\beta = 0$ ), and (c) anticyclonic ( $\alpha_0 = \arctan \sqrt{2}$ ,  $\beta = \frac{1}{4}\pi$ ) cases. The level of the magnitude is represented by colour: red is the highest (7S) and blue is the lowest (i.e. null). Solid curves represent vorticity lines projected on the  $(x_2, x_3)$ -plane. A side length of the domain is 40 in similarity variable  $\eta$ . Two characteristic length scales  $Re^{1/3}$  and  $Re^{1/2}$  are 10 and 31.6, respectively.

FIGURE 14. Same as figure 9 but for spatial distribution of the fluctuation axial vorticity  $\omega'_1$ . The level of  $\omega'_1$  is represented by colour: red is the highest ( $+S^2T$ ) and blue is the lowest ( $-S^2T$ ).

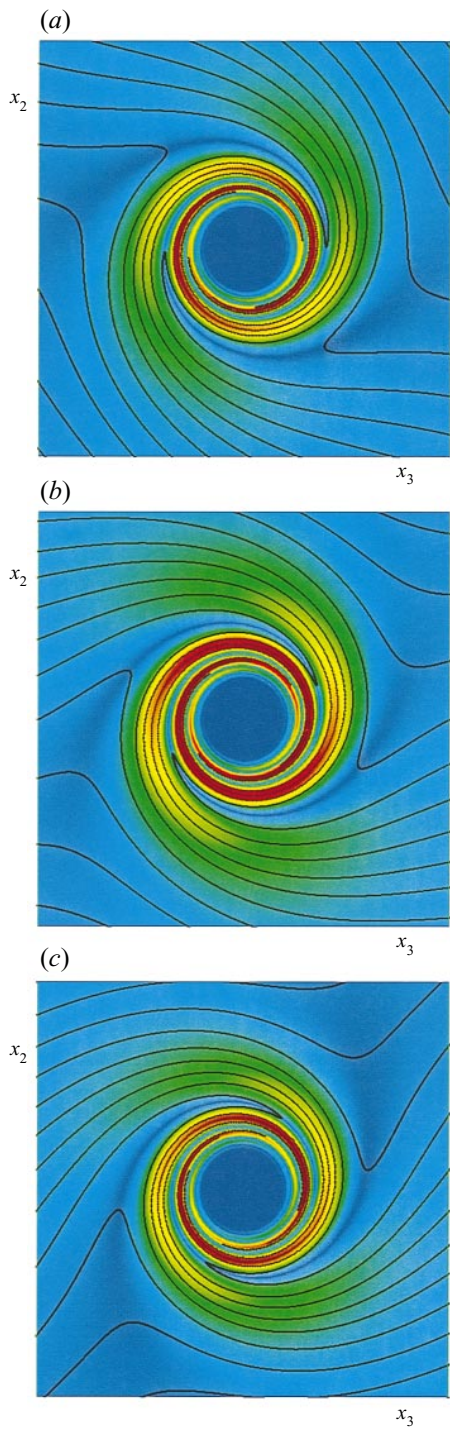


Figure 9

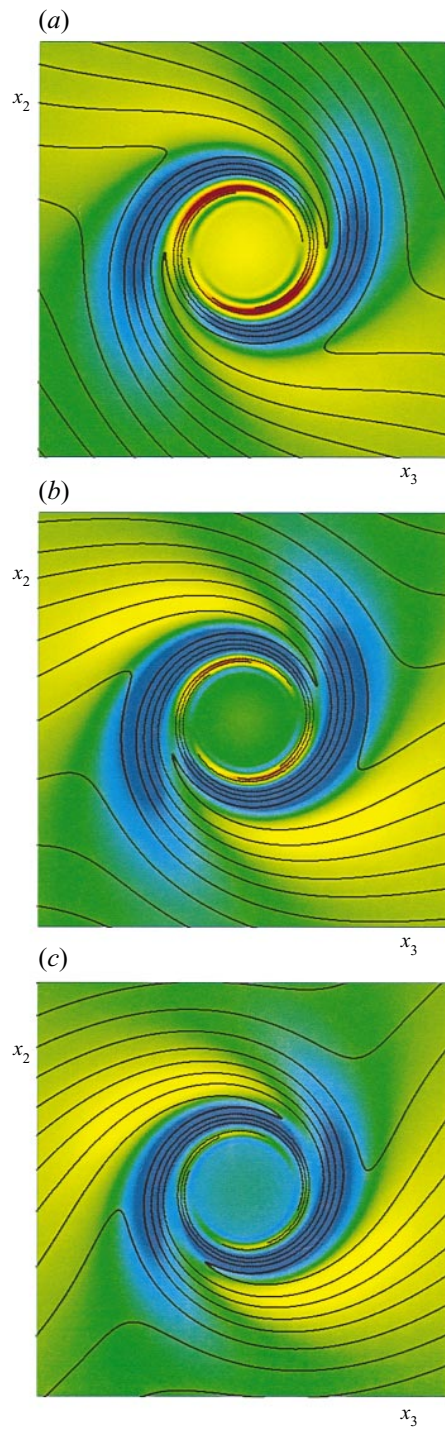


Figure 14

FIGURES 9 AND 14. For caption see facing page.

in opposite directions. One of the most interesting features of the spirals is that the  $x_2$ -component of normal vorticity takes positive values in the outermost spiral layers of strong vorticity. It changes sign every time it makes a half-turn along the spirals. An important consequence of this change of sign will be discussed in §4.3.

In the near region ( $\eta \lesssim Re^{1/4}$ , actually for  $\eta \ll Re^{1/3}$ ), however, the excessive wrapping narrows the spacing of the spiral layers and enhances the viscous diffusion to cancel out their opposite-signed vorticities, which leads to the disappearance of the normal vorticity around the vortex tube and to selective stretch-and-intensification of a cyclonic vortex tube (see §4.2).

The azimuthal vorticity component  $\omega_\theta$ , which dominates the radial component (see the vorticity lines in figure 9), is expressed as

$$\omega_\theta = -\omega_2 \sin \theta + \omega_3 \cos \theta = -A(t)^{-1/2} \operatorname{Re} [i(\eta f)' e^{-i\theta}], \quad (3.69)$$

which takes a local maximum and minimum,

$$\left. \begin{array}{l} \omega_{\theta_{max}} \\ \omega_{\theta_{min}} \end{array} \right\} = \pm A(t)^{-1/2} |(\eta f)'| \quad \text{at} \quad \left. \begin{array}{l} \theta_{max} \\ \theta_{min} \end{array} \right\} = \arg[(\eta f)'] \mp \frac{1}{2}\pi. \quad (3.70)$$

This implies that two spiral vortex layers of high azimuthal vorticity of opposite sign of  $\omega_\theta$  are arranged alternately. It follows from (3.63) and (3.64) that

$$(\eta f)' \approx -D_0 e^{i\varphi_0} \left( 1 - \frac{iRe}{2\eta^2} \right) \exp \left( \frac{iRe}{4\eta^2} - \frac{Re^2}{48\eta^6} \right), \quad (3.71)$$

the magnitude and phase of which are, respectively, written as

$$|(\eta f)'| \approx D_0 \frac{Re}{2\eta^2} \left( 1 + \frac{4\eta^4}{Re^2} \right)^{1/2} \exp \left( -\frac{Re^2}{48\eta^6} \right), \quad (3.72)$$

$$\arg[(\eta f)'] \approx \frac{Re}{4\eta^2} + \arctan \left( \frac{2\eta^2}{Re} \right) + \varphi_0 + \frac{1}{2}\pi. \quad (3.73)$$

In figure 10 we plot  $(\eta f)' Re^{-1/3} / (-D_0 e^{i\varphi_0})$  at  $Re = 1000$ , where solid and dashed curves represent the real and imaginary parts respectively, and thin solid lines  $\pm |(\eta f)'|$ . We can see that magnitude  $|(\eta f)'|$  has a single maximum of  $0.903 D_0 Re^{1/3}$  at  $\eta = 2^{-2/3} Re^{1/3}$  (Moore 1985). It is exponentially small at  $\eta \ll Re^{1/3}$ , while it approaches a constant  $D_0 = (\cos^2 \alpha_0 \sin^2 \beta + \cos^2 \beta)^{1/2}$ , which is the magnitude of the normal component of the simple shear vorticity, as  $\eta$  increases. The phase,  $\arg[(\eta f)']$ , is infinity at  $\eta = 0$  and decreases up to  $\eta = (\frac{1}{2} Re)^{1/2}$  at which it takes a minimum value of  $\frac{1}{2} + \frac{3}{4}\pi + \varphi_0$ , and thereafter it increases monotonically to approach  $\pi + \varphi_0$  at  $\eta \rightarrow \infty$ . Therefore, the spiral form of the layer actually terminates around  $\eta = (\frac{1}{2} Re)^{1/2}$  since beyond this point both  $\theta_{max}$  and  $\theta_{min}$  change only by  $\frac{1}{4}\pi - \frac{1}{2} \approx 0.29$  in the opposite direction.

The distance between adjacent layers of the two spirals is estimated as follows. Let  $\eta_+$  and  $\eta_-$  be successive locations for a fixed value of  $\theta$  of  $\omega_{\theta_{max}}$  and  $\omega_{\theta_{min}}$ , respectively, and let  $\Delta\eta = \eta_- - \eta_+$  be their spacing. Then, it follows from (3.70) and (3.73) that

$$\frac{Re}{\eta_+^2} - \frac{Re}{\eta_-^2} = Re \frac{(\eta_+ + \eta_-)}{\eta_+^2 \eta_-^2} \Delta\eta = 4\pi - 4 \arctan \left( \frac{2\eta_+^2}{Re} \right) + 4 \arctan \left( \frac{2\eta_-^2}{Re} \right), \quad (3.74)$$

and therefore we have

$$\Delta\eta = O \left( \frac{\eta^3}{Re} \right) \quad (3.75)$$

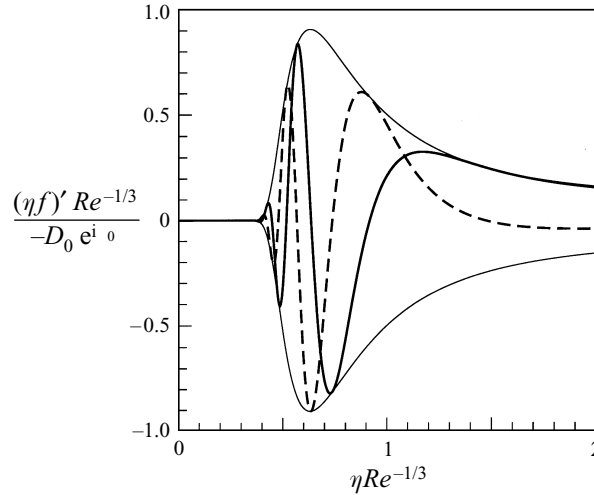


FIGURE 10. Amplitude  $(\eta f)'$  of the circumferential component of vorticity  $\omega_\theta$ . Solid and dashed curves represent the real and imaginary parts respectively. Thin solid lines denote the magnitude,  $\pm|(\eta f)'|$ .

as long as  $\eta < (\frac{1}{2}Re)^{1/2}$ . The spacing between adjacent spiral layers is  $O(1)$  at  $\eta = O(Re^{1/3})$ .<sup>†</sup> It is greater than  $O(1)$  at  $O(Re^{1/3}) < \eta < (\frac{1}{2}Re)^{1/2}$ . For  $Re^{1/3} \gg \eta$  ( $\gg Re^{1/4}$ ), the spacing becomes very small as  $Re \rightarrow \infty$ .

In figure 9, in which a side length of the domain is 40 in similarity variable  $\eta$ , two characteristic length scales  $Re^{1/3}$  and  $Re^{1/2}$  are 10 and 31.6, respectively. It is seen that the double spiral vortex layer is developed at  $\eta = O(Re^{1/3})$  and that it extends up to  $\eta = O(Re^{1/2})$ . Observe also that the spirals are terminated in the near region ( $\eta \ll O(Re^{1/3})$ ).

### 3.5. Higher-order axial vorticity

In this subsection we consider solutions to higher-order equation (3.15) for the axial vorticity. In order to get an explicit analytical solution, we restrict ourselves to early-time evolution ( $T \ll 1$ ). Since time  $T$  does not explicitly appear in the leading-order streamfunction (3.24), the first-order streamfunction and the corresponding axial vorticity may be expanded respectively as

$$\psi^{(1)}(R, \theta, T) = T\psi^{(1,1)}(\eta, \theta) + T^2\psi^{(1,2)}(\eta, \theta) + \dots, \quad (3.76)$$

$$\omega^{(1)}(R, \theta, T) = \omega^{(1,0)}(\eta, \theta) + T\omega^{(1,1)}(\eta, \theta) + \dots, \quad (3.77)$$

where

$$\omega^{(1,j-1)} = -\frac{1}{4}\nabla_\eta^2\psi^{(1,j)} \quad (j = 1, 2, \dots) \quad (3.78)$$

with

$$\nabla_\eta^2 = \partial_\eta^2 + \frac{1}{\eta}\partial_\eta + \frac{1}{\eta^2}\partial_\theta^2. \quad (3.79)$$

For small values of  $T$ , the time-dependent factors,  $\gamma/A$ ,  $\lambda/A$  and  $\xi/A^{5/2}$  in linear operators  $L_1$  and  $L_2$  (see (2.44) and (2.45)) are expanded in power series of  $T$ , using

<sup>†</sup> In dimensional variables, the location of the maximum normal vorticity is  $O(Re^{1/3}(vT)^{1/2})$  and the spacing of the spirals is  $O((vT)^{1/2})$  (see §4.4 and figure 19).

$\alpha = \alpha_0 - T \sin^2 \alpha_0 \cos \beta + \dots$ , as

$$\frac{\gamma}{A} = \frac{\cos \alpha \sin^2 \alpha \sin \beta}{\sin \alpha_0} = \gamma_0 + \gamma_1 T + \dots, \quad (3.80)$$

$$\frac{\lambda}{A} = -\frac{\sin^2 \alpha \sin \beta}{\sin \alpha_0} = \lambda_0 + \lambda_1 T + \dots, \quad (3.81)$$

$$\frac{\xi}{A^{5/2}} = -\frac{2 \sin^{9/2} \alpha \cos \beta}{\sin^{5/2} \alpha_0} = \xi_0 + \xi_1 T + \dots, \quad (3.82)$$

where

$$\gamma_0 = \gamma|_{T=0} = \cos \alpha_0 \sin \alpha_0 \cos \beta, \quad \gamma_1 = \sin^2 \alpha_0 (1 - 3 \cos^2 \alpha_0) \cos^2 \beta, \quad (3.83)$$

$$\lambda_0 = \lambda|_{T=0} = -\sin \alpha_0 \sin \beta, \quad \lambda_1 = 2 \cos \alpha_0 \sin^2 \alpha_0 \cos \beta \sin \beta, \quad (3.84)$$

$$\xi_0 = \xi|_{T=0} = -2 \sin^2 \alpha_0 \cos \beta, \quad \xi_1 = -9 \cos \alpha_0 \sin^3 \alpha_0 \cos^2 \beta. \quad (3.85)$$

Substituting (3.76), (3.77) and (3.80)–(3.82) into (3.15), and equating  $T^{-1}$ -order terms, we obtain

$$-\frac{1}{4\eta} \left[ \frac{\partial(\psi^{(0)}, \omega^{(1,0)})}{\partial(\eta, \theta)} + \frac{\partial(\psi^{(1,1)}, \omega^{(0,0)})}{\partial(\eta, \theta)} \right] - \epsilon \frac{1}{2} \eta \partial_\eta \omega^{(1,0)} - \epsilon \frac{1}{4} \nabla_\eta^2 \omega^{(1,0)} = \epsilon L_{10} \omega^{(0,0)}, \quad (3.86)$$

where

$$L_{10} = \frac{1}{2} [\gamma_0 (-\sin 2\theta \partial_\theta + \eta \cos 2\theta \partial_\eta) + \lambda_0 (\cos 2\theta \partial_\theta + \eta \sin 2\theta \partial_\eta - \partial_\theta)] \quad (3.87)$$

and

$$\omega^{(0,0)}(\eta) = T \omega^{(0)}(\eta) = \frac{1}{4\pi} e^{-\eta^2}. \quad (3.88)$$

At  $T^0$ -order of (3.15), we have

$$\begin{aligned} -\frac{1}{4\eta} \left[ \frac{\partial(\psi^{(0)}, \omega^{(1,1)})}{\partial(\eta, \theta)} + \frac{\partial(\psi^{(1,2)}, \omega^{(0,0)})}{\partial(\eta, \theta)} \right] + \epsilon (1 - \frac{1}{2} \eta \partial_\eta) \omega^{(1,1)} - \epsilon \frac{1}{4} \nabla_\eta^2 \omega^{(1,1)} \\ = \epsilon L_{11} \omega^{(0,0)} + \epsilon^2 L_{20} u^{(0)} + \epsilon^2 2\gamma_0 \lambda_0, \end{aligned} \quad (3.89)$$

where

$$L_{11} = \frac{1}{2} [\gamma_1 (-\sin 2\theta \partial_\theta + \eta \cos 2\theta \partial_\eta) + \lambda_1 (\cos 2\theta \partial_\theta + \eta \sin 2\theta \partial_\eta - \partial_\theta)], \quad (3.90)$$

$$L_{20} = \xi_0 [\cos \theta \partial_\theta + \sin \theta (\eta \partial_\eta + 1)]. \quad (3.91)$$

The right-hand sides of (3.86) and (3.89) represent the effects of the simple shear on the vorticity fluctuation. Asymptotic solutions to (3.86) and (3.89) at large Reynolds numbers ( $Re \gg 1$ ,  $\epsilon \ll 1$ ) are derived in Appendices A and B. Here, we summarize the results.

First, an asymptotic solution to (3.86) is written as

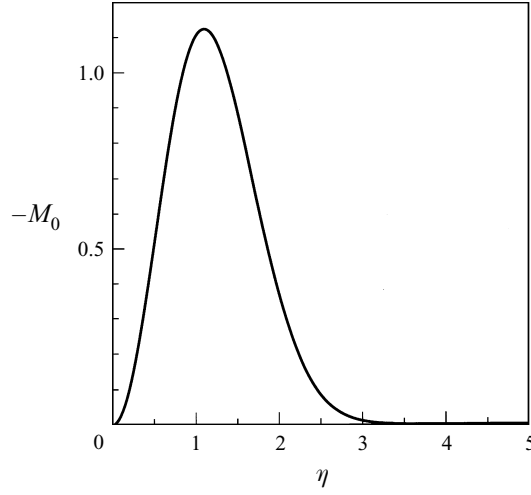
$$\omega^{(1,0)} = -\frac{1}{4} \nabla_\eta^2 \psi^{(1,1)} = \epsilon B_0 M_0(\eta) \sin(2\theta - \phi_0), \quad (3.92)$$

where

$$M_0(\eta) = -\frac{\eta^2}{e^{\eta^2} - 1} (\eta^2 - f_1(\eta)), \quad (3.93)$$

and

$$B_0 = (\gamma_0^2 + \lambda_0^2)^{1/2}, \quad \phi_0 = \arctan \left( \frac{\lambda_0}{\gamma_0} \right) \quad (3.94)$$


 FIGURE 11. Amplitude  $M_0(\eta)$  of the first-order axial vorticity  $\omega^{(1,0)}(\eta, \theta)$ .

(see Appendix A). Real function  $f_1(\eta)$  is determined by differential equation (A 9). As  $\eta \rightarrow 0$ ,  $M_0(\eta)$  approaches zero as

$$M_0(\eta) \approx (a - 1)\eta^2, \quad (3.95)$$

where  $a$  is a constant (see (A 13)). For large values of  $\eta$ , on the other hand,  $M_0(\eta)$  behaves as

$$M_0(\eta) \approx -\eta^4 e^{-\eta^2}, \quad (3.96)$$

and so  $|M_0(\eta)|$  decreases rapidly as  $\eta \rightarrow \infty$ . As shown in figure 11,  $M_0(\eta)$  is negative and has a single peak at  $\eta = O(1)$ .

Next, an asymptotic solution to (3.89) is written as

$$\omega^{(1,1)} = \epsilon M_1(\eta) + \epsilon \operatorname{Re} [M_2(\eta) e^{-i2\theta}], \quad (3.97)$$

where

$$M_1(\eta) = 2\gamma_0\lambda_0 + \frac{8\xi_0 \cos \beta\eta^2}{Re} - \xi_0 D_0 \frac{8\eta^2}{Re} \left[ \sin \left( \frac{Re}{4\eta^2} + \varphi_0 \right) - \frac{Re}{8\eta^2} \cos \left( \frac{Re}{4\eta^2} + \varphi_0 \right) \right] \exp \left( -\frac{Re^2}{48\eta^6} \right) \quad (3.98)$$

and

$$M_2(\eta) = \begin{cases} i e^{-i\phi_1} B_1 M_0(\eta) & \text{for } \eta = O(1), \\ -i \xi_0 D_0 e^{i\varphi_0} \frac{4\eta^2}{Re} \left[ \left( 1 + \frac{iRe}{4\eta^2} \right) \exp \left( \frac{iRe}{4\eta^2} - \frac{Re^2}{48\eta^6} \right) - \exp \left( \frac{iRe}{2\eta^2} - \frac{Re^2}{12\eta^6} \right) \right] & \text{for } \eta \gg 1 \end{cases} \quad (3.99)$$

with

$$B_1 = (\gamma_1^2 + \lambda_1^2)^{1/2}, \quad \phi_1 = \arctan \left( \frac{\lambda_1}{\gamma_1} \right) \quad (3.100)$$

(see Appendix B). Note that for  $\eta = O(1)$ ,  $M_2(\eta)$  is expressed in terms of  $M_0(\eta)$ . For



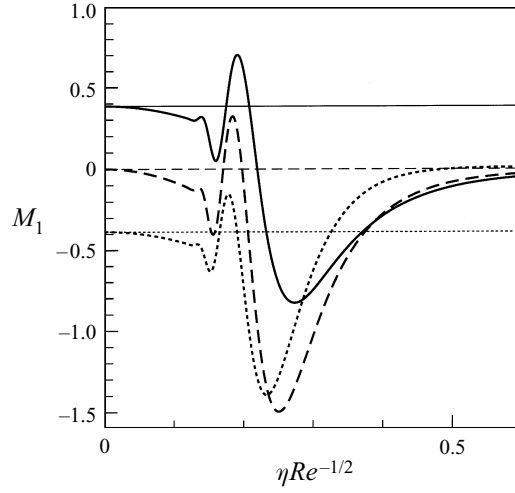


FIGURE 12. The  $\theta$ -independent part  $M_1(\eta)$  of the second-order axial vorticity  $\omega^{(1,1)}(\eta, \theta)$ . Solid, dashed and dotted lines represent the cyclonic, neutral and anticyclonic cases, respectively. Thin straight lines denote the values of  $2\gamma_0\lambda_0$ .

$\eta \ll Re^{1/3}$ ,  $M_1(\eta)$  behaves as

$$M_1 \approx 2\gamma_0\lambda_0 + \frac{8\xi_0 \cos \beta \eta^2}{Re}, \quad (3.101)$$

while  $M_2(\eta)$  is exponentially small. For  $\eta \gg Re^{1/2}$ , on the other hand, both of them approach zero algebraically as

$$M_1 \approx -\frac{\xi_0 Re^2 \cos \alpha_0 \sin \beta}{96\eta^4}, \quad (3.102)$$

$$M_2 \approx -\frac{i\xi_0 Re D_0 e^{i\varphi_0}}{8\eta^2}. \quad (3.103)$$

In figures 12 and 13, we show asymptotic solutions (3.98) and (3.99) at  $Re = 1000$  respectively for the cyclonic ( $\alpha_0 = \arctan \sqrt{2}$ ,  $\beta = -\frac{1}{4}\pi$ ), neutral ( $\alpha_0 = \frac{1}{4}\pi$ ,  $\beta = 0$ ), and anticyclonic ( $\alpha_0 = \arctan \sqrt{2}$ ,  $\beta = \frac{1}{4}\pi$ ) cases. Function  $M_1(\eta)$ , which represents the  $\theta$ -averaged structure of the higher-order axial vorticity, has a significant negative peak at a relatively large value of  $\eta$ . As  $\eta$  decreases, it oscillates and then approaches a constant  $2\gamma_0\lambda_0$ , which is positive, zero or negative according to whether it is a cyclonic, neutral or anticyclonic case. Phases in the oscillation of  $M_1(\eta)$  and  $M_2(\eta)$  depend on the values of  $\varphi_0$ . Peak positions of  $M_1(\eta)$  and  $M_2(\eta)$  in a cyclonic case ( $\varphi_0 > \frac{1}{2}\pi$ ) are located farther away from the vortex tube than those in a neutral case ( $\varphi_0 = \frac{1}{2}\pi$ ) while those in an anticyclonic case ( $\varphi_0 < \frac{1}{2}\pi$ ) are located nearer.

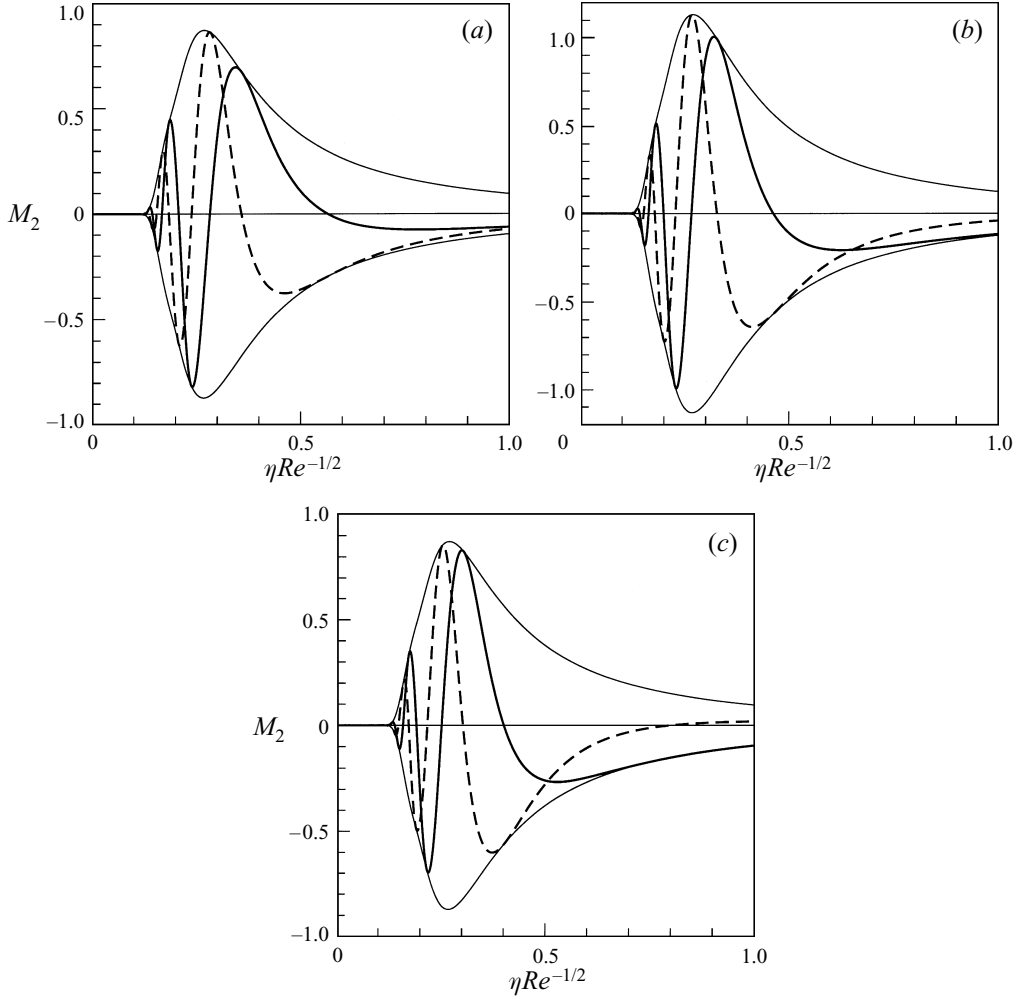


FIGURE 13. Amplitude  $M_2(\eta)$  of the  $\theta$ -dependent part of the second-order axial vorticity  $\omega^{(1,1)}(\eta, \theta)$  for (a) cyclonic, (b) neutral and (c) anticyclonic cases. Solid and dashed bold lines represent real and imaginary parts of  $M_2(\eta)$ . Thin solid lines denote the magnitude,  $\pm|M_2(\eta)|$ .

At this level of approximation, (3.77), (3.92) and (3.97) provide  $\omega^{(1)}$  up to the order of  $T$ . Remembering (2.40) and using an expansion for the dimensionless form of (2.32)

$$A(t) = 1 + \gamma_0 T + \dots, \quad (3.104)$$

we have the higher-order axial vorticity

$$A(t)\omega^{(1)} = \omega^{(1,0)} + (\gamma_0\omega^{(1,0)} + \omega^{(1,1)})T + O(T^2). \quad (3.105)$$

For  $\eta \lesssim Re^{1/4}$ , it follows from (3.92) and (3.97) that

$$A(t)\omega^{(1)} \approx \epsilon M_0(\eta) [B_0 \sin(2\theta - \phi_0) + T d_T (B(t) \sin(2\theta - \phi(t)))|_{T=0}] + \epsilon T 2\gamma_0 \lambda_0, \quad (3.106)$$

where we have used relation

$$B_1 \sin(2\theta - \phi_1) = d_T [B(t) \sin(2\theta - \phi(t))] |_{T=0} - \gamma_0 B_0 \sin(2\theta - \phi_0) \quad (3.107)$$

with

$$B(t) = (\gamma(t)^2 + \lambda(t)^2)^{1/2} = |\sin \alpha| (\cos^2 \alpha \cos^2 \beta + \sin^2 \beta)^{1/2} \quad (3.108)$$

and

$$\phi(t) = \arctan \left( \frac{\lambda(t)}{\gamma(t)} \right) = \arctan \left( \frac{-\sin \beta}{\cos \alpha \cos \beta} \right) \quad (-\pi \leq \phi(t) \leq \pi), \quad (3.109)$$

which represents an angle from the  $x_2$ -axis to a projection of the  $X_2$ -axis on the normal  $(x_2, x_3)$ -plane (see (2.2) and (2.3)). Solution (3.106) represents the leading and first orders of a Taylor expansion of

$$A(t)\omega^{(1)} = \epsilon M_0(\eta)B(t) \sin(2\theta - \phi(t)) + \epsilon(A(t)\lambda_0 - \lambda(t)). \quad (3.110)$$

In the near region  $\eta \lesssim Re^{1/4}$ , we can drop the second term on the right-hand side of (3.15), which is exponentially small as  $Re \rightarrow \infty$  (see §3.4). This tells us the important fact that (3.110) may be obtained only by expanding  $\omega^{(1)}$  in a power series of  $\epsilon$  without the short-time assumption  $T \ll 1$ .

The first term on the right-hand side of (3.110) represents a quadrupole-type distribution, which means a deformation of the vortex core into an elliptical shape by the effect of the simple shear (Moffatt *et al.* 1994). The major and minor axes of the resulting elliptical core are aligned at an angle of  $\theta = \frac{1}{2}\phi(t) + \frac{1}{2}\pi \pm \frac{1}{4}\pi$ , respectively. If the normal velocity components,  $(\bar{u}_2, \bar{u}_3)$ , of the simple shear flow relative to the structural coordinate system are decomposed into symmetric and antisymmetric tensors, we find that the symmetric one, which represents straining flow, has a principal direction with a positive rate of strain at an angle of  $\theta = \frac{1}{2}\phi(t) + \frac{1}{2}\pi$  (the values of rates of strain which are normalized by the uniform shear rate  $S$  are  $\frac{1}{2}(-\gamma(t) \pm B(t))$ ). Therefore, the major and minor axes of the ellipse are inclined away from the direction of strain by  $\pm \frac{1}{4}\pi$ , respectively (Moffatt *et al.* 1994). This quadrupole distribution of vorticity, which does not include the stretch factor  $A(t)$ , is not affected by the axial component of the simple shear stress. In the case of  $\alpha = 0$ , which was considered by Pearson & Abernathy (1984) and Moore (1985), the vortex tube is aligned with the simple shear flow and therefore the vortex core is not deformed ( $B(t) \equiv 0$ ).

The second term on the right-hand side of (3.110) represents the stretching of the axial vorticity component of the simple shear by the axial stress (see §4.2).

#### 4. Physical interpretation

A physical interpretation of the asymptotic solutions derived in §3 will be given in this section to help understand the structure of vorticity field and the physical process. We restore here dimensional variables, following table 1, as

$$T = T^*/S, \quad R = (v/S)^{1/2}R^*, \quad \omega'_1 = \epsilon^{-1}S\omega_1^* \quad u'_1 = (vS)^{1/2}u_1^*. \quad (4.1)$$

A similarity variable  $\eta$  is then  $\eta = (R^*/2T^{*1/2}) = R/2(vT)^{1/2}$ . Recall that the asterisks, which are attached to dimensionless variables, have been dropped in §3.

## 4.1. Structure of the vorticity field

We first express the vorticity field in physical space  $(R, \theta, T)$  and discuss its structure. The axial component of vorticity fluctuation in the near region  $R \lesssim Re^{1/4}(vT)^{1/2}$  is written, using (3.22) and (3.110), as

$$\omega'_1 \approx \frac{\Gamma A(t)}{4\pi v T} \exp\left(-\frac{R^2}{4vT}\right) + SM_0\left(\frac{R}{2(vT)^{1/2}}\right) B(t) \sin(2\theta - \phi(t)) + S(A(t)\lambda_0 - \lambda(t)). \quad (4.2)$$

This equation tells us that the vortex tube is diffused, stretched and deformed by the effects of viscosity and the simple shear. In a stationary coordinate system the vortex tube also rotates toward the streamwise ( $X_1$ ) direction with an angular velocity given by (2.19). The normal components of fluctuation vorticity,  $\omega'_2 \approx \partial_3 u'_{1p}$  and  $\omega'_3 \approx -\partial_2 u'_{1p}$ , on the other hand, cancel out with those of the simple shear vorticity so that the normal component of the total vorticity disappears and vorticity vectors are aligned with the axial direction. This alignment of a vortex tube and vorticity vectors has been observed in homogeneous isotropic turbulence as well as in homogeneously sheared turbulence and near-wall turbulence (see She *et al.* 1990; Kida & Tanaka 1994; Bernard *et al.* 1993). Disappearance of the normal vorticity leads to stretching of the axial component of the simple shear vorticity, which is represented by  $S(A(t)\lambda_0 - \lambda(t))$  (see §4.2 for the mechanism). Note that this effect disappears when the vortex tube is not inclined vertically ( $\alpha = 0$  or  $\beta = \pm\frac{1}{2}\pi$ ), or when it is not inclined in the spanwise ( $X_3$ ) direction ( $\alpha = 0$  or  $\beta = 0$ ). This implies that the asymptotic behaviour of a vortex tube at very large times  $St \gg 1$  is essentially different between a perfectly aligned ( $\alpha \equiv 0$ ) streamwise vortex, which was examined by Pearson & Abernathy (1984) and Moore (1985), and an oblique vortex ( $\alpha_0 \neq 0$ ). In fact, the third term in (4.2) may become comparable with the first at  $St \approx Re^{1/2}$  ( $ST \approx Re$ ) for an oblique case.

In the far region  $R \gg Re^{1/4}(vT)^{1/2}$ , the axial vorticity of the vortex tube itself is exponentially small, and therefore it follows from (3.105), (3.97)–(3.99) that

$$\begin{aligned} \omega'_1 \approx & 2S^2 T \gamma_0 \lambda_0 + S^2 \xi_0 \frac{4\pi R^2}{\Gamma} \cos \beta \\ & + S^2 \xi_0 D_0 \left[ 2T \cos\left(\frac{\Gamma T}{2\pi R^2} + \varphi_0 - \theta\right) \cos \theta - \frac{4\pi R^2}{\Gamma} \sin\left(\frac{\Gamma T}{2\pi R^2} + \varphi_0\right) \right. \\ & \left. + \frac{2\pi R^2}{\Gamma} \sin\left(\frac{\Gamma T}{2\pi R^2} + \varphi_0 - 2\theta\right) \right] \exp\left[-\frac{8\pi v}{3\Gamma} \left(\frac{\Gamma T}{2\pi R^2}\right)^3\right] \\ & - S^2 \xi_0 D_0 \frac{2\pi R^2}{\Gamma} \sin\left(\frac{\Gamma T}{\pi R^2} + \varphi_0 - 2\theta\right) \exp\left[-\frac{32\pi v}{3\Gamma} \left(\frac{\Gamma T}{2\pi R^2}\right)^3\right]. \end{aligned} \quad (4.3)$$

The normal components of the vorticity fluctuation, on the other hand, are expressed, using (3.67) and (3.68), as

$$\begin{aligned} \omega'_2 \approx & S \cos \alpha_0 \sin \beta \\ & - SD_0 \left[ \cos\left(\frac{\Gamma T}{2\pi R^2} + \varphi_0\right) + \frac{\Gamma T}{\pi R^2} \cos\left(\frac{\Gamma T}{2\pi R^2} + \varphi_0 - \theta\right) \sin \theta \right] \exp\left[-\frac{8\pi v}{3\Gamma} \left(\frac{\Gamma T}{2\pi R^2}\right)^3\right] \end{aligned} \quad (4.4)$$

and

$$\omega'_3 \approx S \cos \beta - SD_0 \left[ \sin \left( \frac{\Gamma T}{2\pi R^2} + \varphi_0 \right) - \frac{\Gamma T}{\pi R^2} \cos \left( \frac{\Gamma T}{2\pi R^2} + \varphi_0 - \theta \right) \cos \theta \right] \exp \left[ -\frac{8\pi\nu}{3\Gamma} \left( \frac{\Gamma T}{2\pi R^2} \right)^3 \right]. \quad (4.5)$$

The spatial distributions on the normal plane of axial vorticity fluctuation (4.3) are drawn in figure 14 (see p. 133) for (a) cyclonic ( $\alpha_0 = \arctan \sqrt{2}$ ,  $\beta = -\frac{1}{4}\pi$ ), (b) neutral ( $\alpha_0 = \frac{1}{4}\pi$ ,  $\beta = 0$ ) and (c) anticyclonic ( $\alpha_0 = \arctan \sqrt{2}$ ,  $\beta = \frac{1}{4}\pi$ ) cases at  $Re = 1000$  together with projected vorticity lines. The vorticity of the vortex tube itself is not shown in these figures. Along vorticity lines at the outermost double spirals of high azimuthal vorticity there are two crescent-shaped regions of strong negative axial vorticity, which is opposite to the vorticity of the vortex tube (cf. figure 9). Also commonly observed is relatively weak positive vorticity inside the crescent-shaped regions of negative vorticity. Further inside, i.e. in a circular domain in the vicinity of a cyclonic (or anticyclonic) vortex tube the axial vorticity takes positive (or negative) values. The appearance of negative vorticity in the far region as well as different signs of vorticity in the central region between cyclonic and anticyclonic vortex tubes were also observed in the  $\theta$ -averaged structure (see figure 12).

#### 4.2. Expulsion of normal vorticity and stretch of axial vorticity

Here, we discuss briefly a close relation between expulsion of normal vorticity around a vortex tube and stretch and intensification of axial vorticity. In a stationary frame the vorticity equation is written as

$$\frac{D\boldsymbol{\omega}}{Dt} \equiv [\partial_t + (\mathbf{u} \cdot \nabla)] \boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla) \mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}. \quad (4.6)$$

Only the tilting-and-stretching term, the first term on the right-hand side of this equation, is responsible for the vorticity intensification. For a simple shear flow ( $\mathbf{U} = SX_2 \hat{\mathbf{X}}_1$  and  $\nabla \times \mathbf{U} = -S \hat{\mathbf{X}}_3$ ), it vanishes identically, which means no vortex stretching. As was shown in §3.4, in the near field ( $r \ll (\Gamma/\nu)^{1/3}(\nu t)^{1/2}$ ) of a straight vortex tube the normal component of the simple shear vorticity is expelled by viscous cancellation of tightly wrapped vorticity lines of opposite directions and only the axial component  $(\nabla \times \mathbf{U}) \cdot \hat{\mathbf{x}}_1 = -S(\hat{\mathbf{X}}_3 \cdot \hat{\mathbf{x}}_1)$  survives there. The axial component of the tilting-and-stretching term is then estimated as

$$(\nabla \times \mathbf{U}) \cdot \hat{\mathbf{x}}_1 \partial_1 U_1 = -S^2 (\hat{\mathbf{X}}_3 \cdot \hat{\mathbf{x}}_1) (\hat{\mathbf{X}}_2 \cdot \hat{\mathbf{x}}_1) (\hat{\mathbf{X}}_1 \cdot \hat{\mathbf{x}}_1), \quad (4.7)$$

which is proportional to the product of the three direction cosines of the axial vector  $\hat{\mathbf{x}}_1$ . It takes a maximum (or minimum) value ( $\pm S^2/3^{3/2}$ ) for

$$\hat{\mathbf{x}}_1 = \frac{1}{\sqrt{3}} (\hat{\mathbf{X}}_1 + \hat{\mathbf{X}}_2 \mp \hat{\mathbf{X}}_3). \quad (4.8)$$

The upper (or lower) sign represents a cyclonic (or anticyclonic) vortex tube whose spanwise component of vorticity has the same (or opposite) sign as the simple shear vorticity. As a consequence, a cyclonic vortex tube is intensified while an anticyclonic one is weakened (see the near regions of cyclonic and anticyclonic vortex tubes in figure 14). In both homogeneous shear turbulence and near-wall turbulence, streamwise vortex tubes often have a cyclonic inclination with respect to the mean

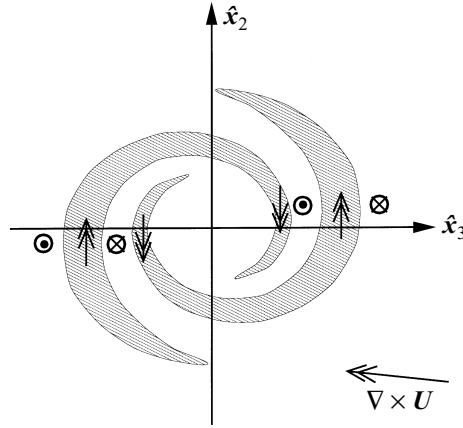


FIGURE 15. Generation mechanism of axial vorticity along spiral layers of high azimuthal vorticity which are represented by crescent-shaped shadow regions. Double arrows denote the direction of normal vorticity.  $\odot$  and  $\otimes$  denote the direction of axial velocity induced by the spiral vorticity layers by which the simple shear vorticity  $\nabla \times U$  is tilted toward the axial direction.

shear vorticity (see Kida & Tanaka 1994; Miyake & Tsujimoto 1996), which may be connected with the above mechanism of selective intensification of a cyclonic vortex.

#### 4.3. Wrap and tilt of vorticity lines and generation of axial vorticity

We discuss here the generation process of axial vorticity through wrapping and tilting of vorticity lines in the far region  $R \gg Re^{1/4}(\nu T)^{1/2}$ . Recall that the dominant contributor to the production of the axial vorticity is the tilting of the  $x_3$ -component of the simple shear vorticity by the velocity fluctuation (which is the second term on the right-hand side of (2.23)). Vorticity lines are wrapped around a vortex tube by a swirling motion to form spiral layers of high azimuthal vorticity, which in turn induce axial shear flows which tilt the simple shear vorticity toward the axial direction. If the velocity gradient  $\partial_3 u'_1 (= \omega'_2)$  in the spirals is positive (or negative), the  $x_3$ -component of the simple shear vorticity, the sign of which is negative, is tilted in the positive (or negative) axial direction to generate negative (or positive) axial vorticity (see figure 15). As was mentioned in relation to figure 9,  $\omega'_2$  is positive in the outermost spirals of intense azimuthal vorticity, which leads to the generation of the crescent-shaped regions of strong negative axial vorticity in figure 14.

The spatial distributions of the axial and normal components of fluctuation vorticity  $\omega'_1$  and  $-\omega'_2$  (NB the minus sign) are compared in figure 16 for the neutral case ( $\alpha_0 = \frac{1}{4}\pi$ ,  $\beta = 0$ ). Here, two regions, (a, b)  $R = O(Re^{1/2}(\nu T)^{1/2})$  and (c, d)  $R = O(Re^{1/3}(\nu T)^{1/2})$ , are shown at  $Re = 1000$ . We can see that the two components vary almost in phase in the far region (cf. a and b), whereas the phase of the axial component is nearly  $\frac{1}{2}\pi$  in advance clockwise compared with that of the normal one in the near region (cf. c and d). This phase difference is due to the advection by the swirling flow induced by a vortex tube (see below).

In the far region  $R \gg Re^{1/2}(\nu T)^{1/2}$  (or  $\Gamma T/R^2 \ll 1$ ) the three components of fluctuation vorticity (4.3)–(4.5) are written as

$$\omega'_1 \approx S^2 \xi_0 D_0 \frac{\Gamma T^2}{4\pi R^2} \sin(\varphi_0 - 2\theta), \quad (4.9)$$

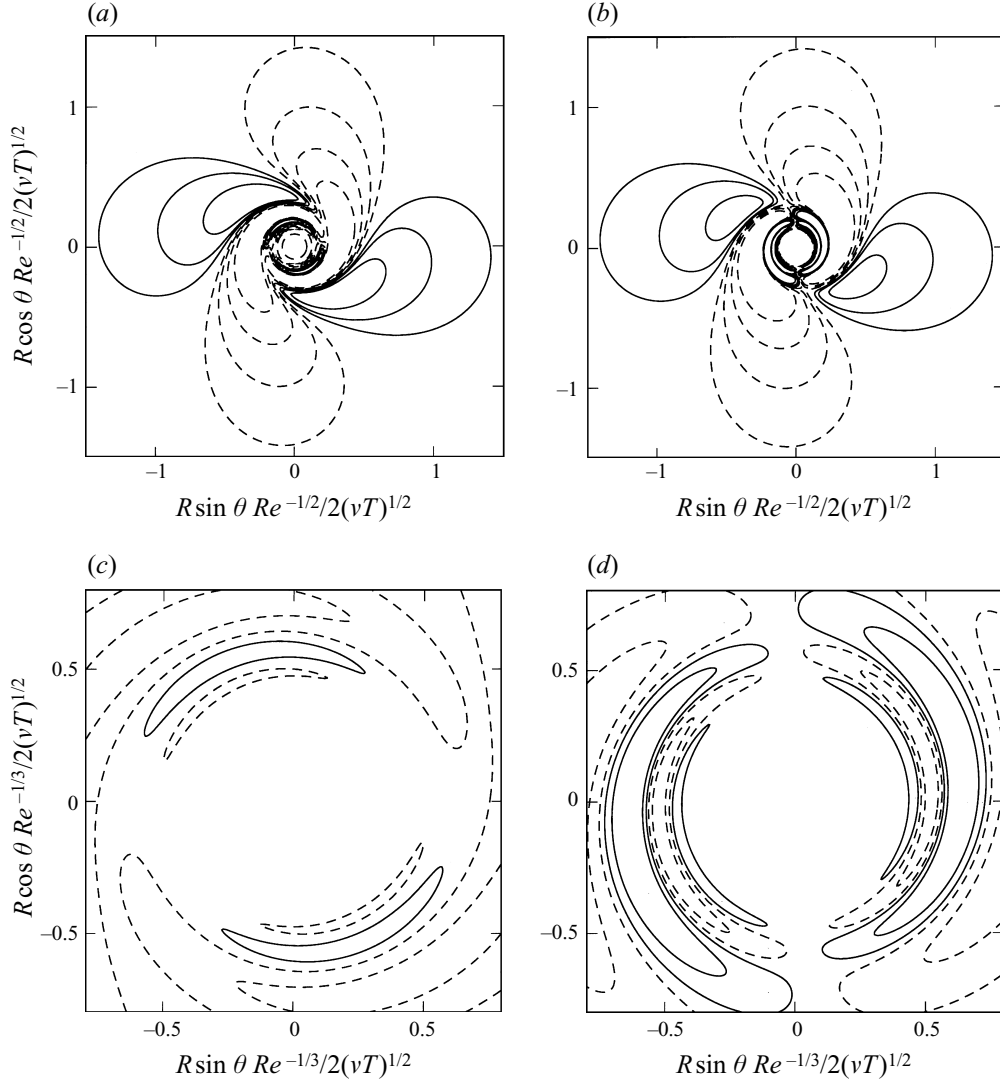


FIGURE 16. Spatial distributions of the fluctuation vorticity field around a neutral vortex tube. (a)  $\omega'_1/(S^2T|\xi_0|D_0)$  at  $R = O(Re^{1/2}(vT)^{1/2})$ , (b)  $-\omega'_2/(2SD_0)$  at  $R = O(Re^{1/2}(vT)^{1/2})$ , (c)  $\omega'_1/(S^2T|\xi_0|D_0)$  at  $R = O(Re^{1/3}(vT)^{1/2})$ , (d)  $-\omega'_2/(2SD_0)$  at  $R = O(Re^{1/3}(vT)^{1/2})$ . The levels of contour lines are (a, b)  $\pm\frac{1}{16}$ ,  $\pm\frac{1}{8}$ ,  $\pm\frac{1}{4}$ ,  $\pm\frac{1}{2}$  and (c, d)  $\pm\frac{1}{2}$ ,  $\pm 2$ . Dashed lines represent negative values.

$$\omega'_2 \approx SD_0 \frac{\Gamma T}{2\pi R^2} \sin(\varphi_0 - 2\theta), \quad (4.10)$$

$$\omega'_3 \approx SD_0 \frac{\Gamma T}{2\pi R^2} \cos(\varphi_0 - 2\theta). \quad (4.11)$$

Equations (4.9) and (4.10) show that the phases of  $\omega'_1$  and  $-\omega'_2$  coincide since  $\xi_0 \leq 0$ . This is because in this region the effect of advection by the swirling flow is negligibly small and thus the time-derivative term of  $\omega'_1$  should be balanced with the production term  $S\xi\omega'_2$  (see (4.26) below). For  $Re^{1/2}(vT)^{1/2} \gg R \gg Re^{1/4}(vT)^{1/2}$  (or  $\Gamma T/R^2 \gg 1$ ),

on the other hand, we have

$$\omega'_1 \approx 2S^2 T \gamma_0 \lambda_0 + 2S^2 T \xi_0 D_0 \cos\left(\frac{\Gamma T}{2\pi R^2} + \varphi_0 - \theta\right) \cos\theta \exp\left[-\frac{8\pi\nu}{3\Gamma} \left(\frac{\Gamma T}{2\pi R^2}\right)^3\right]. \quad (4.12)$$

$$\omega'_2 \approx -SD_0 \frac{\Gamma T}{\pi R^2} \cos\left(\frac{\Gamma T}{2\pi R^2} + \varphi_0 - \theta\right) \sin\theta \exp\left[-\frac{8\pi\nu}{3\Gamma} \left(\frac{\Gamma T}{2\pi R^2}\right)^3\right], \quad (4.13)$$

$$\omega'_3 \approx +SD_0 \frac{\Gamma T}{\pi R^2} \cos\left(\frac{\Gamma T}{2\pi R^2} + \varphi_0 - \theta\right) \cos\theta \exp\left[-\frac{8\pi\nu}{3\Gamma} \left(\frac{\Gamma T}{2\pi R^2}\right)^3\right]. \quad (4.14)$$

By comparing (4.12) and (4.13), we can see that along a spiral of  $(\Gamma T/2\pi R^2) + \varphi_0 - \theta = \text{const.}$  there is the phase advance of an angle of  $\frac{1}{2}\pi$  in  $\omega'_1$  relative to  $-\omega'_2$  (or  $S\xi\omega'_2$ , i.e. the production term). This phase difference is a result of the advection by the axisymmetric swirling motion  $(\Gamma/2\pi R^2)\partial_\theta\omega'_1$  which is effective at  $R \lesssim Re^{1/2}(\nu T)^{1/2}$  (see (4.26) below).

The generation of  $\omega'_2 > 0$  and  $\omega'_1 < 0$  in the crescent-shaped regions of strong vorticity implies that in the stationary coordinate system  $OX_1X_2X_3$  the streamwise component of vorticity, which is equal to  $(\omega'_1 \cos\alpha - \omega'_2 \sin\alpha)$  and is opposite to that of the vortex tube, is generated along the outermost spiral vortex layers. Recently, Sendstad & Moin (1992) and Miyake & Tsujimoto (1996) observed that streamwise vorticity of opposite sign appears around the near-wall streamwise vortex tube and it develops into a new streamwise vortex. The present wrapping and tilting mechanism of vorticity lines by a vortex tube is expected to express the regeneration process of streamwise vortices in near-wall turbulence.

The three-dimensional structure of wrapped vorticity lines around a vortex tube in a simple shear is drawn in figure 17 for (a) cyclonic ( $\alpha_0 = \arctan\sqrt{2}$ ,  $\beta = -\frac{1}{4}\pi$ ), (b) neutral ( $\alpha_0 = \frac{1}{4}\pi$ ,  $\beta = 0$ ) and (c) anticyclonic ( $\alpha_0 = \arctan\sqrt{2}$ ,  $\beta = \frac{1}{4}\pi$ ) cases at  $ST = 1$  and  $Re = 1000$ . They are obtained by numerical integration of the total vorticity given by adding the corresponding components of the simple shear vorticity to (4.3)–(4.5). The vortex tubes are represented by isosurfaces of  $R = (\nu T)^{1/2}$ , and vorticity lines which starts at different points are labelled with letters A–E. Contour lines of  $(\omega'_2 + \omega'_3)^{1/2} = 2S$ , which represent the double spirals of high azimuthal vorticity, are also drawn in the normal  $(x_2, x_3)$ -plane.

In the far region the vortex tube wraps and stretches vorticity lines to intensify the azimuthal component of vorticity on the normal planes so that wrapped vorticity lines tend to be perpendicular to the vortex tube. As vorticity lines become more wrapped, the negative axial vorticity is generated and thus vorticity lines are tilted toward the axial direction. In the near region, on the other hand, the normal vorticity component is rapidly dissipated owing to the viscous cancellation of tightly wrapped vorticity of alternate sign, and eventually only the axial component of the simple shear is left and is stretched. Therefore, vorticity lines near a cyclonic (or anticyclonic) vortex turn in the axial direction and tend to become parallel (or anti-parallel) to the vortex tube.

In the stationary coordinate system  $OX_1X_2X_3$ , vorticity vectors at the outermost spirals of the strong vorticity region line up in the direction of  $X_1 < 0$  and  $X_3 < 0$  both on the upper and lower (the upstream and downstream) sides of a cyclonic vortex (figure 18). This is because, as mentioned above, vorticity lines tend to be perpendicular to the vortex tube and tilted toward the axial direction due to the generation of



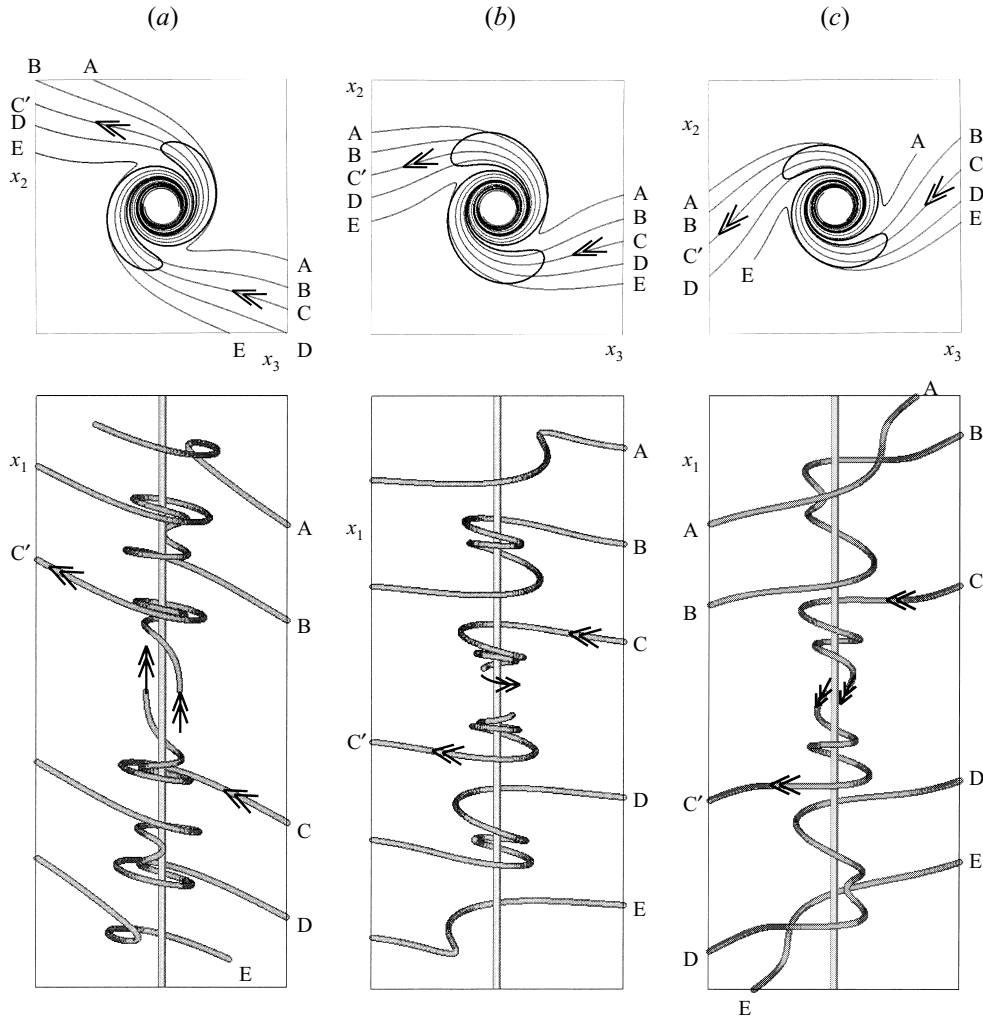


FIGURE 17. Vorticity lines in the structural coordinate system  $Ox_1x_2x_3$  for (a) the cyclonic, (b) the neutral and (c) the anticyclonic cases. A rectangular domain of  $140 \times 60 \times 60$  (in similarity variable  $\eta$ ) is viewed from the negative  $x_1$ - and the positive  $x_2$ -directions in the upper and lower panels, respectively. Thin curves in upper panels represent contour lines of  $(\omega_2^2 + \omega_3^2)^{1/2} = 2S$ .

negative axial vorticity. Therefore, the vorticity vector of the cyclonic vortex tube, which has positive streamwise ( $X_1$ ) and negative spanwise ( $X_3$ ) components, and the resulting vorticity vectors on the upstream and downstream sides of it, have a zig-zag arrangement in the streamwise direction. Recently, Miyake & Tsujimoto (1996) observed that the near-wall cyclonic streamwise vortices actually have this type of arrangement in their regeneration process.

#### 4.4. Spiral vortex layers

The asymptotic analysis at  $St \ll 1$  described in §3 is exact but it is not so simple to understand the physical process involved. In this subsection, we focus on only the region  $(\Gamma/\nu)^{1/4}(\nu t)^{1/2} \ll r \ll (\Gamma/S)^{1/2}$  to discuss intuitively the generation of spiral vortex layers without the short-time assumption that  $St \ll 1$ .

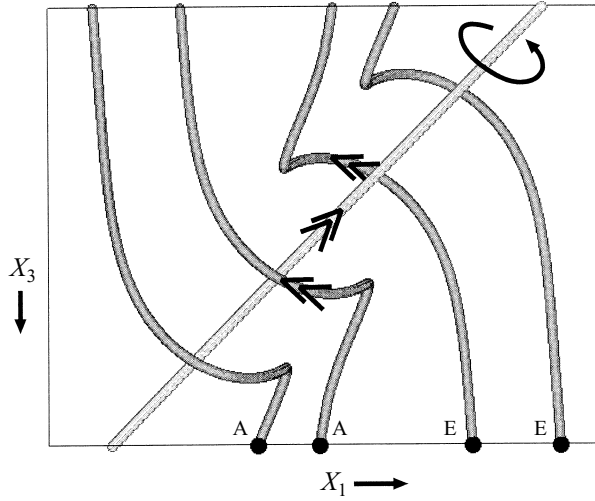


FIGURE 18. Vorticity lines distorted by a cyclonic vortex tube in the stationary coordinate system  $OX_1X_2X_3$ . Lines A and E are the same as the corresponding ones in figure 17. Note that the streamwise ( $X_1$ ) component of vorticity of distorted vorticity lines (which is negative) is opposite to that of the vortex tube (which is positive).

The flow structure in the far region  $r \gg (\Gamma/\nu)^{1/4}(\nu t)^{1/2}$  may be analysed more simply by replacing the vortex tube with a filament. Separating the contribution from the vortex filament in the streamfunction as

$$\psi = -\frac{\Gamma}{2\pi} \ln r + \psi', \quad (4.15)$$

and substituting it into (2.23) and (2.24), we obtain

$$\begin{aligned} \partial_t \omega'_1 - \frac{\partial(\psi', \omega'_1)}{\partial(x_2, x_3)} + \frac{\Gamma}{2\pi r^2} \partial_\theta \omega'_1 - S(\gamma(t)x_2 + \lambda(t)x_3) \partial_2 \omega'_1 \\ = S\gamma(t)\omega'_1 + S\xi(t)\partial_3 u'_1 + \nu \nabla_\perp^2 \omega'_1 \end{aligned} \quad (4.16)$$

and

$$\begin{aligned} \partial_t u'_1 - \frac{\partial(\psi', u'_1)}{\partial(x_2, x_3)} + \frac{\Gamma}{2\pi r^2} \partial_\theta u'_1 - S(\gamma(t)x_2 + \lambda(t)x_3) \partial_2 u'_1 \\ = -S\gamma(t)u'_1 + \frac{\Gamma S}{2\pi r^2} (\cos \alpha \sin \beta x_2 + \cos \beta x_3) - S(\cos \alpha \sin \beta \partial_2 + \cos \beta \partial_3) \psi' + \nu \nabla_\perp^2 u'_1. \end{aligned} \quad (4.17)$$

These equations are supplemented by initial conditions,  $\omega'_1 = 0$  and  $u'_1 = 0$ .

We first consider the behaviour of the velocity field deformed by a strong vortex filament, and suppose that

$$\frac{\Gamma}{\nu} \gg 1. \quad (4.18)$$

Within  $r \ll (\Gamma/S)^{1/2}$ , the velocity induced by the filament dominates the simple shear velocity. It is the first and third terms on the left-hand side and the second term on the right-hand side that are dominant in (4.17) except for the viscous term which will

be taken into account below. Then we have

$$\partial_t u'_1 + \frac{\Gamma}{2\pi r^2} \partial_\theta u'_1 = \frac{\Gamma S}{2\pi r^2} (\cos \alpha \sin \beta x_2 + \cos \beta x_3). \quad (4.19)$$

This has a solution

$$u'_1 = SD_0 r \sin \left( \frac{\Gamma t}{2\pi r^2} + \varphi_0 - \theta \right) - SD(t)r \sin(\varphi(t) - \theta), \quad (4.20)$$

which represents spiral vortex layers, where  $D_0$  and  $\varphi_0$ , and  $D(t)$  and  $\varphi(t)$  are given by (3.4) and (2.38), respectively.

In order to take account of the effect of viscous diffusion, we introduce unknown functions  $s_1(r, t)$  and  $s_2(r, t)$  as

$$u'_1 = s_1(r, t) \cos \left( \theta - \frac{\Gamma t}{2\pi r^2} \right) + s_2(r, t) \sin \left( \theta - \frac{\Gamma t}{2\pi r^2} \right) - SD(t)r \sin(\varphi(t) - \theta). \quad (4.21)$$

In the region  $r \gg (\Gamma/\nu)^{1/4}(\nu t)^{1/2}$  (which can exist if  $S t \ll (\Gamma/\nu)^{1/2}$ ) the viscous term may be approximated by

$$\nu \nabla_\perp^2 u'_1 \approx -4\nu \left( \frac{\Gamma t}{2\pi r^3} \right)^2 \left[ s_1 \cos \left( \theta - \frac{\Gamma t}{2\pi r^2} \right) + s_2 \sin \left( \theta - \frac{\Gamma t}{2\pi r^2} \right) \right]. \quad (4.22)$$

Substitution of (4.21) and (4.22) into (4.19) plus the viscous term gives a set of differential equations for  $s_1$  and  $s_2$ , which are solved easily under initial condition  $u'_1 = 0$ , i.e.  $s_1|_{t=0} = Sr \cos \beta$ ,  $s_2|_{t=0} = -Sr \cos \alpha \sin \beta$  to yield

$$u'_1 = SD_0 r \sin \left( \frac{\Gamma t}{2\pi r^2} + \varphi_0 - \theta \right) \exp \left[ -\frac{8\pi\nu}{3\Gamma} \left( \frac{\Gamma t}{2\pi r^2} \right)^3 \right] - SD(t)r \sin(\varphi(t) - \theta). \quad (4.23)$$

This is a viscous version of (4.20).

The  $x_2$ - and  $x_3$ -components of the fluctuation vorticity are then calculated to be

$$\begin{aligned} \omega'_2 = \partial_3 u'_1 &= S \cos \alpha \sin \beta \\ &- SD_0 \left[ \cos \left( \frac{\Gamma t}{2\pi r^2} + \varphi_0 \right) + \frac{\Gamma t}{\pi r^2} \cos \left( \frac{\Gamma t}{2\pi r^2} + \varphi_0 - \theta \right) \sin \theta \right] \exp \left[ -\frac{8\pi\nu}{3\Gamma} \left( \frac{\Gamma t}{2\pi r^2} \right)^3 \right] \end{aligned} \quad (4.24)$$

and

$$\begin{aligned} \omega'_3 = -\partial_2 u'_1 &= S \cos \beta \\ &- SD_0 \left[ \sin \left( \frac{\Gamma t}{2\pi r^2} + \varphi_0 \right) - \frac{\Gamma t}{\pi r^2} \cos \left( \frac{\Gamma t}{2\pi r^2} + \varphi_0 - \theta \right) \cos \theta \right] \exp \left[ -\frac{8\pi\nu}{3\Gamma} \left( \frac{\Gamma t}{2\pi r^2} \right)^3 \right], \end{aligned} \quad (4.25)$$

which agree with (4.4) and (4.5) for  $S t \ll 1$ . The  $x_2$ -component of the fluctuation vorticity (4.24) appears in axial-vorticity equation (4.16) as a source term.

In the near-field  $r \ll (\Gamma/S)^{1/2}$  and during the period  $S t \ll (\Gamma/\nu)^{1/2}$  the dominant terms in (4.16) are the first and third terms on the left-hand side and the second and third terms on the right-hand side. Then we have

$$\partial_t \omega'_1 + \frac{\Gamma}{2\pi r^2} \partial_\theta \omega'_1 = S \xi(t) \partial_3 u'_1 + \nu \nabla_\perp^2 \omega'_1. \quad (4.26)$$

A solution to this equation under initial condition  $\omega'_1|_{t=0} = 0$  is

$$\begin{aligned} \omega'_1 = & S(A(t)\lambda_0 - \lambda(t)) + \left[1 + O\left(\frac{v}{\Gamma}\right)\right] S^2 \xi(t) \frac{4\pi r^2}{\Gamma} \cos \beta \\ & + S^2 \xi(t) D_0 \left[2t \cos\left(\frac{\Gamma t}{2\pi r^2} + \varphi_0 - \theta\right) \cos \theta - \frac{4\pi r^2}{\Gamma} \sin\left(\frac{\Gamma t}{2\pi r^2} + \varphi_0\right)\right. \\ & \left. + \frac{2\pi r^2}{\Gamma} \sin\left(\frac{\Gamma t}{2\pi r^2} + \varphi_0 - 2\theta\right)\right] \exp\left[-\frac{8\pi v}{3\Gamma} \left(\frac{\Gamma t}{2\pi r^2}\right)^3\right] \\ & - S^2 \xi(t) D_0 \frac{2\pi r^2}{\Gamma} \sin\left(\frac{\Gamma t}{\pi r^2} + \varphi_0 - 2\theta\right) \exp\left[-\frac{32\pi v}{3\Gamma} \left(\frac{\Gamma t}{2\pi r^2}\right)^3\right], \end{aligned} \quad (4.27)$$

which agrees with (4.3) for  $St \ll 1$ . If we restrict ourselves in a period  $1 \lesssim St \ll (\Gamma/v)^{1/2}$  (and so the region  $(\Gamma/v)^{1/4}(vt)^{1/2} \ll r \ll (\Gamma/v)^{1/2}(vt)^{1/2}$ ), four terms containing  $O(S^2 r^2/\Gamma)$  may be neglected in (4.27). Incidentally, a spiral vortex solution which represents the wrapping of the axial component of vorticity by an axisymmetric swirling flow was obtained by Lundgren (1982). Our solution (4.27), on the other hand, describes the conversion of normal vorticity to the axial direction by shearing motion induced by wrapped spiral vortex layers of high azimuthal vorticity. Thus, these two are essentially different from each other.

It should be emphasized that solutions (4.24), (4.25) and (4.27) have been obtained for  $(\Gamma/v)^{1/4}(vt)^{1/2} \ll r \ll (\Gamma/S)^{1/2}$  without the short-time assumption that  $St \ll 1$ . A similarity in the functional forms between asymptotic solutions (4.3)–(4.5) and (4.24), (4.25) and (4.27) suggests that the former may be approximated well even for  $St \approx 1$ .

The position of the maximum normal vorticity and spacing of the spirals are intuitively estimated as follows. The spirals of high azimuthal vorticity are primarily generated by a strong swirling potential velocity induced by the vortex tube in the form (see the first term of (4.20))

$$\theta - \frac{\Gamma t}{2\pi r^2} = \text{const.} \quad (4.28)$$

The spacing  $\Delta r$  of the spiral is given by the change of  $r$  when angle  $\theta$  changes by  $2\pi$ , so that it follows from (4.28) that

$$\frac{\Gamma t}{r^3} \Delta r \approx 1. \quad (4.29)$$

The region in which viscous effects play a role is estimated by equating the viscous and the time-derivative terms

$$\frac{1}{t} \approx v \left(\frac{\Gamma t}{r^3}\right)^2, \quad (4.30)$$

or

$$r \approx \left(\frac{\Gamma}{v}\right)^{1/3} (vt)^{1/2}. \quad (4.31)$$

Combination of (4.29) and (4.31) yields

$$\Delta r \approx (vt)^{1/2}. \quad (4.32)$$

Finally, we discuss intuitively the order of magnitude of the axial vorticity generated through the tilting of the  $x_3$ -component of the simple shear vorticity by the spiral vortex layers. The gradient of axial velocity across the spirals may be estimated, by

using (4.29), as

$$\frac{u'_1}{\Delta r} \approx S \frac{\Gamma t}{r^2}, \quad (4.33)$$

which works to tilt the simple shear vorticity toward the axial direction. In the region  $(\Gamma/\nu)^{1/2}(\nu t)^{1/2} \gg r \gg (\Gamma/\nu)^{1/4}(\nu t)^{1/2}$  the time scale of the time-derivative term is much smaller than that of the advection term because of the swirling flow,

$$\frac{1}{t} \ll \frac{\Gamma}{r^2}, \quad (4.34)$$

so that the former may dominate the latter and balance the production term in (4.26). Thus we have

$$\frac{\Gamma}{r^2} \omega'_1 \approx S^2 \frac{\Gamma t}{r^2}, \quad (4.35)$$

that is, the axial vorticity increases in time as

$$\omega'_1 \approx S^2 t. \quad (4.36)$$

In the region  $r \approx (\Gamma/\nu)^{1/2}(\nu t)^{1/2}$  (which can exist if  $S t \ll 1$ ), on the other hand, the two time scales are comparable,

$$\frac{1}{t} \approx \frac{\Gamma}{r^2}, \quad (4.37)$$

and we have

$$\omega'_1 \approx S^2 t \approx S^2 \frac{\Gamma t^2}{r^2} \approx S^2 \frac{r^2}{\Gamma}, \quad (4.38)$$

which increases with radial distance. Therefore, the magnitude of  $\omega'_1$  can be strongest near the outermost spirals at least during the initial time  $S t \ll 1$  of evolution because there is no spiral and so  $\omega'_1$  decays at  $r \gtrsim (\Gamma/\nu)^{1/2}(\nu t)^{1/2}$ .

## 5. Concluding remarks

Asymptotic solutions describing the interactions between a straight vortex tube of circulation  $\Gamma$  starting with a vortex filament and a simple shear flow of shear rate  $S$  have been obtained at a large Reynolds number  $\Gamma/\nu \gg 1$  and at an early time  $S t \ll 1$  of evolution as well as  $S t \ll (\Gamma/\nu)^{1/2}$  for  $(\Gamma/\nu)^{1/4}(\nu t)^{1/2} \ll r \ll (\Gamma/S)^{1/2}$  (see §4.4). These solutions are expected to be useful for understanding interactions of tube-like structures with background vorticity in real turbulent flows.

Let us now summarize the global structure of the present vorticity field in terms of the original physical coordinate  $r$  and time  $t$ . A thin strong straight vortex tube which starts with a filament is diffused by viscosity and the core size increases in time as  $(\nu t)^{1/2}$  (see figure 19). This vortex tube induces an axisymmetric swirling flow around it, which dominates the simple shear velocity in the near region where  $r \ll (\Gamma/S)^{1/2}$ . The simple shear vorticity (denoted by double arrows) is stretched and wrapped by this swirling flow to make double spirals around the vortex tube. The stretching (in planes normal to the vortex tube) of the vorticity lines is more effective and the intensity of the vorticity is enhanced more near the vortex tube because the swirling motion is more rapid there. Therefore the magnitude of the azimuthal components of the vorticity increases as it approaches the vortex tube. The rapid swirling motion around the vortex tube also makes the spirals wind so tightly that viscous effects may become important there. Because the azimuthal vorticity changes direction alternately on neighbouring layers of spirals, viscous diffusion smooths out

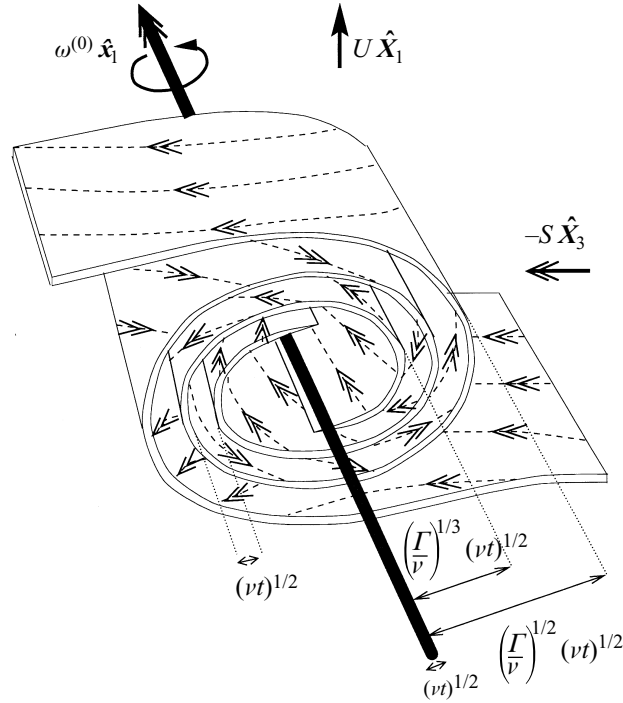


FIGURE 19. An illustration of the structure of the vorticity field around a straight vortex tube in a simple shear. Double arrows on the double helical layers denote the direction of vorticity.

the vorticity in the region where  $r \ll (\Gamma/v)^{1/3}(vt)^{1/2}$ . This expulsion of normal vorticity around the vortex tube leads to a stretch and intensification of the axial component of vorticity. This partial cancellation of vorticity by viscous diffusion has a counter effect against enhancement of normal vorticity due to stretching by the swirling motion mentioned above. As a result, the magnitude of the normal vorticity has a peak at  $r = O((\Gamma/v)^{1/3}(vt)^{1/2})$ , where the distance between neighbouring layers of spirals is of  $O((vt)^{1/2})$  (see (4.32)). The spirals of high azimuthal vorticity induce axial shear velocity which changes direction alternately in each successive spiral layer. This axial shear velocity works to tilt (the  $x_3$ -component of) the simple shear vorticity toward the axial direction. It is remarkable that axial vorticity opposed to that of the vortex tube is being induced in the outermost layers of the spirals where the axial vorticity is strongest.

The structure of energy dissipation around a vortex tube may be worth commenting on. The spatial distribution of the high energy dissipation rate on a normal plane is composed of two main contributions. One originates from the swirling flow induced by the vortex tube itself. The energy dissipation rate per unit mass takes large values of  $O(v(\Gamma/vt)^2)$  in a circular ring region of both radius and thickness of  $O((vt)^{1/2})$  centred at the vortex axis. The other contribution comes from two spiral regions associated with the wrapped spiral vortex layers of high azimuthal vorticity. The energy dissipation rate takes values of  $O(vS^2(\Gamma/v)^{2/3})$  over a ring region of both radius and thickness of  $O((\Gamma/v)^{1/3}(vt)^{1/2})$ . Hence, the dissipation rates per unit axial length are estimated to be  $O(\Gamma^2/t)$  and  $O(v^{2/3}S^2\Gamma^{4/3}t)$  for the vortex tube and the spirals, respectively. The former dominates the latter during the initial time  $St \ll 1$  of evolution. They would become comparable at  $St \approx (\Gamma/v)^{1/3}$ .

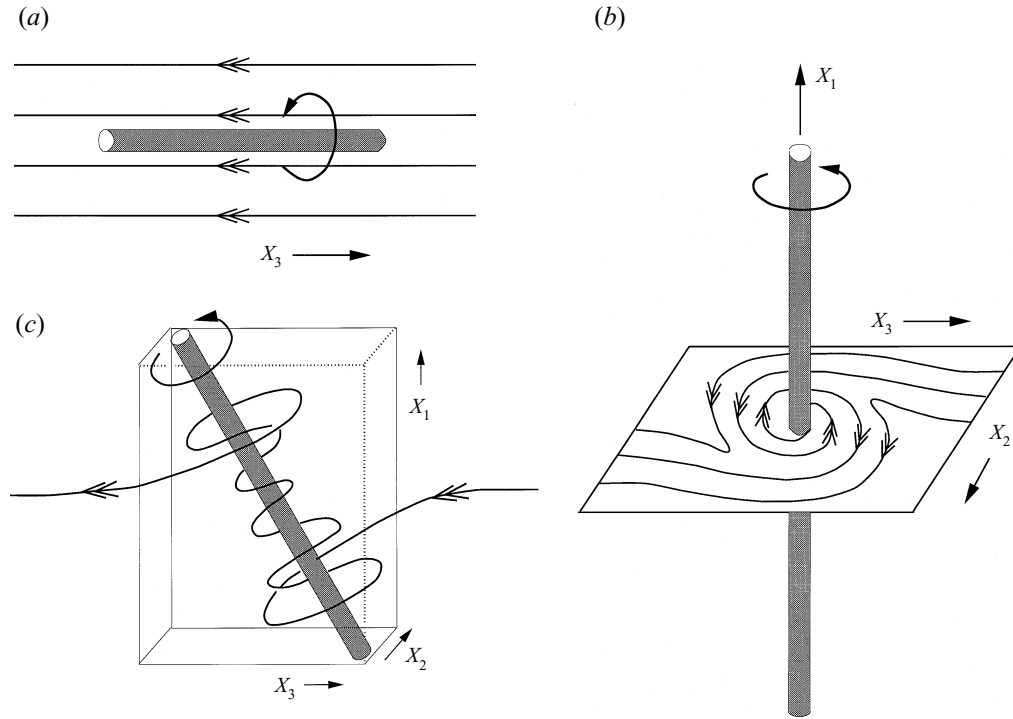


FIGURE 20. Wrap, tilt and stretch of vorticity lines around a straight vortex tube in a simple shear flow. (a) Vorticity lines are neither wrapped, tilted nor stretched around a *spanwise* vortex tube; (b) they are wrapped on normal planes but are neither tilted nor stretched toward the axial direction around a *streamwise* one; (c) they are wrapped, tilted and stretched around an *oblique* one.

Applications of this analysis to real flows are restricted by the assumption of uniformity of the fluctuation flow field along a vortex tube and by the initial condition (3.1). However, tube-like structures of concentrated intense vorticity in many turbulent flows are likely to be elongated rather straight so that the essential process of their vorticity interactions may be approximately described in terms of the present asymptotic solutions at  $r \gg (vt)^{1/2}$ . In the following we discuss a few aspects of the flow structure of the present asymptotic solutions relevant to real turbulent flows.

First, it has been often observed that vorticity vectors align quite well with tube-like structures in the core region in spite of the presence of background vorticity in isotropic turbulence, homogeneous shear turbulence and near-wall turbulence (see She *et al.* 1990; Kida & Tanaka 1994; Bernard *et al.* 1993). The expulsion of vorticity normal to a vortex tube (see §3.4) may explain this alignment of tube-like structures and vorticity vectors therein. Secondly, helically winding vorticity lines similar to those in figure 17 were observed in She *et al.* (1990). In their numerical analysis of homogeneous isotropic turbulence, they showed that vorticity lines are wrapped around the vortex tube forming a spiral. The helical wrapping of vorticity lines was also observed around streamwise vortex tubes in near-wall turbulence by Jiménez & Moin (1991). Thirdly, cyclonic vortex tubes, which are tilted toward the direction of the mean shear vorticity, have often been observed in both homogeneous shear turbulence and near-wall turbulence (see Kida & Tanaka 1994; Miyake & Tsujimoto 1996). The selective intensification of a cyclonic vortex tube (see §4.2) is conjectured to explain the dominance of this type of vortical structure in real shear flows. Fourthly,

it has also been observed in near-wall turbulence that streamwise vorticity of the opposite sign is generated around a streamwise vortex tube and develops into a new streamwise vortex (Sendstad & Moin 1992; Miyake & Tsujimoto 1996) and that the streamwise vortices have a zig-zag arrangement in the streamwise direction in their regeneration process (Miyake & Tsujimoto 1996). In our analytical solution, as shown in figure 18, the streamwise component of the vorticity vectors in the most-intensified part have an opposite sign to that of the vortex tube, that is, they are oriented in the same manner as the above zig-zag arrangement. All of these similarities between our analytical solution and the observations suggest that the wrapping, tilting and stretching mechanisms of vorticity lines associated with an oblique vortex tube may play a key role in the dynamics of vortex tubes in various kinds of turbulence.

Finally, we would like to stress that a vortex tube in a simple shear flow should be inclined away from the streamwise direction both vertically and horizontally in order that all the three fundamental processes, i.e. wrapping, tilting and stretching, may take part in the vortex dynamics. In figure 20 are illustrated typical vorticity lines around (a) a spanwise, (b) a streamwise and (c) an oblique vortex tube in a simple shear flow. If a vortex tube is aligned with the spanwise direction ( $\alpha = \frac{1}{2}\pi$ ,  $\beta = \pm\frac{1}{2}\pi$ ), it is parallel to the background vorticity lines and thus they are neither wrapped, tilted nor stretched. If a vortex tube is aligned with the streamwise direction ( $\alpha = 0$ ), it is stationary and perpendicular to background vorticity lines, so that vorticity lines are wrapped on planes normal to the tube but they are neither tilted nor stretched to the axial direction. Only in the case of an oblique vortex tube inclined away from the streamwise direction both vertically and horizontally, are vorticity lines wrapped around the vortex tube, and tilted and stretched in the axial direction so that the simple shear vorticity may be converted to the axial direction as well as the streamwise direction (see §§4.2 and 4.3). When a vortex tube is inclined only horizontally ( $\alpha \neq \frac{1}{2}\pi$ ,  $\beta = \pm\frac{1}{2}\pi$ ), the simple shear vorticity can be converted to the streamwise direction via the wrapping on the normal plane but not to the axial direction (see (2.23)). When a tube is not inclined to the spanwise direction ( $\alpha \neq 0$ ,  $\beta = 0$ ), the stretch and intensification of the axial component of the simple shear vorticity (see §4.2) is not observed.

Obviously, it is interesting and important to investigate whether the present asymptotic two-dimensional straight vortex tube is structurally stable or not. This problem is left for a future study.

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### Appendix A. Solution to (3.86)

Since the leading-order solutions,  $\omega^{(0,0)}$  and  $\psi^{(0)}$ , are independent of  $\theta$ , (3.86) can be rewritten, using (3.78), as

$$\begin{aligned} & \frac{1}{4\eta} \partial_\theta \left[ \frac{1}{4} (\partial_\eta \psi^{(0)}) \nabla_\eta^2 \psi^{(1,1)} + (\partial_\eta \omega^{(0,0)}) \psi^{(1,1)} \right] \\ & = -\epsilon \frac{1}{8} \eta \partial_\eta \nabla_\eta^2 \psi^{(1,1)} - \epsilon \frac{1}{16} \nabla_\eta^4 \psi^{(1,1)} + \epsilon \frac{1}{2} B_0 \eta (\partial_\eta \omega^{(0,0)}) \operatorname{Re} [e^{i(2\theta - \phi_0)}], \end{aligned} \quad (\text{A } 1)$$

where  $B_0$  and  $\phi_0$  are given by (3.94). We then seek a solution to (A 1) in the form

$$\psi^{(1,1)} = \operatorname{Re} [F_1(\eta) e^{i(2\theta - \phi_0)}]. \quad (\text{A } 2)$$



Substituting (A 2) into (A 1), and using (3.24) and (3.88), we obtain

$$\epsilon \left[ F_1^{\text{iv}} + 2 \left( \eta + \frac{1}{\eta} \right) F_1''' + \left( 2 - \frac{9}{\eta^2} \right) F_1'' - \frac{1}{\eta} \left( 10 - \frac{9}{\eta^2} \right) F_1' + \frac{16}{\eta^2} F_1 \right] - \frac{i}{\pi} \left[ \frac{1 - e^{-\eta^2}}{\eta^2} \left( F_1'' + \frac{1}{\eta} F_1' - \frac{4}{\eta^2} F_1 \right) + 4e^{-\eta^2} F_1 \right] = -\epsilon \frac{4}{\pi} B_0 \eta^2 e^{-\eta^2}, \quad (\text{A } 3)$$

where the right-hand side originates from the deformation of the spatial distribution of the axial vorticity in the  $(x_2, x_3)$ -plane by the simple shear (the first term in (3.7)). Boundary conditions to be imposed are that  $TF_1(\eta)$  is regular at  $\eta = 0$  and  $TF_1(\eta) \rightarrow 0$  as  $\eta \rightarrow \infty$  (see (3.18)). For  $\epsilon \ll 1$ , we expand a solution to (A 3) in a power series of  $\epsilon$  as

$$F_1 = F_{10} + \epsilon F_{11} + \dots \quad (\text{A } 4)$$

Substituting (A 4) into (A 3), and equating  $\epsilon^0$ -order terms, we find

$$F_{10}'' + \frac{1}{\eta} F_{10}' - \frac{4}{\eta^2} F_{10} + \frac{4\eta^2}{e^{\eta^2} - 1} F_{10} = 0. \quad (\text{A } 5)$$

This has the solution regular at  $\eta = 0$

$$F_{10} \approx a_2 \eta^2 \quad \text{as } \eta \rightarrow 0, \quad (\text{A } 6)$$

where  $a_2$  is a constant. However, the asymptotic behaviour at large  $\eta$  of this solution can be numerically shown to be

$$F_{10} \approx b_2 \eta^2 \quad \text{as } \eta \rightarrow \infty, \quad (\text{A } 7)$$

where  $b_2$  is a zero or non-zero constant according to whether  $a_2$  is zero or non-zero. Equation (A 7) implies that  $TF_{10} \approx b_2 R^2 / 4$  as  $\eta \rightarrow \infty$ , which is inconsistent with the boundary condition unless  $b_2 = 0$ . Hence, we conclude that

$$F_{10}(\eta) \equiv 0. \quad (\text{A } 8)$$

At order  $\epsilon$ , (A 3) is written as

$$f_1'' + \frac{1}{\eta} f_1' - \frac{4}{\eta^2} f_1 = \frac{4\eta^2}{e^{\eta^2} - 1} (\eta^2 - f_1) \quad (\equiv -4M_0(\eta), \quad \text{say}), \quad (\text{A } 9)$$

where

$$\text{Re} [F_{11}(\eta)] \equiv 0, \quad \text{Im} [F_{11}(\eta)] = -B_0 f_1(\eta) \quad (\text{A } 10)$$

and (A 8) has been used. Recall that (A 9) is identical to the first-order equation obtained by Moffatt *et al.* (1994) in their analysis of a large-Reynolds-number asymptotic solution for a non-axisymmetric Burgers vortex tube. For small  $\eta$  a solution regular at  $\eta = 0$  is expanded as

$$f_1(\eta) = a\eta^2 + \frac{1}{3}(1-a) \left( \eta^4 - \frac{5}{16}\eta^6 + \dots \right), \quad (\text{A } 11)$$

where  $a$  is a constant. For  $\eta \gg 1$ , it decays as

$$f_1(\eta) = \frac{b}{\eta^2} + O(e^{-\eta^2}). \quad (\text{A } 12)$$

Constants  $a$  and  $b$  were numerically determined to be

$$a = -1.5259\dots, \quad b = -4.3680\dots \quad (\text{A } 13)$$

by Moffatt *et al.* (1994). By using (3.78), (A 2) and (A 9), we obtain (3.92) in §3.5 up to order  $\epsilon$ .

### Appendix B. Solution to (3.89)

Substitution of (3.25) and (3.78) into (3.89) gives

$$\begin{aligned} \frac{1}{4\eta} \partial_\theta \left[ \frac{1}{4} (\partial_\eta \psi^{(0)}) \nabla_\eta^2 \psi^{(1,2)} + (\partial_\eta \omega^{(0,0)}) \psi^{(1,2)} \right] \\ = \epsilon \frac{1}{4} \left( 1 - \frac{1}{2} \eta \partial_\eta \right) \nabla_\eta^2 \psi^{(1,2)} - \epsilon \frac{1}{16} \nabla_\eta^4 \psi^{(1,2)} + \epsilon \frac{1}{2} B_1 \eta (\partial_\eta \omega^{(0,0)}) \operatorname{Re} [e^{-i(2\theta - \phi_1)}] \\ - \epsilon^2 \frac{1}{2} \xi_0 \eta \operatorname{Re} [f' e^{-i2\theta}] + \epsilon^2 \frac{1}{2} \xi_0 \operatorname{Re} [2f + \eta f'] + \epsilon^2 2\gamma_0 \lambda_0, \end{aligned} \quad (\text{B } 1)$$

where  $B_1$  and  $\phi_1$  are given by (3.100). Suggested by the source terms in (B 1), we seek a solution in the form

$$\psi^{(1,2)} = G_2(\eta) + \operatorname{Re} [F_2(\eta) e^{-i2\theta}]. \quad (\text{B } 2)$$

Substituting (B 2) into (B 1), and using (3.24) and (3.88), we obtain, for the  $\theta$ -independent part,

$$M_1'' + \left( 2\eta + \frac{1}{\eta} \right) M_1' - 4M_1 = -2\xi_0 \operatorname{Re} [2f + \eta f'] - 8\gamma_0 \lambda_0, \quad (\text{B } 3)$$

where

$$\epsilon M_1(\eta) = -\frac{1}{4} \nabla_\eta^2 G_2 = -\frac{1}{4} \left( G_2'' + \frac{1}{\eta} G_2' \right) \quad (\text{B } 4)$$

is the  $\theta$ -independent part of the vorticity associated with streamfunction  $G_2$ . The right-hand side of (B 3) originates from the production of the axial vorticity through the tilting of the  $x_2$ -component of the fluctuation vorticity by the simple shear, the tilting of the  $x_3$ -component of the simple shear vorticity via the velocity fluctuation, and the effect of frame rotation. On the other hand, for the  $\theta$ -dependent part, we obtain

$$\begin{aligned} \epsilon \left[ F_2^{\text{iv}} + 2 \left( \eta + \frac{1}{\eta} \right) F_2''' - \left( 2 + \frac{9}{\eta^2} \right) F_2'' - \frac{1}{\eta} \left( 14 - \frac{9}{\eta^2} \right) F_2' + \frac{32}{\eta^2} F_2 \right] \\ + \frac{i}{\pi} \left[ \frac{1 - e^{-\eta^2}}{\eta^2} \left( F_2'' + \frac{1}{\eta} F_2' - \frac{4}{\eta^2} F_2 \right) + 4e^{-\eta^2} F_2 \right] = -\epsilon \frac{4}{\pi} B_1 e^{-i\phi_1} \eta^2 e^{-\eta^2} - \epsilon^2 8\xi_0 \eta f'. \end{aligned} \quad (\text{B } 5)$$

There exist two different types of source terms on the right-hand side of (B 5). They come from the deformation and the production of the axial vorticity by the simple shear, respectively.

#### B.1. $\theta$ -independent part

We consider here a solution to (B 3). Boundary conditions to be imposed are that  $TM_1(\eta)$  is regular at  $\eta = 0$  and that  $TM_1(\eta) \rightarrow 0$  as  $\eta \rightarrow \infty$  (see (3.17)). The asymptotic expansion of the solution to (B 3) for small and large values of  $\eta$  may be obtained using (3.28) and (3.27). For  $\eta \ll Re^{-1/2}$ , we have

$$M_1(\eta) = 2\gamma_0 \lambda_0 + m_0 + (m_0 - \xi_0 \operatorname{Re}(c_0)) \eta^2 - \frac{\operatorname{Re} \xi_0 \operatorname{Im}(c_0)}{16} \eta^4 + \dots \quad (\text{B } 6)$$

with some constant  $m_0$ , where  $c_0$  is an exponentially small constant given by (3.57). For  $\eta \gg Re^{1/2}$ , using (3.4), we find

$$M_1(\eta) = -\xi_0 \left( \frac{Re^2 \cos \alpha_0 \sin \beta}{96\eta^4} + \frac{18Re^2 \cos \alpha_0 \sin \beta - Re^3 \cos \beta}{768\eta^6} + \dots \right). \quad (\text{B } 7)$$

The homogeneous equation associated with (B 3) has two linearly independent solutions

$$M_{1a} = 1 + \eta^2, \quad M_{1b} = (1 + \eta^2) \int_{\eta}^{\infty} \frac{e^{-s^2}}{s(1+s^2)^2} ds. \quad (\text{B } 8)$$

It can be shown that

$$\begin{aligned} & \int_{\eta}^{\infty} \frac{e^{-s^2}}{s(1+s^2)^2} ds \\ &= -\ln \eta + \int_1^{\infty} \frac{e^{-s^2}}{s(1+s^2)^2} ds + \int_0^1 \left[ \frac{e^{-s^2}}{s(1+s^2)^2} - \frac{1}{s} \right] ds + O(\eta^2) \quad (\text{for } \eta \ll 1), \end{aligned} \quad (\text{B } 9)$$

and that

$$\int_{\eta}^{\infty} \frac{e^{-s^2}}{s(1+s^2)^2} ds = \frac{e^{-\eta^2}}{2\eta^6} \left[ 1 - \frac{5}{\eta^2} + O\left(\frac{1}{\eta^4}\right) \right] \quad (\text{for } \eta \gg 1). \quad (\text{B } 10)$$

By making use of (B 8), we can write a solution to inhomogeneous equation (B 3) as

$$M_1 = 2\gamma_0\lambda_0 + (c_{1a} + \widetilde{M}_{1a})M_{1a} + (c_{1b} + \widetilde{M}_{1b})M_{1b}, \quad (\text{B } 11)$$

where

$$\widetilde{M}_{1a}(\eta) = -2\xi_0 \int_0^{\eta} s e^{s^2} \operatorname{Re} [2f(s) + sf'(s)] M_{1b}(s) ds, \quad (\text{B } 12)$$

$$\widetilde{M}_{1b}(\eta) = 2\xi_0 \int_0^{\eta} s e^{s^2} \operatorname{Re} [2f(s) + sf'(s)] M_{1a}(s) ds. \quad (\text{B } 13)$$

Constants  $c_{1a}$  and  $c_{1b}$  are determined by invoking boundary conditions at  $\eta = 0$  and  $\infty$ . For  $\eta \ll 1$ , since  $f(\eta)$  and therefore the first term on the right-hand side of (B 3) are very small, we obtain, using (B 9) and (B 11), that

$$\begin{aligned} M_1 &\approx 2\gamma_0\lambda_0 + c_{1a}M_{1a} + c_{1b}M_{1b} \\ &\approx 2\gamma_0\lambda_0 + c_{1a} + c_{1b} \left\{ -\ln \eta + \int_1^{\infty} \frac{e^{-s^2}}{s(1+s^2)^2} ds + \int_0^1 \left[ \frac{e^{-s^2}}{s(1+s^2)^2} - \frac{1}{s} \right] ds \right\}. \end{aligned} \quad (\text{B } 14)$$

Thus, the regularity of  $M_1$  at  $\eta = 0$  requires

$$c_{1b} = 0. \quad (\text{B } 15)$$

By comparing (B 14) with (B 6), we find

$$m_0 = c_{1a}. \quad (\text{B } 16)$$

Since  $f$  is very small up to  $\eta = O(Re^{1/4})$  (see §3.4),  $\widetilde{M}_{1a}$  and  $\widetilde{M}_{1b}$  may be evaluated to be

$$\begin{aligned} & \widetilde{M}_{1a}(\eta) \\ &\approx +2\xi_0 D_0 \int_0^{\eta} \frac{1}{s^3} \left( 1 - \frac{4}{s^2} \right) \left[ \cos\left(\frac{Re}{4s^2} + \varphi_0\right) + \frac{Re}{4s^2} \sin\left(\frac{Re}{4s^2} + \varphi_0\right) \right] \exp\left(-\frac{Re^2}{48s^6}\right) ds, \end{aligned} \quad (\text{B } 17)$$

$$\begin{aligned} & \widetilde{M}_{1b}(\eta) \\ & \approx -4\xi_0 D_0 \int_0^\eta s^3 \left(1 + \frac{1}{s^2}\right) \left[ \cos\left(\frac{Re}{4s^2} + \varphi_0\right) + \frac{Re}{4s^2} \sin\left(\frac{Re}{4s^2} + \varphi_0\right) \right] \exp\left(s^2 - \frac{Re^2}{48s^6}\right) ds \quad (\text{B 18}) \end{aligned}$$

by using (3.63), (3.64), (B 8) and (B 10). After elementary calculation, we arrive at

$$\begin{aligned} \widetilde{M}_{1a}(\eta) \approx & -\xi_0 D_0 \left\{ \frac{8}{Re} \left[ \sin\left(\frac{Re}{4\eta^2} + \varphi_0\right) - \frac{Re}{8\eta^2} \cos\left(\frac{Re}{4\eta^2} + \varphi_0\right) \right] \right. \\ & - \frac{16}{Re^2} \left[ \left(\frac{Re^2}{16\eta^4} + 2\right) \cos\left(\frac{Re}{4\eta^2} + \varphi_0\right) \right. \\ & \left. \left. + \frac{Re}{4\eta^2} \left(\frac{Re^2}{16\eta^4} + 2\right) \sin\left(\frac{Re}{4\eta^2} + \varphi_0\right) + \dots \right] \right\} \exp\left(-\frac{Re^2}{48\eta^6}\right), \quad (\text{B 19}) \end{aligned}$$

$$\begin{aligned} \widetilde{M}_{1b}(\eta) \approx & -\xi_0 D_0 \left\{ \eta^2 \left[ 2 \cos\left(\frac{Re}{4\eta^2} + \varphi_0\right) + \frac{Re}{2\eta^2} \sin\left(\frac{Re}{4\eta^2} + \varphi_0\right) \right] \exp\left(-\frac{Re^2}{48\eta^6}\right) \right. \\ & \left. + \left[ \left(4 - \frac{Re^2}{8\eta^4}\right) \cos\left(\frac{Re}{4\eta^2} + \varphi_0\right) + \frac{Re}{\eta^2} \sin\left(\frac{Re}{4\eta^2} + \varphi_0\right) \right] \exp\left(-\frac{Re^2}{48\eta^6}\right) + \dots \right\} e^{\eta^2}. \quad (\text{B 20}) \end{aligned}$$

For  $\eta \gg Re^{1/2}$ , (B 19) and (B 20) are reduced to

$$\widetilde{M}_{1a}(\eta) \approx -\xi_0 D_0 \left[ \frac{8 \sin \varphi_0}{Re} - \frac{32 \cos \varphi_0}{Re^2} + \frac{\cos \varphi_0}{\eta^2} - \frac{2 \cos \varphi_0}{\eta^4} + \frac{Re^2 \cos \varphi_0}{96\eta^6} + \dots \right], \quad (\text{B 21})$$

$$\widetilde{M}_{1b}(\eta) \approx -2\xi_0 D_0 \eta^2 e^{\eta^2} \left[ \cos \varphi_0 + \frac{2 \cos \varphi_0}{\eta^2} + \frac{Re^2 \cos \varphi_0}{32\eta^4} + \dots \right]. \quad (\text{B 22})$$

Then,  $\widetilde{M}_{1a}M_{1a}$  and  $\widetilde{M}_{1b}M_{1b}$  can be written, using (B 8), (B 10), (B 21) and (B 22), as

$$\widetilde{M}_{1a}M_{1a} \approx -\xi_0 D_0 \left\{ \frac{8 \sin \varphi_0}{Re} [1 + O(Re^{-1})](1 + \eta^2) + \cos \varphi_0 - \frac{\cos \varphi_0}{\eta^2} + \frac{Re^2 \cos \varphi_0}{96\eta^4} \right\}, \quad (\text{B 23})$$

$$\widetilde{M}_{1b}M_{1b} \approx -\xi_0 D_0 \frac{\cos \varphi_0}{\eta^2}. \quad (\text{B 24})$$

It follows from (3.4), (B 8), (B 10), (B 11), (B 15), (B 23) and (B 24) that

$$M_1 \approx \left[ c_{1a} - \frac{8\xi_0 \cos \beta}{Re} + O(Re^{-2}) \right] (1 + \eta^2) - \frac{\xi_0 Re^2 \cos \alpha \sin \beta}{96\eta^4} \quad (\text{for } \eta \gg Re^{1/2}). \quad (\text{B 25})$$

Since  $M_1(\eta)$  must approach zero as  $\eta \rightarrow \infty$ , we find

$$m_0 = c_{1a} \approx \frac{8\xi_0 \cos \beta}{Re}. \quad (\text{B 26})$$

Then the last term in (B 25) gives the leading order, which is consistent with the leading term in asymptotic expansion (B 7). Equations (B 8), (B 11), (B 17) and (B 18) provide a solution to (B 3) for given constants  $c_{1a}$  and  $c_{1b}$ .

Consider now an asymptotic form of  $M_1(\eta)$  at large Reynolds numbers. For  $\eta \lesssim Re^{1/4}$ , the first term on the right-hand side of (B 3) is very small, and so using (B 15) we have

$$M_1 \approx 2\gamma_0 \lambda_0 + c_{1a} M_{1a} \approx 2\gamma_0 \lambda_0 + \frac{8\xi_0 \cos \beta \eta^2}{Re}. \quad (\text{B 27})$$

For  $\eta \gg Re^{1/4}$ , on the other hand, (B 8), (B 10), (B 19) and (B 20) give asymptotic

forms of  $\widetilde{M}_{1a}M_{1a}$  and  $\widetilde{M}_{1b}M_{1b}$  as

$$\widetilde{M}_{1a}M_{1a} \approx -\xi_0 D_0 \left[ \frac{8\eta^2}{Re} \sin\left(\frac{Re}{4\eta^2} + \varphi_0\right) - \cos\left(\frac{Re}{4\eta^2} + \varphi_0\right) \right] \exp\left(-\frac{Re^2}{48\eta^6}\right), \quad (\text{B } 28)$$

$$\widetilde{M}_{1b}M_{1b} \approx -\xi_0 D_0 \left[ \frac{1}{\eta^2} \cos\left(\frac{Re}{4\eta^2} + \varphi_0\right) + \frac{Re}{4\eta^4} \sin\left(\frac{Re}{4\eta^2} + \varphi_0\right) \right] \exp\left(-\frac{Re^2}{48\eta^6}\right). \quad (\text{B } 29)$$

Since  $\widetilde{M}_{1a}M_{1a} \gg \widetilde{M}_{1b}M_{1b}$  for  $\eta \gg Re^{1/4}$ , we can express an asymptotic form of  $M_1(\eta)$  for  $\eta \gg Re^{1/4}$ , using (B 11), (B 26) and (B 28), as (3.98) in §3.5. Equation (3.98) matches (B 27) for  $\eta \ll Re^{1/3}$ , and thus (3.98) turns out to be available for any value of  $\eta$ .

### B.2. $\theta$ -dependent part with zero mean

Solutions to (B 5) are considered here. Boundary conditions to be imposed are that  $T^2F_2(\eta)$  is regular at  $\eta = 0$  and that  $T^2F_2(\eta) \rightarrow 0$  as  $\eta \rightarrow \infty$  (see (3.18)). Two source terms on the right-hand side of (B 5) have peak values at extremely different locations for  $Re \gg 1$ . The first term has a peak value of  $O(Re^{-1})$  at  $\eta = 1$  and rapidly decreases as  $\eta$  increases. The second is, on the other hand, exponentially small up to  $\eta = O(Re^{1/4})$ , and it has a peak value of  $O(Re^{-5/3})$  at  $\eta = 2^{-2/3}Re^{1/3}$  (see §3.4). Hence, for  $Re \gg 1$ , we consider two regions of values of  $\eta$  separately, that is,  $\eta = O(1)$  and  $\eta \gg 1$ .

First, in the region  $\eta = O(1)$  we take only the first term on the right-hand side of (B 5). By substituting an  $\epsilon$ -expansion,

$$F_2 = F_{20} + \epsilon F_{21} + \dots \quad (\text{B } 30)$$

into (B 5), we obtain, at order  $\epsilon^0$ ,

$$F_{20}'' + \frac{1}{\eta} F_{20}' - \frac{4}{\eta^2} F_{20} + \frac{4\eta^2}{e^{\eta^2} - 1} F_{20} = 0, \quad (\text{B } 31)$$

which is the same as (A 5), and therefore

$$F_{20}(\eta) \equiv 0. \quad (\text{B } 32)$$

At order  $\epsilon$  of (B 5), by putting

$$F_{21}(\eta) = iB_1 e^{-i\phi_1} f_2(\eta), \quad (\text{B } 33)$$

we obtain

$$f_2'' + \frac{1}{\eta} f_2' - \frac{4}{\eta^2} f_2 = \frac{4\eta^2}{e^{\eta^2} - 1} (\eta^2 - f_2), \quad (\text{B } 34)$$

which is identical to (A 9). Hence, we can take

$$\text{Re}[f_2(\eta)] \equiv f_1(\eta), \quad \text{Im}[f_2(\eta)] \equiv 0, \quad (\text{B } 35)$$

so that

$$F_{21}(\eta) = iB_1 e^{-i\phi_1} f_1(\eta). \quad (\text{B } 36)$$

Combination of (A 9), (B 30), (B 32) and (B 36) leads to the first equation of (3.99) in §3.5.

By introducing

$$\epsilon M_2(\eta) = -\frac{1}{4} \left( F_2'' + \frac{1}{\eta} F_2' - \frac{4}{\eta^2} F_2 \right) \quad (\text{B } 37)$$

and using (3.78) we may express the  $\theta$ -dependent part of vorticity associated with streamfunction  $\text{Re}[F_2(\eta)e^{-i2\theta}]$  as

$$-\frac{1}{4}\nabla_\eta^2 \text{Re}[F_2(\eta)e^{-i2\theta}] = \epsilon \text{Re}[M_2(\eta)e^{-i2\theta}]. \quad (\text{B } 38)$$

Second, we consider the region  $\eta \gg 1$  in which all the terms that include  $e^{-\eta^2}$  may be neglected. By dropping them and using (B 37), we can rewrite (B 5) as

$$M_2'' + \left(2\eta + \frac{1}{\eta}\right) M_2' + 2 \left(\frac{iRe - 2}{\eta^2} - 2\right) M_2 = 2\xi_0 \eta f'. \quad (\text{B } 39)$$

Boundary conditions to be imposed are that  $TM_2(\eta)$  is regular at  $\eta = 0$  and that  $TM_2(\eta) \rightarrow 0$  as  $\eta \rightarrow \infty$  (see (3.17)). Using (3.28) and (3.27), we can expand a solution to (B 39) as

$$M_2(\eta) = -\xi_0 c_0 \left[ \frac{iRe}{4(iRe + 6)} \eta^4 - \frac{24iRe + (iRe + 6)(8iRe - Re^2)}{48(iRe + 6)(iRe + 16)} \eta^6 + \dots \right] \quad (\text{for } \eta \ll Re^{-1/2}), \quad (\text{B } 40)$$

and

$$M_2(\eta) = -\xi_0 D_0 e^{i\varphi_0} \left[ \frac{iRe}{8\eta^2} - \frac{Re^2}{24\eta^4} - \frac{(11iRe + 72)Re^2}{1536\eta^6} + \dots \right] \quad (\text{for } \eta \gg Re^{1/2}), \quad (\text{B } 41)$$

where  $c_0$  is an exponentially small constant given by (3.57).

Now, following the asymptotic analysis performed for (3.26) in §3.4, we employ the WKB method in order to construct solutions to the homogeneous equation associated with (B 39). Let us introduce a new dependent variable  $g_2(\eta)$  by

$$M_2(\eta) = \eta^{-1/2} e^{-\eta^2/2} g_2(\eta). \quad (\text{B } 42)$$

By substituting (B 42) into (B 39), we obtain

$$g_2'' + \left(\frac{2iRe}{\eta^2} - \eta^2 - 6 - \frac{15}{4\eta^2}\right) g_2 = 2\xi_0 \eta^{3/2} e^{\eta^2/2} f'. \quad (\text{B } 43)$$

We consider the region  $\eta = O(Re^{1/4})$  and put  $\eta = Re^{1/4}\chi$ . Then, (B 43) becomes

$$g_2'' + Re \left[ \frac{2i}{\chi^2} - \chi^2 - 6Re^{-1/2} + O(Re^{-1}) \right] g_2 = 2Re^{3/4} \xi_0 \chi^{3/2} \exp\left(\frac{1}{2}Re^{1/2}\chi^2\right) f'. \quad (\text{B } 44)$$

The homogeneous equation associated with (B 44) has two linearly independent WKB solutions

$$g_{2a} = \chi^{1/2} (\chi^4 - 2i)^{-1/4} (\chi^2 + (\chi^4 - 2i)^{1/2})^{-3/2} \exp(-Re^{1/2}\sigma_1(\chi)), \quad (\text{B } 45)$$

$$g_{2b} = \chi^{1/2} (\chi^4 - 2i)^{-1/4} (\chi^2 + (\chi^4 - 2i)^{1/2})^{3/2} \exp(Re^{1/2}\sigma_1(\chi)), \quad (\text{B } 46)$$

where

$$\sigma_1(\chi) = 2^{-1/2} e^{\pi i/4} \left[ e^{-\pi i/4} (\frac{1}{2}\chi^4 - i)^{1/2} - \arctan(e^{-\pi i/4} (\frac{1}{2}\chi^4 - i)^{1/2}) + \frac{1}{2}\pi \right]. \quad (\text{B } 47)$$

For small values of  $\chi$ , function  $\sigma_1(\chi)$  may be expressed asymptotically as

$$\sigma_1 = \sqrt{2} e^{-\pi i/4} \ln \chi + \rho_1 + O(\chi^4), \quad (\text{B } 48)$$

where

$$\rho_1 = \frac{3}{2} 2^{-1/2} e^{3\pi i/4} \ln 2 + \frac{1}{2} \left[ \frac{1}{4}\pi + 1 + i \left( \frac{1}{4}\pi - 1 \right) \right]. \quad (\text{B } 49)$$

For large  $\chi$ , on the other hand, it is expanded as

$$\sigma_1 = \frac{1}{2}\chi^2 + \frac{i}{2\chi^2} - \frac{1}{12\chi^6} + \cdots. \quad (\text{B } 50)$$

Then, we use (B 48) and (B 50), and restore the original variable  $\eta$  to obtain, for  $\eta \ll Re^{1/4}$  ( $\chi \ll 1$ ),

$$g_{2a}(\eta) \approx Re^{-1/8} 2^{-1} \eta^{1/2} \exp(-Re^{1/2}(\sqrt{2}e^{-\pi i/4} \ln \eta + \sqrt{2}e^{3\pi i/4} \ln Re^{1/4} + \rho_1)), \quad (\text{B } 51)$$

$$g_{2b}(\eta) \approx Re^{-1/8} \sqrt{2}e^{-\pi i/4} \eta^{1/2} \exp(Re^{1/2}(\sqrt{2}e^{-\pi i/4} \ln \eta + \sqrt{2}e^{3\pi i/4} \ln Re^{1/4} + \rho_1)), \quad (\text{B } 52)$$

and, for  $\eta \gg Re^{1/4}$  ( $\chi \gg 1$ ),

$$g_{2a}(\eta) \approx 2^{-3/2} Re^{7/8} \eta^{-7/2} \exp\left(-\frac{1}{2}\eta^2 - \frac{iRe}{2\eta^2} + \frac{Re^2}{12\eta^6}\right), \quad (\text{B } 53)$$

$$g_{2b}(\eta) \approx 2^{3/2} Re^{-5/8} \eta^{5/2} \exp\left(\frac{1}{2}\eta^2 + \frac{iRe}{2\eta^2} - \frac{Re^2}{12\eta^6}\right). \quad (\text{B } 54)$$

In order to solve the full equation (B 44), we write a solution in terms of (B 45) and (B 46) as

$$g_2(\eta) = (c_{2a} + \tilde{g}_{2a}(\eta))g_{2a}(\eta) + (c_{2b} + \tilde{g}_{2b}(\eta))g_{2b}(\eta), \quad (\text{B } 55)$$

where

$$\tilde{g}_{2a}(\eta) = -2\xi_0 \int_0^\eta \frac{s^{3/2} e^{1/2s^2} f'(s) g_{2b}(s)}{W(s)} ds, \quad (\text{B } 56)$$

$$\tilde{g}_{2b}(\eta) = -2\xi_0 \int_\eta^\infty \frac{s^{3/2} e^{1/2s^2} f'(s) g_{2a}(s)}{W(s)} ds, \quad (\text{B } 57)$$

with

$$W(\eta) = g_{2a}(\eta)g'_{2b}(\eta) - g'_{2a}(\eta)g_{2b}(\eta). \quad (\text{B } 58)$$

It can be shown from (B 51)–(B 54) that

$$W(\eta) \approx 2Re^{1/4}. \quad (\text{B } 59)$$

Constants  $c_{2a}$  and  $c_{2b}$  are determined by the boundary conditions at  $\eta = 0$  and  $\infty$ . For  $\eta \ll Re^{1/4}$ , since  $f(\eta)$  and therefore the right-hand side of (B 44) are very small, (B 55) is written, using (B 51) and (B 52), as

$$g_2 \approx c_{2a}g_{2a} + c_{2b}g_{2b} \approx c_{2a}2^{-1}Re^{-1/8}e^{\pi i/2}q_1^{-1}\eta^{-q_2+\frac{1}{2}} + c_{2b}\sqrt{2}Re^{-1/8}e^{-\pi i/4}q_1\eta^{q_2+\frac{1}{2}}, \quad (\text{B } 60)$$

where

$$q_1 = \exp(Re^{1/2}(\sqrt{2}e^{3\pi i/4} \ln Re^{1/4} + \rho_1)), \quad (\text{B } 61)$$

$$q_2 = Re^{1/2} \sqrt{2}e^{-\pi i/4}. \quad (\text{B } 62)$$

Since  $\text{Re}(q_2) = Re^{1/2}$ , and  $q_1$  is exponentially small for  $Re \gg 1$ , we obtain, using definition (B 42) of  $g_2$ ,

$$M_2 \approx c_{2a}2^{-1}Re^{-1/8}q_1^{-1}\eta^{-q_2} \quad (\text{at } \eta \ll Re^{1/4}). \quad (\text{B } 63)$$

Hence, in order that  $M_2$  may be connected smoothly with a regular solution at  $\eta = 0$  we must take

$$c_{2a} = 0. \quad (\text{B } 64)$$

For  $\eta \gg Re^{1/4}$ , it can be shown by using (3.64), (B 53) and (B 54) that

$$\tilde{g}_{2a} \approx -i\sqrt{2}\xi_0 D_0 e^{i\varphi_0} Re^{1/8} \int_0^\eta s \exp\left(s^2 + \frac{3iRe}{4s^2} - \frac{5Re^2}{48s^6}\right) ds, \quad (\text{B } 65)$$

$$\tilde{g}_{2b} \approx -i2^{-5/2}\xi_0 D_0 e^{i\varphi_0} Re^{13/8} \int_\eta^\infty \frac{1}{s^5} \exp\left(-\frac{iRe}{4s^2} + \frac{Re^2}{16s^6}\right) ds \quad (\text{B } 66)$$

because  $f$  is very small up to  $\eta = O(Re^{1/4})$  (see §3.4). Integrations by parts of the integrals in (B 65) and (B 66) respectively lead to

$$\tilde{g}_{2a} \approx -i2^{-1/2}\xi_0 D_0 e^{i\varphi_0} Re^{1/8} \exp\left(\eta^2 + \frac{3iRe}{4\eta^2} - \frac{5Re^2}{48\eta^6}\right), \quad (\text{B } 67)$$

$$\tilde{g}_{2b} \approx -i\sqrt{2}\xi_0 D_0 e^{i\varphi_0} Re^{-3/8} \left[ \left(1 + \frac{iRe}{4\eta^2}\right) \exp\left(-\frac{iRe}{4\eta^2} + \frac{Re^2}{16\eta^6}\right) - 1 \right]. \quad (\text{B } 68)$$

Thus, it follows from (B 42), (B 53)–(B 55), (B 64), (B 67) and (B 68) that

$$\begin{aligned} M_2(\eta) &\approx c_{2b} 2^{3/2} Re^{-5/8} \eta^2 \exp\left(\frac{iRe}{2\eta^2} - \frac{Re^2}{12\eta^6}\right) \\ &\quad - i\xi_0 D_0 e^{i\varphi_0} \frac{4\eta^2}{Re} \left[ \left(1 + \frac{iRe}{4\eta^2}\right) \exp\left(\frac{iRe}{4\eta^2} - \frac{Re^2}{48\eta^6}\right) - \exp\left(\frac{iRe}{2\eta^2} - \frac{Re^2}{12\eta^6}\right) \right] \\ &\quad - i\xi_0 D_0 e^{i\varphi_0} \frac{Re}{4\eta^4} \exp\left(\frac{iRe}{4\eta^2} - \frac{Re^2}{48\eta^6}\right). \end{aligned} \quad (\text{B } 69)$$

Furthermore, for  $\eta \gg Re^{1/2}$ , we expand (B 69) in a series of inverse powers of  $\eta$  to obtain

$$M_2 \approx c_{2b} 2^{3/2} Re^{-5/8} \eta^2 - \xi_0 D_0 e^{i\varphi_0} \frac{iRe}{8\eta^2}. \quad (\text{B } 70)$$

Function  $M_2(\eta)$  must vanish as  $\eta \rightarrow \infty$ . This requires that

$$c_{2b} = 0, \quad (\text{B } 71)$$

and then the last term in (B 70) gives the leading order, which is consistent with the leading term in asymptotic expansion (B 41). Given constants  $c_{2a}$  and  $c_{2b}$ , then (B 55), (B 65) and (B 66) provide a solution to (B 44), and  $M_2(\eta)$  is determined by (B 42).

Let us summarize an asymptotic form of  $M_2(\eta)$  at a large Reynolds number. For  $1 \ll \eta \lesssim Re^{1/4}$ , the right-hand side of (B 39) is very small, and so is  $M_2(\eta)$  because  $c_{2a} = c_{2b} = 0$ . For  $\eta \gg Re^{1/4}$ , on the other hand, the second term on the left-hand side of (B 69) dominates the third (the first term has already vanished), and then an asymptotic form of  $M_2(\eta)$  is given by the second equation of (3.99) in §3.5.

#### REFERENCES

- BERNARD, P. S., THOMAS, J. M. & HANDLER, R. A. 1993 Vortex dynamics and the production of Reynolds stress. *J. Fluid Mech.* **253**, 385–419.
- BROOKE, J. W. & HANRATTY, T. J. 1993 Origin of turbulence-producing eddies in a channel flow. *Phys. Fluids A* **5**, 1011–1022.
- CHOI, H., MOIN, P. & KIM, J. 1993 Direct numerical simulation of turbulent flow over riblets. *J. Fluid Mech.* **255**, 503–539.
- DOUADY, D., COUDER, Y. & BRACHET, M. E. 1991 Direct observation of the intermittency of intense vorticity filaments in turbulence. *Phys. Rev. Lett.* **67**, 983–986.



- HOSOKAWA, I. & YAMAMOTO, K. 1989 Fine structure of a directly simulated isotropic turbulence. *J. Phys. Soc. Japan* **58**, 20–23.
- HUSSAIN, A. K. M. F. 1986 Coherent structures and turbulence. *J. Fluid Mech.* **173**, 303–356.
- JIMÉMEZ, J., MOFFATT, H. K. & VASCO, C. 1996 The structure of the vortices in freely decaying two-dimensional turbulence. *J. Fluid Mech.* **313**, 209–222.
- JIMÉMEZ, J. & MOIN, P. 1991 The minimal flow unit in near-wall turbulence. *J. Fluid Mech.* **225**, 213–240.
- JIMÉMEZ, J., WRAY, A. A., SAFFMAN, P. G. & ROGALLO, R. S. 1993 The structure of intense vorticity in homogeneous isotropic turbulence. *J. Fluid Mech.* **255**, 65–90.
- KERR, R. M. 1985 Higher-order derivative correlation and the alignment of small scale structures in isotropic turbulence. *J. Fluid Mech.* **153**, 31–58.
- KIDA, S. 1993 Tube-like structures in turbulence. *Lecture Notes in Numerical Applied Analysis*, **12**, 137–159.
- KIDA, S. & OHKITANI, K. 1992 Spatio-temporal intermittency and instability of a forced turbulence. *Phys. Fluids A* **4**, 1018–1027.
- KIDA, S. & TANAKA, M. 1992 Reynolds stress and vortical structure in a uniformly sheared turbulence. *J. Phys. Soc. Japan* **61**, 4400–4417.
- KIDA, S. & TANAKA, M. 1994 Dynamics of vortical structures in a homogeneous shear flow. *J. Fluid Mech.* **274**, 43–68.
- KRAVCHENKO, A. G., CHOI, H. & MOIN, P. 1993 On the relation of near-wall streamwise vortices to wall skin friction in turbulent boundary layers. *Phys. Fluids A* **5**, 3307–3309.
- LUNDGREN, T. S. 1982 Strained spiral vortex model for turbulent fine structure. *Phys. Fluids* **25**, 2193–2203.
- MIYAKE, Y. & TSUJIMOTO, K. 1996 Regeneration and self-sustenance of quasi-streamwise vortices. Submitted to *Phys. Fluids*.
- MOFFATT, H. K., KIDA, S. & OHKITANI, K. 1994 Stretched vortices – the sinews of turbulence; large-Reynolds-number asymptotics. *J. Fluid Mech.* **259**, 241–264.
- MOORE, D. W. 1985 The interaction of a diffusing line vortex and aligned shear flow. *Proc. R. Soc. Lond. A* **399**, 367–375.
- PEARSON, C. F. & ABERNATHY, F. H. 1984 Evolution of the flow field associated with a streamwise diffusing vortex. *J. Fluid Mech.* **146**, 271–283.
- ROBINSON, S. K. 1991 The kinematics of turbulent boundary layer structure. *NASA Tech. Mem.* 103859.
- ROBINSON, S. K., KLINE, S. J. & SPALART, P. R. 1988 Quasi-coherent structures in the turbulent boundary layer: Part II. Verification and new information from a numerically simulated flat-plate layer. In *Near-Wall turbulence* (ed. S. J. Kline & N. H. Afgan), pp. 218–247. Hemisphere.
- RUETSCH, G. R. & MAXEY, M. R. 1991 Small scale features of vorticity and passive scalar fields in homogeneous isotropic turbulence. *Phys. Fluids A* **3**, 1587–1597.
- SENDSTAD, O. & MOIN, P. 1992 The near-wall mechanics of three-dimensional boundary layers. *Rep. TF-57*. Thermoscience Division, Department of Mechanical Engineering, Stanford University, Stanford, CA.
- SHE, Z.-S., JACKSON, E. & ORSZAG, S. A. 1990 Intermittent vortex structures in homogeneous isotropic turbulence. *Nature* **344**, 226–228.
- SIGGIA, E. D. 1981 Numerical study of small scale intermittency in three-dimensional turbulence. *J. Fluid Mech.* **107**, 375–406.
- VINCENT, A. & MENEGUZZI, M. 1991 The spatial structure and statistical properties of homogeneous turbulence. *J. Fluid Mech.* **225**, 1–25.